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Hadamard k-fractional inequalities of Fejér type for GA-s-convex mappings and applications

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Abstract

The main aim of this paper is to establish some Fejér-type inequalities involving hypergeometric functions in terms of GA-s-convexity. For this purpose, we construct a Hadamard k-fractional identity related to geometrically symmetric mappings. Moreover, we give the upper and lower bounds for the weighted inequalities via products of two different mappings. Some applications of the presented results to special means are also provided.

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1 Introduction

Let $f:[\mu,\nu]\to\mathbb{R}$ be a convex mapping with $\mu<\nu$, and let $g:[\mu,\nu]\to\mathbb{R}$ be a non-negative, integrable and symmetric mapping corresponding to $\frac{\mu+\nu}{2}$. Then one has

$$f\left(\frac{\mu+\nu}{2}\right)\int_\mu^\nu g(x)\,\mathrm{d}x\le\int_\mu^\nu f(x)g(x)\,\mathrm{d}x\le\frac{f(\mu)+f(\nu)}{2}\int_\mu^\nu g(x)\,\mathrm{d}x, \tag{1.1}$$

which is called a Fejér-type inequality.

If we take $g(x)=1$ in (1.1), then inequality (1.1) reduces to the Hermite–Hadamard inequality,

$$f\left(\frac{\mu+\nu}{2}\right)\le\frac{1}{\nu-\mu}\int_\mu^\nu f(x)\,\mathrm{d}x\le\frac{f(\mu)+f(\nu)}{2}. \tag{1.2}$$

To see more recent results and the related extensions corresponding to (1.1) and (1.2), we refer the interested reader to [1, 2, 6, 7, 10–12, 19, 21, 23–25, 28, 29, 34] and the references therein.

Let us recall that Niculescu [26] introduced and considered a class of mappings, called GA-convex mappings, as follows: A mapping $f:\mathcal{I}\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ is said to be GA-convex on $\mathcal{I}$ if

$$f(\mu^{\lambda}\nu^{1-\lambda}) \le \lambda f(\mu) + (1-\lambda)f(\nu) \tag{1.3}$$

for all $\mu, \nu \in \mathcal{I}$ and $\lambda \in [0,1]$.

Using mappings whose first derivative in absolute value are GA-convex, Latif et al. [22] proved the following estimation-type result for the right-middle part of Fejér-type inequality (1.1).

Theorem 1.1 *Let $f : \mathcal{I} \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ and $\mu, \nu \in \mathcal{I}^\circ$ with $\mu < \nu$ satisfying that $f' \in L^1([\mu,\nu])$, and let $g : [\mu,\nu] \to [0,\infty)$ be a continuous positive mapping geometrically symmetric with respect to $\sqrt{\mu\nu}$, i.e. $g(\frac{\mu\nu}{x}) = g(x)$. If $|f'|^q$ is GA-convex on $[\mu,\nu]$ for $q > 1$, then the following inequality holds*:

$$\begin{aligned} &\left|\frac{f(\mu)+f(\nu)}{2}\int_{\mu}^{\nu}\frac{g(x)}{x}\,\mathrm{d}x - \int_{\mu}^{\nu}\frac{f(x)g(x)}{x}\,\mathrm{d}x\right| \\ &\quad\le \frac{(\ln\nu-\ln\mu)^{2-1/q}\|g\|_\infty}{4\cdot q^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \\ &\qquad\times \Big\{\nu^{1/2}\Big(\big[L\big(\mu^{q/2},\nu^{q/2}\big)-\mu^{q/2}\big]\big|f'(\mu)\big|^q + \big[2\nu^{q/2}-\mu^{q/2}-L\big(\mu^{q/2},\nu^{q/2}\big)\big]\big|f'(\nu)\big|^q\Big)^{1/q} \\ &\qquad+ \mu^{1/2}\Big(\big[L\big(\mu^{q/2},\nu^{q/2}\big)+\nu^{q/2}-2\mu^{q/2}\big]\big|f'(\mu)\big|^q + \big[\nu^{q/2}-L\big(\mu^{q/2},\nu^{q/2}\big)\big]\big|f'(\nu)\big|^q\Big)^{1/q}\Big\}, \end{aligned} \tag{1.4}$$

where

$$\|g\|_\infty = \sup_{x\in[\mu,\nu]} g(x) < \infty$$

and

$$L(\rho,\varrho) = \frac{\rho-\varrho}{\ln\rho-\ln\varrho}$$

for $\rho, \varrho > 0$ with $\rho \ne \varrho$.

More integral inequalities considering GA-convex mappings can be found in [4, 5, 13, 16].

Motivated by the research going on this dynamic field, Shuang et al. [30] presented a new class of GA-convex mappings, which is named the GA-s-convex mapping. For the recent results and details, the interested reader is directed to [14, 17] and the references cited therein.

Definition 1.1 ([30]) A mapping $f : \mathcal{I} \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is named GA-s-convex mapping on $\mathcal{I}$ if

$$f(\mu^{\lambda}\nu^{1-\lambda}) \le \lambda^s f(\mu) + (1-\lambda)^s f(\nu)$$

holds for all $\mu, \nu \in \mathcal{I}$, $\lambda \in [0,1]$ and for certain fixed $s \in (0,1]$.

Fractional calculus, as a very useful tool, shows its significance to implement differentiation and integration of real or complex number orders. This topic has attracted much attention from researchers during the last few decades. Among a lot of the fractional integral operators that appeared, because of applications in many fields of sciences, the Riemann–Liouville fractional integral operator and Hadamard fractional integral operator have been extensively studied.

An important generalization of Hadamard fractional integrals was considered by Iqbal et al. in [15] which is named the Hadamard k-fractional integral operators.

Definition 1.2 Let $f \in L^1[a,b]$, then the left-sided and right-sided Hadamard k-fractional integrals of order $\alpha \in \mathbb{R}^+$ and $k, a \in \mathbb{R}^+$ are defined as

$$ {}_k\mathcal{H}^{\alpha}_{a^+}\{f(t)\} = \frac{1}{k\Gamma_k(\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\frac{\alpha}{k}-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad (0 < a < t \le b) $$

and

$$ {}_k\mathcal{H}^{\alpha}_{b^-}\{f(t)\} = \frac{1}{k\Gamma_k(\alpha)} \int_t^b \left(\ln\left(\frac{\tau}{t}\right)\right)^{\frac{\alpha}{k}-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad (0 < a \le t < b) $$

respectively, where $\Gamma_k(\alpha)$ is the k-gamma function defined by $\Gamma_k(\alpha) = \int_0^\infty \tau^{\alpha-1}e^{-\frac{\tau^k}{k}}\,\mathrm{d}\tau$. Furthermore, $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$.

Some important inequalities pertaining Hadamard k-fractional integrals can be found in [3, 27, 32].

The following theorem, involving Hadamard-type k-fractional integral operators, is a direct generalization of Theorem 2.6 established by Kunt et al. in [20].

Theorem 1.2 *Let $f : [\mu,\nu] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a GA-convex mapping, $\alpha > 0$ and $f \in L^1[\mu,\nu]$. If $g : [\mu,\nu] \to \mathbb{R}$ is non-negative, integral and geometrically symmetric corresponding to $\sqrt{\mu\nu}$, then the following inequality for Hadamard-type k-fractional integral operators holds*:

$$ \begin{aligned} & f(\sqrt{\mu\nu})\left[{}_k\mathcal{H}^{\alpha}_{\sqrt{\mu\nu}^-}\{g(\mu)\} + {}_k\mathcal{H}^{\alpha}_{\sqrt{\mu\nu}^+}\{g(\nu)\}\right] \\ &\quad \le \left[{}_k\mathcal{H}^{\alpha}_{\sqrt{\mu\nu}^-}\{(fg)(\mu)\} + {}_k\mathcal{H}^{\alpha}_{\sqrt{\mu\nu}^+}\{(fg)(\nu)\}\right] \\ &\quad \le \frac{f(\mu)+f(\nu)}{2}\left[{}_k\mathcal{H}^{\alpha}_{\sqrt{\mu\nu}^-}\{g(\mu)\} + {}_k\mathcal{H}^{\alpha}_{\sqrt{\mu\nu}^+}\{g(\nu)\}\right]. \end{aligned} \tag{1.5} $$

It is easy to observe that, for $k = 1$ in Definition 1.2, we have the definition of the left-sided and right-sided Hadamard fractional integrals, i.e.

$$ \mathcal{H}^{\alpha}_{a^+}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad (0 < a < t \le b) $$

and

$$ \mathcal{H}^{\alpha}_{b^-}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln\left(\frac{\tau}{t}\right)\right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad (0 < a \le t < b). $$

For more details corresponding to the Hadamard fractional integral inequalities, the interested reader is directed to Refs. [8, 31, 33] and the references cited therein.

Consider the Hadamard fractional inequality of the Fejér type with respect to GA-convexity, Kunt [20] obtained the following theorem related to the right-middle part of inequality (1.1).

Theorem 1.3 *Let $f:\mathcal{I}\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable mapping on $\mathcal{I}^\circ$ satisfying that $f\in L^1[\mu,\nu]$ where $\mu,\nu\in\mathcal{I}$ with $\mu<\nu$ and $\alpha>0$. If $|f'|$ is GA-convex on $[\mu,\nu]$ and $g:[\mu,\nu]\to\mathbb{R}$ is continuous and geometrically symmetric corresponding to $\sqrt{\mu\nu}$, then the following inequality via fractional integrals holds*:

$$\begin{aligned}&\left|\frac{f(\mu)+f(\nu)}{2}\left[\mathcal{H}^{\alpha}_{\mu^+}g(\nu)+\mathcal{H}^{\alpha}_{\nu^-}g(\mu)\right]-\left[\mathcal{H}^{\alpha}_{\mu^+}(fg)(\nu)+\mathcal{H}^{\alpha}_{\nu^-}(fg)(\mu)\right]\right|\\&\quad\leq\frac{\|g\|_\infty\ln^{\alpha+1}(\frac{\nu}{\mu})}{\Gamma(\alpha+1)}\left[\mathbb{A}_1(\alpha)\left|f'(\mu)\right|+\mathbb{A}_2(\alpha)\left|f'(\nu)\right|\right],\end{aligned}\tag{1.6}$$

where

$$\mathbb{A}_1(\alpha)=\int_0^{\frac{1}{2}}\left[(1-\xi)^\alpha-\xi^\alpha\right]\left[(1-\xi)a^{1-\xi}b^{\xi}+\xi a^{\xi}b^{1-\xi}\right]\mathrm{d}\xi$$

and

$$\mathbb{A}_2(\alpha)=\int_0^{\frac{1}{2}}\left[(1-\xi)^\alpha-\xi^\alpha\right]\left[\xi a^{1-\xi}b^{\xi}+(1-\xi)a^{\xi}b^{1-\xi}\right]\mathrm{d}\xi.$$

Different from [22], our purpose in this paper is to obtain, using the Hadamard k-fractional integrals, certain estimation-type results for the left-middle part of a Fejér-type inequality in terms of GA-s-convexity. We also get the upper and lower bounds for the weighted Hadamard-type inequalities via product of two different mappings.

2 Some preliminary lemmas

In this section, we state the following lemmas, which are useful in the proofs of our main results.

Lemma 2.1 *For $\mathcal{U},\mathcal{V}>0$, we have*

(i)

$$\Phi(\mathcal{U},\mathcal{V}):=\int_0^1\mathcal{U}^{1-\frac{t}{2}}\mathcal{V}^{\frac{t}{2}}\,\mathrm{d}t=\begin{cases}\frac{2(\mathcal{U}-\sqrt{\mathcal{U}\mathcal{V}})}{\ln\mathcal{U}-\ln\mathcal{V}}, & \mathcal{U}\neq\mathcal{V},\\ \mathcal{U}, & \mathcal{U}=\mathcal{V},\end{cases}\tag{2.1}$$

(ii)

$$\Psi(\mathcal{U},\mathcal{V}):=\frac{1}{2}\int_0^1 t\mathcal{U}^{1-\frac{t}{2}}\mathcal{V}^{\frac{t}{2}}\,\mathrm{d}t=\begin{cases}\frac{\sqrt{\mathcal{U}}[2\sqrt{\mathcal{U}}+\sqrt{\mathcal{V}}(\ln\mathcal{V}-\ln\mathcal{U}-2)]}{(\ln\mathcal{U}-\ln\mathcal{V})^2}, & \mathcal{U}\neq\mathcal{V},\\ \frac{1}{4}\mathcal{U}, & \mathcal{U}=\mathcal{V},\end{cases}\tag{2.2}$$

(iii)

$$\int_0^1 t^\sigma \mathcal{U}^{1-\frac{t}{2}}\mathcal{V}^{\frac{t}{2}}\,\mathrm{d}t \le \frac{\sqrt{\mathcal{U}\mathcal{V}}(\sigma+1)+\mathcal{U}}{(\sigma+1)(\sigma+2)} := \Upsilon(\mathcal{U},\mathcal{V},\sigma), \quad \sigma \neq -1,-2. \tag{2.3}$$

Proof The proofs of (i) and (ii) follow from a straightforward computation.

The proof of (iii) is as follows.

Using the inequality of $u^s \le (u-1)s+1$ for all $0 \le s \le 1$ with $u > 0$, we have

$$\begin{aligned}\int_0^1 t^\sigma \mathcal{U}^{1-\frac{t}{2}}\mathcal{V}^{\frac{t}{2}}\,\mathrm{d}t &= \mathcal{U}\int_0^1 t^\sigma \left(\mathcal{U}^{-\frac{1}{2}}\mathcal{V}^{\frac{1}{2}}\right)^t \mathrm{d}t \\ &\le \mathcal{U}\int_0^1 t^\sigma \left[\left(\mathcal{U}^{-\frac{1}{2}}\mathcal{V}^{\frac{1}{2}}-1\right)t+1\right]\mathrm{d}t \\ &= \frac{\sqrt{\mathcal{U}\mathcal{V}}(\sigma+1)+\mathcal{U}}{(\sigma+1)(\sigma+2)} := \Upsilon(\mathcal{U},\mathcal{V},\sigma).\end{aligned} \tag{2.4}$$

This ends the proof. □

Lemma 2.2 *If $w:[a,b]\subset(0,\infty)\to\mathbb{R}$ is integrable and geometrically symmetric corresponding to $\sqrt{ab}$ with $a<b$ and $k,\alpha>0$, then we have*

$${}_k\mathcal{H}^\alpha_{\sqrt{ab}^+}\{w(b)\} = {}_k\mathcal{H}^\alpha_{\sqrt{ab}^-}\{w(a)\} = \frac{1}{2}\left[{}_k\mathcal{H}^\alpha_{\sqrt{ab}^-}\{w(a)\} + {}_k\mathcal{H}^\alpha_{\sqrt{ab}^+}\{w(b)\}\right]. \tag{2.5}$$

Proof Using the geometrically symmetry of w with respect to $\sqrt{ab}$, we have $w(\frac{ab}{x}) = w(x)$, for all $x\in[a,b]$. If we set $x=\frac{ab}{t}$, then we have

$$\begin{aligned}{}_k\mathcal{H}^\alpha_{\sqrt{ab}^-}\{w(a)\} &= \frac{1}{k\Gamma_k(\alpha)}\int_a^{\sqrt{ab}}\left(\ln\left(\frac{x}{a}\right)\right)^{\frac{\alpha}{k}-1} w(x)\frac{\mathrm{d}x}{x} \\ &= \frac{1}{k\Gamma_k(\alpha)}\int_{\sqrt{ab}}^b\left(\ln\left(\frac{b}{t}\right)\right)^{\frac{\alpha}{k}-1} w\left(\frac{ab}{t}\right)\frac{\mathrm{d}t}{t} \\ &= \frac{1}{k\Gamma_k(\alpha)}\int_{\sqrt{ab}}^b\left(\ln\left(\frac{b}{t}\right)\right)^{\frac{\alpha}{k}-1} w(t)\frac{\mathrm{d}t}{t} \\ &= {}_k\mathcal{H}^\alpha_{\sqrt{ab}^+}\{w(b)\}.\end{aligned}$$

□

Lemma 2.3 *Let $f:\mathcal{I}\subseteq\mathbb{R}^+=(0,\infty)\to\mathbb{R}$, be a differentiable mapping on $\mathcal{I}^\circ$ (the interior of $\mathcal{I}$), $a,b\in\mathcal{I}$ with $a<b$, and let $g:[a,b]\to\mathbb{R}$ be a continuous positive mapping geometrically symmetric to $\sqrt{ab}$. If $f'\in L^1([a,b])$, then the following equality for Hadamard k-fractional integral operators with $k,\alpha>0$ holds*:

$$\begin{aligned}&f(\sqrt{ab})\left[{}_k\mathcal{H}^\alpha_{\sqrt{ab}^-}\{g(a)\} + {}_k\mathcal{H}^\alpha_{\sqrt{ab}^+}\{g(b)\}\right] - \left[{}_k\mathcal{H}^\alpha_{\sqrt{ab}^-}\{(fg)(a)\} + {}_k\mathcal{H}^\alpha_{\sqrt{ab}^+}\{(fg)(b)\}\right] \\ &\quad= \frac{1}{k\Gamma_k(\alpha)}\frac{\ln b-\ln a}{2}\int_0^1\left[\int_a^{L(t)}\left(\ln\frac{s}{a}\right)^{\frac{\alpha}{k}-1} g(s)\frac{\mathrm{d}s}{s}\right]\left[L(t)f'\big(L(t)\big) - U(t)f'\big(U(t)\big)\right]\mathrm{d}t,\end{aligned} \tag{2.6}$$

where $L(t)=a^{1-\frac{t}{2}}b^{\frac{t}{2}}$ and $U(t)=a^{\frac{t}{2}}b^{1-\frac{t}{2}}$.

Proof Let

$$I_1 = \frac{1}{k\Gamma_k(\alpha)} \frac{\ln b - \ln a}{2} \int_0^1 \left[\int_a^{L(t)} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} \right] L(t) f'\big(L(t)\big) \,\mathrm{d}t \tag{2.7}$$

and

$$I_2 = \frac{1}{k\Gamma_k(\alpha)} \frac{\ln b - \ln a}{2} \int_0^1 \left[\int_a^{L(t)} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} \right] U(t) f'\big(U(t)\big) \,\mathrm{d}t. \tag{2.8}$$

Integrating by parts, we have

$$\begin{aligned}
I_1 &= \frac{1}{k\Gamma_k(\alpha)} \frac{\ln b - \ln a}{2} \int_0^1 \left[\int_a^{L(t)} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} \right] L(t) f'\big(L(t)\big) \,\mathrm{d}t \\
&= \frac{1}{k\Gamma_k(\alpha)} \int_0^1 \left[\int_a^{L(t)} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} \right] \mathrm{d}\big[f\big(L(t)\big)\big] \\
&= \frac{1}{k\Gamma_k(\alpha)} \left\{ \left[\int_a^{L(t)} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} \right] f\big(L(t)\big) \Big|_0^1 \right. \\
&\quad \left. - \frac{\ln b - \ln a}{2} \int_0^1 \left(\ln \frac{L(t)}{a} \right)^{\frac{\alpha}{k}-1} f\big(L(t)\big) g\big(L(t)\big) \,\mathrm{d}t \right\} \\
&= \frac{1}{k\Gamma_k(\alpha)} \left\{ f(\sqrt{ab}) \int_a^{\sqrt{ab}} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} - \int_a^{\sqrt{ab}} \left(\ln \frac{x}{a} \right)^{\frac{\alpha}{k}-1} f(x) g(x) \frac{\mathrm{d}x}{x} \right\} \\
&= f(\sqrt{ab})\, {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-} \big\{g(a)\big\} - {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-} \big\{(fg)(a)\big\}.
\end{aligned} \tag{2.9}$$

Since g is geometrically symmetric to $\sqrt{ab}$, one has

$$\int_a^{L(t)} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} = \int_{U(t)}^{b} \left(\ln \frac{b}{s} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s}.$$

By this, we have

$$\begin{aligned}
-I_2 &= -\frac{1}{k\Gamma_k(\alpha)} \frac{\ln b - \ln a}{2} \int_0^1 \left[\int_a^{L(t)} \left(\ln \frac{s}{a} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} \right] U(t) f'\big(U(t)\big) \,\mathrm{d}t \\
&= -\frac{1}{k\Gamma_k(\alpha)} \frac{\ln b - \ln a}{2} \int_0^1 \left[\int_{U(t)}^{b} \left(\ln \frac{b}{s} \right)^{\frac{\alpha}{k}-1} g(s) \frac{\mathrm{d}s}{s} \right] U(t) f'\big(U(t)\big) \,\mathrm{d}t \\
&= f(\sqrt{ab})\, {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+} \big\{g(b)\big\} - {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+} \big\{(fg)(b)\big\}.
\end{aligned} \tag{2.10}$$

Adding (2.9) and (2.10), we get the required identity in (2.6). This ends the proof. □

3 Inequalities involving hypergeometric functions

Our first main result is given by the following theorem. For this purpose, we note that the gamma function, the beta function and the incomplete beta function are defined by

$$\Gamma(x) = \int_0^\infty \eta^{x-1} e^{-\eta} \,\mathrm{d}\eta, \quad x > 0,$$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 \eta^{x-1} (1-\eta)^{y-1} \,\mathrm{d}\eta, \quad x, y > 0,$$

and

$$\beta(\delta;x,y)=\int_0^\delta \eta^{x-1}(1-\eta)^{y-1}\,\mathrm{d}\eta,\quad x,y>0,\ 0<\delta<1.$$

The integral form of the hypergeometric function is

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 \eta^{b-1}(1-\eta)^{c-b-1}(1-z\eta)^{-a}\,\mathrm{d}\eta,\quad c>b>0,|z|<1.$$

For the sake of simplicity, we also denote

$$\begin{aligned}\mathcal{T}_{f,g}(k,\alpha;a,b):=&f(\sqrt{ab})\bigl[{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\{g(a)\}+{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\{g(b)\}\bigr]\\&-\bigl[{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\{(fg)(a)\}+{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\{(fg)(b)\}\bigr],\end{aligned}$$

unless otherwise specified.

Theorem 3.1 *Let $f:\mathcal{I}\subseteq\mathbb{R}^+=(0,\infty)\to\mathbb{R}$, be a differentiable mapping on $\mathcal{I}^\circ$ (the interior of $\mathcal{I}$), $a,b\in\mathcal{I}^\circ$ with $a<b$, and let $g:[a,b]\to\mathbb{R}$ be a continuous positive mapping geometrically symmetric to $\sqrt{ab}$ such that $f'\in L^1([a,b])$. If $|f'|^q$ for $q\geq 1$ is GA-s-convex on $[a,b]$, then the following Hadamard k-fractional inequality with $k,\alpha>0$ holds*:

$$\begin{aligned}&\bigl|\mathcal{T}_{f,g}(k,\alpha;a,b)\bigr|\\&\quad\leq\frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\biggl\{\biggl[\Upsilon\biggl(a,b,\frac{\alpha}{k}\biggr)\biggr]^{1-\frac{1}{q}}\bigl(\mathbb{B}_1(k,\alpha,s)\bigl|f'(a)\bigr|^q+\mathbb{B}_2(k,\alpha,s)\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\\&\qquad+\biggl[\Upsilon\biggl(b,a,\frac{\alpha}{k}\biggr)\biggr]^{1-\frac{1}{q}}\bigl(\mathbb{B}_3(k,\alpha,s)\bigl|f'(a)\bigr|^q+\mathbb{B}_4(k,\alpha,s)\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\biggr\},\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\|g\|_\infty&=\sup_{x\in[a,b]}g(x)<\infty,\\\mathbb{B}_1(k,\alpha,s)&=\frac{\sqrt{ab}-a}{\frac{\alpha}{k}+2}{}_2F_1\biggl(-s,\frac{\alpha}{k}+2;\frac{\alpha}{k}+3;\frac{1}{2}\biggr)\\&\quad+\frac{a}{\frac{\alpha}{k}+1}{}_2F_1\biggl(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{2}\biggr),\\\mathbb{B}_2(k,\alpha,s)&=\frac{\sqrt{ab}-a}{2^s(\frac{\alpha}{k}+s+2)}+\frac{a}{2^s(\frac{\alpha}{k}+s+1)},\\\mathbb{B}_3(k,\alpha,s)&=\frac{\sqrt{ab}-b}{2^s(\frac{\alpha}{k}+s+2)}+\frac{b}{2^s(\frac{\alpha}{k}+s+1)},\\\mathbb{B}_4(k,\alpha,s)&=\frac{\sqrt{ab}-b}{\frac{\alpha}{k}+2}{}_2F_1\biggl(-s,\frac{\alpha}{k}+2;\frac{\alpha}{k}+3;\frac{1}{2}\biggr)\\&\quad+\frac{b}{\frac{\alpha}{k}+1}{}_2F_1\biggl(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{2}\biggr),\end{aligned}$$

and $\Upsilon(\mathcal{U},\mathcal{V},\sigma)$ is defined by (2.3) in Lemma 2.1.

Proof If we use Lemma 2.3 and Hölder's inequality, then we have

$$\begin{aligned}
&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\
&\quad\leq \frac{1}{k\Gamma_k(\alpha)}\frac{\ln b-\ln a}{2} \\
&\qquad\times\int_0^1\left[\int_a^{L(t)}\left(\ln\frac{s}{a}\right)^{\frac{\alpha}{k}-1}g(s)\frac{\mathrm{d}s}{s}\right]\left[L(t)\left|f'\bigl(L(t)\bigr)\right|+U(t)\left|f'\bigl(U(t)\bigr)\right|\right]\mathrm{d}t \\
&\quad\leq \frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\int_0^1\left[t^{\frac{\alpha}{k}}L(t)\left|f'\bigl(L(t)\bigr)\right|+t^{\frac{\alpha}{k}}U(t)\left|f'\bigl(U(t)\bigr)\right|\right]\mathrm{d}t \\
&\quad\leq \frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\left\{\left(\int_0^1 t^{\frac{\alpha}{k}}L(t)\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}L(t)\left|f'\bigl(L(t)\bigr)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_0^1 t^{\frac{\alpha}{k}}U(t)\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}U(t)\left|f'\bigl(U(t)\bigr)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}.
\end{aligned}\tag{3.2}$$

Using (iii) of Lemma 2.1, we have

$$\int_0^1 t^{\frac{\alpha}{k}}L(t)\,\mathrm{d}t=\int_0^1 t^{\frac{\alpha}{k}}a^{1-\frac{t}{2}}b^{\frac{t}{2}}\,\mathrm{d}t\leq\Upsilon\left(a,b,\frac{\alpha}{k}\right)\tag{3.3}$$

and

$$\int_0^1 t^{\frac{\alpha}{k}}U(t)\,\mathrm{d}t=\int_0^1 t^{\frac{\alpha}{k}}a^{\frac{t}{2}}b^{1-\frac{t}{2}}\,\mathrm{d}t\leq\Upsilon\left(b,a,\frac{\alpha}{k}\right).\tag{3.4}$$

Considering GA-s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}}L(t)\left|f'\bigl(L(t)\bigr)\right|^q\mathrm{d}t \\
&\quad=\int_0^1 t^{\frac{\alpha}{k}}a\bigl(a^{-\frac{1}{2}}b^{\frac{1}{2}}\bigr)^t\left|f'\bigl(a^{1-\frac{t}{2}}b^{\frac{t}{2}}\bigr)\right|^q\mathrm{d}t \\
&\quad\leq\left|f'(a)\right|^q\int_0^1 t^{\frac{\alpha}{k}}\left(1-\frac{t}{2}\right)^s a\bigl(a^{-\frac{1}{2}}b^{\frac{1}{2}}\bigr)^t\,\mathrm{d}t+\left|f'(b)\right|^q\int_0^1 t^{\frac{\alpha}{k}}\left(\frac{t}{2}\right)^s a\bigl(a^{-\frac{1}{2}}b^{\frac{1}{2}}\bigr)^t\,\mathrm{d}t.
\end{aligned}$$

Utilizing the inequality of $u^\theta\leq(u-1)\theta+1$ for all $0\leq\theta\leq1$ with $u>0$, we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}}L(t)\left|f'\bigl(L(t)\bigr)\right|^q\mathrm{d}t \\
&\quad\leq\left|f'(a)\right|^q\int_0^1 t^{\frac{\alpha}{k}}\left(1-\frac{t}{2}\right)^s a\left[\bigl(a^{-\frac{1}{2}}b^{\frac{1}{2}}-1\bigr)t+1\right]\mathrm{d}t \\
&\qquad+\left|f'(b)\right|^q\int_0^1 t^{\frac{\alpha}{k}}\left(\frac{t}{2}\right)^s a\left[\bigl(a^{-\frac{1}{2}}b^{\frac{1}{2}}-1\bigr)t+1\right]\mathrm{d}t \\
&\quad=\left|f'(a)\right|^q\left[\frac{\sqrt{ab}-a}{\frac{\alpha}{k}+2}{}_2F_1\left(-s,\frac{\alpha}{k}+2;\frac{\alpha}{k}+3;\frac{1}{2}\right)+\frac{a}{\frac{\alpha}{k}+1}{}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{2}\right)\right] \\
&\qquad+\left|f'(b)\right|^q\left[\frac{\sqrt{ab}-a}{2^s(\frac{\alpha}{k}+s+2)}+\frac{a}{2^s(\frac{\alpha}{k}+s+1)}\right].
\end{aligned}\tag{3.5}$$

Similarly,

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}} U(t)\left|f'\bigl(U(t)\bigr)\right|^q \,\mathrm{d}t \\
&\quad\le \left|f'(a)\right|^q \left[\frac{\sqrt{ab}-b}{2^s(\frac{\alpha}{k}+s+2)} + \frac{b}{2^s(\frac{\alpha}{k}+s+1)}\right] \\
&\qquad + \left|f'(b)\right|^q \left[\frac{\sqrt{ab}-b}{\frac{\alpha}{k}+2}\,{}_2F_1\left(-s, \frac{\alpha}{k}+2; \frac{\alpha}{k}+3; \frac{1}{2}\right)\right. \\
&\qquad \left. + \frac{b}{\frac{\alpha}{k}+1}\,{}_2F_1\left(-s, \frac{\alpha}{k}+1; \frac{\alpha}{k}+2; \frac{1}{2}\right)\right]. \qquad (3.6)
\end{aligned}$$

Using (3.3)–(3.6) in (3.2), we get the required inequality in (3.1). This ends the proof. □

Corollary 3.1 *If we take* $q = 1$ *in Theorem* 3.1, *then the following inequality holds*:

$$\begin{aligned}
&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\
&\quad\le \frac{(\ln b - \ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty \\
&\qquad \times \left[\bigl(\mathbb{B}_1(k,\alpha,s)+\mathbb{B}_3(k,\alpha,s)\bigr)\left|f'(a)\right| + \bigl(\mathbb{B}_2(k,\alpha,s)+\mathbb{B}_4(k,\alpha,s)\bigr)\left|f'(b)\right|\right],
\end{aligned}$$

where $\mathbb{B}_i(k,\alpha,s)$, $i = 1,2,3,4$, *are defined in Theorem* 3.1.

Corollary 3.2 *If we take* $s = 1$ *in Theorem* 3.1, *then the following inequality holds*:

$$\begin{aligned}
&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\
&\quad\le \frac{(\ln b - \ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty \\
&\qquad \times \left\{\left[\Upsilon\left(a,b,\frac{\alpha}{k}\right)\right]^{1-\frac{1}{q}} \left(\mathbb{B}_1(k,\alpha,1)\left|f'(a)\right|^q + \mathbb{B}_2(k,\alpha,1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left[\Upsilon\left(b,a,\frac{\alpha}{k}\right)\right]^{1-\frac{1}{q}} \left(\mathbb{B}_3(k,\alpha,1)\left|f'(a)\right|^q + \mathbb{B}_4(k,\alpha,1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\mathbb{B}_1(k,\alpha,1) = \frac{(\frac{\alpha}{k}+4)(\sqrt{ab}-a)}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)} + \frac{(\frac{\alpha}{k}+3)a}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)},$$

$$\mathbb{B}_2(k,\alpha,1) = \frac{\sqrt{ab}}{2(\frac{\alpha}{k}+3)} + \frac{a}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)},$$

$$\mathbb{B}_3(k,\alpha,1) = \frac{\sqrt{ab}}{2(\frac{\alpha}{k}+3)} + \frac{b}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)},$$

and

$$\mathbb{B}_4(k,\alpha,1) = \frac{(\frac{\alpha}{k}+4)(\sqrt{ab}-b)}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)} + \frac{(\frac{\alpha}{k}+3)b}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}.$$

Corollary 3.3 *If we take $k = 1$, $\alpha = 1$ and $s = 1$ in Theorem 3.1, then the following inequality holds*:

$$\begin{aligned}
&\left|f(\sqrt{ab})\int_a^b g(x)\frac{\mathrm{d}x}{x} - \int_a^b (fg)(x)\frac{\mathrm{d}x}{x}\right| \\
&\quad\le \frac{(\ln b-\ln a)^2}{2^{3+\frac{2}{q}}\cdot 3}\|g\|_\infty \\
&\qquad\times\Big\{\big(2a^{\frac12}b^{\frac12}+a\big)^{1-\frac1q}\big[\big(5a^{\frac12}b^{\frac12}+3a\big)\big|f'(a)\big|^q+\big(3a^{\frac12}b^{\frac12}+a\big)\big|f'(b)\big|^q\big]^{\frac1q} \\
&\qquad+\big(2a^{\frac12}b^{\frac12}+b\big)^{1-\frac1q}\big[\big(3a^{\frac12}b^{\frac12}+b\big)\big|f'(a)\big|^q+\big(5a^{\frac12}b^{\frac12}+3b\big)\big|f'(b)\big|^q\big]^{\frac1q}\Big\}.
\end{aligned}$$

Theorem 3.2 *Let $f:\mathcal{I}\subseteq\mathbb{R}^+=(0,\infty)\to\mathbb{R}$, be a differentiable mapping on $\mathcal{I}^\circ$, $a,b\in\mathcal{I}$ with $a<b$, and let $g:[a,b]\to\mathbb{R}$ be a continuous positive mapping geometrically symmetric to $\sqrt{ab}$ such that $f'\in L^1([a,b])$. If $|f'|^q$ for $q>1$ is GA-s-convex on $[a,b]$, then the following inequality for Hadamard k-fractional integral operators with $k,\alpha>0$ holds*:

$$\begin{aligned}
&\big|\mathcal{T}_{f,g}(k,\alpha;a,b)\big| \\
&\quad\le \frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\left(\frac{kq-k}{(k+\alpha)q-k}\right)^{1-\frac1q} \\
&\qquad\times\Big\{\big[\mathbb{C}_1(q,s)\big|f'(a)\big|^q+\mathbb{C}_2(q,s)\big|f'(b)\big|^q\big]^{\frac1q}+\big[\mathbb{C}_3(q,s)\big|f'(a)\big|^q+\mathbb{C}_4(q,s)\big|f'(b)\big|^q\big]^{\frac1q}\Big\},
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
\mathbb{C}_1(q,s)&=\frac12\big(a^{\frac q2}b^{\frac q2}-a^q\big)_2F_1\left(-s,2;3;\frac12\right)+\frac{2a^q}{s+1}\left(1-\frac{1}{2^{s+1}}\right),\\
\mathbb{C}_2(q,s)&=\frac{a^{\frac q2}b^{\frac q2}}{2^s(s+2)}+\frac{a^q}{2^s(s+1)(s+2)},\\
\mathbb{C}_3(q,s)&=\frac{a^{\frac q2}b^{\frac q2}}{2^s(s+2)}+\frac{b^q}{2^s(s+1)(s+2)},
\end{aligned}$$

and

$$\mathbb{C}_4(q,s)=\frac12\big(a^{\frac q2}b^{\frac q2}-b^q\big)_2F_1\left(-s,2;3;\frac12\right)+\frac{2b^q}{s+1}\left(1-\frac{1}{2^{s+1}}\right).$$

Proof Utilizing Lemma 2.3 and Hölder's inequality, we have

$$\begin{aligned}
&\big|\mathcal{T}_{f,g}(k,\alpha;a,b)\big| \\
&\quad\le\frac{1}{k\Gamma_k(\alpha)}\frac{\ln b-\ln a}{2} \\
&\qquad\times\int_0^1\left[\int_a^{L(t)}\left(\ln\frac{s}{a}\right)^{\frac{\alpha}{k}-1}g(s)\frac{\mathrm{d}s}{s}\right]\big[L(t)\big|f'\big(L(t)\big)\big|+U(t)\big|f'\big(U(t)\big)\big|\big]\,\mathrm{d}t \\
&\quad\le\frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\int_0^1\big[t^{\frac{\alpha}{k}}L(t)\big|f'\big(L(t)\big)\big|+t^{\frac{\alpha}{k}}U(t)\big|f'\big(U(t)\big)\big|\big]\,\mathrm{d}t
\end{aligned}$$

$$\leq \frac{(\ln b - \ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\left(\int_0^1 (t^{\frac{\alpha}{k}})^{\frac{q}{q-1}}\,\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 [L(t)]^q|f'(L(t))|^q\,\mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^1 [U(t)]^q|f'(U(t))|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right\}. \tag{3.8}$$

Considering GA-s-convexity of $|f'|^q$, and using the inequality of $u^\theta \leq (u-1)\theta + 1$ for all $0 \leq \theta \leq 1$ with $u > 0$, we have

$$\begin{aligned}
&\int_0^1 [L(t)]^q|f'(L(t))|^q\,\mathrm{d}t \\
&\leq |f'(a)|^q\int_0^1\left(1-\frac{t}{2}\right)^s\left[a^q(a^{-\frac{q}{2}}b^{\frac{q}{2}})^t\right]\mathrm{d}t + |f'(b)|^q\int_0^1\left(\frac{t}{2}\right)^s\left[a^q(a^{-\frac{q}{2}}b^{\frac{q}{2}})^t\right]\mathrm{d}t \\
&\leq |f'(a)|^q\int_0^1\left(1-\frac{t}{2}\right)^s a^q\left[(a^{-\frac{q}{2}}b^{\frac{q}{2}}-1)t+1\right]\mathrm{d}t \\
&\quad + |f'(b)|^q\int_0^1\left(\frac{t}{2}\right)^s a^q\left[(a^{-\frac{q}{2}}b^{\frac{q}{2}}-1)t+1\right]\mathrm{d}t \\
&= |f'(a)|^q\left[\frac{1}{2}(a^{\frac{q}{2}}b^{\frac{q}{2}}-a^q)\,{}_2F_1\left(-s,2;3;\frac{1}{2}\right) + \frac{2a^q}{s+1}\left(1-\frac{1}{2^{s+1}}\right)\right] \\
&\quad + |f'(b)|^q\left[\frac{a^{\frac{q}{2}}b^{\frac{q}{2}}}{2^s(s+2)} + \frac{a^q}{2^s(s+1)(s+2)}\right]
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\int_0^1 [U(t)]^q|f'(U(t))|^q\,\mathrm{d}t \\
&\leq |f'(a)|^q\left[\frac{a^{\frac{q}{2}}b^{\frac{q}{2}}}{2^s(s+2)} + \frac{b^q}{2^s(s+1)(s+2)}\right] \\
&\quad + |f'(b)|^q\left[\frac{1}{2}(a^{\frac{q}{2}}b^{\frac{q}{2}}-b^q)\,{}_2F_1\left(-s,2;3;\frac{1}{2}\right) + \frac{2b^q}{s+1}\left(1-\frac{1}{2^{s+1}}\right)\right].
\end{aligned} \tag{3.10}$$

Also,

$$\int_0^1 (t^{\frac{\alpha}{k}})^{\frac{q}{q-1}}\,\mathrm{d}t = \frac{kq-k}{(k+\alpha)q-k}. \tag{3.11}$$

The inequality (3.7) is proved by using (3.9), (3.10) and (3.11) in inequality (3.8). This ends the proof. □

Corollary 3.4 *If we take* $k = 1$ *and* $g(x) = 1$ *in Theorem* 3.2, *then the following inequality holds*:

$$\begin{aligned}
&\left|f(\sqrt{ab}) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln b - \ln a)^\alpha}\left[\mathcal{H}^\alpha_{\sqrt{ab}^-}\{f(a)\} + \mathcal{H}^\alpha_{\sqrt{ab}^+}\{f(b)\}\right]\right| \\
&\leq \frac{\ln b - \ln a}{4}\left(\frac{q-1}{(1+\alpha)q-1}\right)^{1-\frac{1}{q}} \\
&\quad \times\left\{\left(\mathbb{C}_1(q,s)|f'(a)|^q + \mathbb{C}_2(q,s)|f'(b)|^q\right)^{\frac{1}{q}} + \left(\mathbb{C}_3(q,s)|f'(a)|^q + \mathbb{C}_4(q,s)|f'(b)|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\mathbb{C}_i(q,s)$, $i = 1,2,3,4$, *are defined in Theorem* 3.2.

Corollary 3.5 *If we take $k = 1$, $\alpha = 1$ and $s = 1$ in Theorem 3.2, then the following inequality holds*:

$$\begin{aligned}
&\left|f(\sqrt{ab})\int_a^b g(x)\frac{dx}{x} - \int_a^b (fg)(x)\frac{dx}{x}\right| \\
&\quad\le \frac{(\ln b-\ln a)^2}{4}\|g\|_\infty\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \\
&\qquad\times\left\{\left(\left[\frac{1}{3}a^{\frac{q}{2}}b^{\frac{q}{2}}+\frac{5}{12}a^q\right]\left|f'(a)\right|^q+\left[\frac{a^{\frac{q}{2}}b^{\frac{q}{2}}}{6}+\frac{a^q}{12}\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\left[\frac{a^{\frac{q}{2}}b^{\frac{q}{2}}}{6}+\frac{b^q}{12}\right]\left|f'(a)\right|^q+\left[\frac{1}{3}a^{\frac{q}{2}}b^{\frac{q}{2}}+\frac{5}{12}b^q\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 3.3 *Let $f:\mathcal{I}\subseteq\mathbb{R}^+=(0,\infty)\to\mathbb{R}$, be a differentiable mapping on $\mathcal{I}^\circ$, $a,b\in\mathcal{I}^\circ$ with $a<b$, and let $g:[a,b]\to\mathbb{R}$ be a continuous positive mapping geometrically symmetric to $\sqrt{ab}$ such that $f'\in L^1([a,b])$. If $|f'|^q$ for $q>1$ is GA-s-convex on $[a,b]$, then the following inequality for Hadamard k-fractional integral operators with $k,\alpha>0$ holds*:

$$\begin{aligned}
&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\
&\quad\le \frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{(\frac{\alpha}{k}+\frac{1}{q})}\Gamma_k(\alpha+k)}\|g\|_\infty\left(\frac{kq-k}{(k+\alpha)q-k}\right)^{1-\frac{1}{q}} \\
&\qquad\times\left\{\left[\mathbb{C}_1(q,s)+\mathbb{C}_3(q,s)\right]\left|f'(a)\right|^q+\left[\mathbb{C}_2(q,s)+\mathbb{C}_4(q,s)\right]\left|f'(b)\right|^q\right\}^{1/q},
\end{aligned}\tag{3.12}$$

where $\mathbb{C}_i(q,s)$, $i=1,2,3,4$ are defined in Theorem 3.2.

Proof If we use the inequality $\mu^r+\nu^r\le 2^{1-r}(\mu+\nu)^r$ for $\mu>0$, $\nu>0$ and $r<1$, then we have

$$\begin{aligned}
&\left(\int_0^1[L(t)]^q\left|f'(L(t))\right|^q dt\right)^{1/q}+\left(\int_0^1[U(t)]^q\left|f'(U(t))\right|^q dt\right)^{1/q} \\
&\quad\le 2^{1-1/q}\left(\int_0^1[L(t)]^q\left|f'(L(t))\right|^q dt+\int_0^1[U(t)]^q\left|f'(U(t))\right|^q dt\right)^{1/q}.
\end{aligned}\tag{3.13}$$

Using inequalities (3.9) and (3.10) in (3.13), we have

$$\begin{aligned}
&\left(\int_0^1[L(t)]^q\left|f'(L(t))\right|^q dt\right)^{1/q}+\left(\int_0^1[U(t)]^q\left|f'(U(t))\right|^q dt\right)^{1/q} \\
&\quad\le 2^{1-1/q}\left\{\left[\mathbb{C}_1(q,s)+\mathbb{C}_3(q,s)\right]\left|f'(a)\right|^q+\left[\mathbb{C}_2(q,s)+\mathbb{C}_4(q,s)\right]\left|f'(b)\right|^q\right\}^{1/q}.
\end{aligned}\tag{3.14}$$

Applying (3.14) and (3.11) to inequality (3.8) in the proof of Theorem 3.2, we obtain the required inequality in (3.12). This ends the proof. □

Corollary 3.6 *If we take $k = 1$ and $g(x) = 1$ in Theorem 3.2, then the following inequality holds*:

$$\begin{aligned}&\left|f(\sqrt{ab}) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha}}\left[\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\{f(a)\} + \mathcal{H}^{\alpha}_{\sqrt{ab}+}\{f(b)\}\right]\right| \\ &\quad\le \frac{\ln b - \ln a}{2^{(1+\frac{1}{q})}}\left(\frac{q-1}{(1+\alpha)q-1}\right)^{1-\frac{1}{q}} \\ &\quad\times\left\{\left[\mathbb{C}_1(q,s) + \mathbb{C}_3(q,s)\right]\left|f'(a)\right|^q + \left[\mathbb{C}_2(q,s) + \mathbb{C}_4(q,s)\right]\left|f'(b)\right|^q\right\}^{1/q},\end{aligned}$$

where $\mathbb{C}_i(q,s)$, $i = 1,2,3,4$ are defined in Theorem 3.2.

Corollary 3.7 *If we take $k = 1$, $\alpha = 1$ and $s = 1$ in Theorem 3.2, then the following inequality holds*:

$$\begin{aligned}&\left|f(\sqrt{ab})\int_a^b g(x)\frac{\mathrm{d}x}{x} - \int_a^b (fg)(x)\frac{\mathrm{d}x}{x}\right| \\ &\quad\le \frac{(\ln b - \ln a)^2}{2^{(1+\frac{1}{q})}}\|g\|_\infty\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \\ &\quad\times\left\{\left(\frac{1}{2}a^{\frac{q}{2}}b^{\frac{q}{2}} + \frac{5}{12}a^q + \frac{1}{12}b^q\right)\left|f'(a)\right|^q + \left(\frac{1}{2}a^{\frac{q}{2}}b^{\frac{q}{2}} + \frac{5}{12}b^q + \frac{1}{12}a^q\right)\left|f'(b)\right|^q\right\}^{1/q}.\end{aligned}$$

Theorem 3.4 *Let $f : \mathcal{I} \subseteq \mathbb{R}^+ = (0,\infty) \to \mathbb{R}$, be a differentiable mapping on $\mathcal{I}^\circ$, $a,b \in \mathcal{I}$ with $a < b$, and let $g : [a,b] \to \mathbb{R}$ be a continuous positive mapping geometrically symmetric to $\sqrt{ab}$ such that $f' \in L^1([a,b])$. If $|f'|$ is GA-s-convex on $[a,b]$, then for $q > 1$ the following Hadamard k-fractional inequality with $k,\alpha > 0$ holds*:

$$\begin{aligned}&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\ &\quad\le \frac{(\ln b - \ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty \\ &\quad\times\left\{\left(\Phi\left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left[\mathbb{D}_1^{\frac{1}{q}}(k,\alpha,q,s)\left|f'(a)\right| + \mathbb{D}_2^{\frac{1}{q}}(k,\alpha,q,s)\left|f'(b)\right|\right]\right. \\ &\quad\left.+\left(\Phi\left(b^{\frac{q}{q-1}}, a^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left[\mathbb{D}_2^{\frac{1}{q}}(k,\alpha,q,s)\left|f'(a)\right| + \mathbb{D}_1^{\frac{1}{q}}(k,\alpha,q,s)\left|f'(b)\right|\right]\right\},\end{aligned} \tag{3.15}$$

where

$$\mathbb{D}_1(k,\alpha,q,s) = \frac{1}{\frac{\alpha}{k}q+1}\,{}_2F_1\left(-sq, \frac{\alpha}{k}q+1; \frac{\alpha}{k}q+2; \frac{1}{2}\right)$$

and

$$\mathbb{D}_2(k,\alpha,q,s) = \frac{1}{2^{sq}[(\frac{\alpha}{k}+s)q+1]}.$$

Proof From Lemma 2.3 and the GA-s-convexity of $|f'|$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\
&\quad\le \frac{1}{k\Gamma_k(\alpha)}\frac{\ln b-\ln a}{2} \\
&\qquad\times\int_0^1\left[\int_a^{L(t)}\left(\ln\frac{s}{a}\right)^{\frac{\alpha}{k}-1}g(s)\frac{\mathrm{d}s}{s}\right]\left[L(t)\left|f'\left(L(t)\right)\right|+U(t)\left|f'\left(U(t)\right)\right|\right]\mathrm{d}t \\
&\quad\le \frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\int_0^1\left[t^{\frac{\alpha}{k}}L(t)\left|f'\left(L(t)\right)\right|+t^{\frac{\alpha}{k}}U(t)\left|f'\left(U(t)\right)\right|\right]\mathrm{d}t \\
&\quad\le \frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\left\{\int_0^1 a^{1-\frac{t}{2}}b^{\frac{t}{2}}\left[t^{\frac{\alpha}{k}}\left(1-\frac{t}{2}\right)^s\left|f'(a)\right|+t^{\frac{\alpha}{k}}\left(\frac{t}{2}\right)^s\left|f'(b)\right|\right]\mathrm{d}t\right. \\
&\qquad\left.+\int_0^1 a^{\frac{t}{2}}b^{1-\frac{t}{2}}\left[t^{\frac{\alpha}{k}}\left(\frac{t}{2}\right)^s\left|f'(a)\right|+t^{\frac{\alpha}{k}}\left(1-\frac{t}{2}\right)^s\left|f'(b)\right|\right]\mathrm{d}t\right\}.
\end{aligned}\tag{3.16}$$

Using Hölder's integral inequality, we have

$$\begin{aligned}
&\int_0^1 a^{1-\frac{t}{2}}b^{\frac{t}{2}}\left[t^{\frac{\alpha}{k}}\left(1-\frac{t}{2}\right)^s\left|f'(a)\right|+t^{\frac{\alpha}{k}}\left(\frac{t}{2}\right)^s\left|f'(b)\right|\right]\mathrm{d}t \\
&\quad\le\left(\int_0^1\left(a^{1-\frac{t}{2}}b^{\frac{t}{2}}\right)^{\frac{q}{q-1}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left\{\left|f'(a)\right|\left(\int_0^1 t^{\frac{\alpha}{k}q}\left(1-\frac{t}{2}\right)^{sq}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left|f'(b)\right|\left(\int_0^1 t^{\frac{\alpha}{k}q}\left(\frac{t}{2}\right)^{sq}\mathrm{d}t\right)^{\frac{1}{q}}\right\} \\
&\quad=\left(\Phi\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left\{\left|f'(a)\right|\left[\frac{1}{\frac{\alpha}{k}q+1}{}_2F_1\left(-sq,\frac{\alpha}{k}q+1;\frac{\alpha}{k}q+2;\frac{1}{2}\right)\right]^{\frac{1}{q}}\right. \\
&\qquad\left.+\left|f'(b)\right|\left(\frac{1}{2^{sq}[(\frac{\alpha}{k}+s)q+1]}\right)^{\frac{1}{q}}\right\}.
\end{aligned}\tag{3.17}$$

Similarly, one has

$$\begin{aligned}
&\int_0^1 a^{\frac{t}{2}}b^{1-\frac{t}{2}}\left[t^{\frac{\alpha}{k}}\left(\frac{t}{2}\right)^s\left|f'(a)\right|+t^{\frac{\alpha}{k}}\left(1-\frac{t}{2}\right)^s\left|f'(b)\right|\right]\mathrm{d}t \\
&\quad\le\left(\Phi\left(b^{\frac{q}{q-1}},a^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{2^{sq}[(\frac{\alpha}{k}+s)q+1]}\right)^{\frac{1}{q}}\left|f'(a)\right|\right. \\
&\qquad\left.+\left[\frac{1}{\frac{\alpha}{k}q+1}{}_2F_1\left(-sq,\frac{\alpha}{k}q+1;\frac{\alpha}{k}q+2;\frac{1}{2}\right)\right]^{\frac{1}{q}}\left|f'(b)\right|\right\}.
\end{aligned}\tag{3.18}$$

Using (3.17) and (3.18) in (3.16), we obtain the required inequality (3.15). This ends the proof. □

Corollary 3.8 *If we take $k = 1$, $s = 1$ and $g(x) = 1$ in Theorem 3.4, then the following inequality holds*:

$$\begin{aligned}&\left|f(\sqrt{ab}) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha}}\left[\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\{f(a)\} + \mathcal{H}^{\alpha}_{\sqrt{ab}^+}\{f(b)\}\right]\right| \\ &\quad\leq \frac{\ln b - \ln a}{4}\left\{\left(\Phi\left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left[\mathbb{D}_1^{\frac{1}{q}}(1,\alpha,q,1)\left|f'(a)\right| + \mathbb{D}_2^{\frac{1}{q}}(1,\alpha,q,1)\left|f'(b)\right|\right]\right. \\ &\quad\quad\left. + \left(\Phi\left(b^{\frac{q}{q-1}}, a^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left[\mathbb{D}_2^{\frac{1}{q}}(1,\alpha,q,1)\left|f'(a)\right| + \mathbb{D}_1^{\frac{1}{q}}(1,\alpha,q,1)\left|f'(b)\right|\right]\right\}.\end{aligned}$$

Specially, taking $\alpha = 1$, we have

$$\begin{aligned}&\left|f(\sqrt{ab}) - \frac{1}{\ln b - \ln a}\int_a^b f(x)\frac{\mathrm{d}x}{x}\right| \\ &\quad\leq \frac{\ln b - \ln a}{4}\left\{\left(\Phi\left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left[\mathbb{D}_1^{\frac{1}{q}}(1,1,q,1)\left|f'(a)\right| + \mathbb{D}_2^{\frac{1}{q}}(1,1,q,1)\left|f'(b)\right|\right]\right. \\ &\quad\quad\left. + \left(\Phi\left(b^{\frac{q}{q-1}}, a^{\frac{q}{q-1}}\right)\right)^{1-\frac{1}{q}}\left[\mathbb{D}_2^{\frac{1}{q}}(1,1,q,1)\left|f'(a)\right| + \mathbb{D}_1^{\frac{1}{q}}(1,1,q,1)\left|f'(b)\right|\right]\right\}.\end{aligned}$$

Theorem 3.5 *Let $f : \mathcal{I} \subseteq \mathbb{R}^+ = (0,\infty) \to \mathbb{R}$, be a differentiable mapping on $\mathcal{I}^\circ$, $a, b \in \mathcal{I}$ with $a < b$, and let $g : [a,b] \to \mathbb{R}$ be a continuous positive mapping geometrically symmetric to $\sqrt{ab}$ such that $f' \in L^1([a,b])$. If $|f'|^q$ for $q > 1$ is GA-s-convex on $[a,b]$ with $p^{-1} + q^{-1} = 1$, then the following inequality for Hadamard k-fractional integral operators with $k, \alpha > 0$ holds*:

$$\begin{aligned}&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\ &\quad\leq \frac{(\ln b - \ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty \\ &\quad\quad\times\left\{\left(\Upsilon\left(a^p, b^p, \frac{\alpha}{k}\right)\right)^{\frac{1}{p}}\left(\mathbb{E}_1(k,\alpha,s)\left|f'(a)\right|^q + \mathbb{E}_2(k,\alpha,s)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\ &\quad\quad\left. + \left(\Upsilon\left(b^p, a^p, \frac{\alpha}{k}\right)\right)^{\frac{1}{p}}\left(\mathbb{E}_2(k,\alpha,s)\left|f'(a)\right|^q + \mathbb{E}_1(k,\alpha,s)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\mathbb{E}_1(k,\alpha,s) &= \frac{1}{\frac{\alpha}{k}+1}\,{}_2F_1\left(-s, \frac{\alpha}{k}+1; \frac{\alpha}{k}+2; \frac{1}{2}\right), \\ \mathbb{E}_2(k,\alpha,s) &= \frac{1}{2^s(\frac{\alpha}{k}+s+1)},\end{aligned}$$

and $\Upsilon(\mathcal{U},\mathcal{V},\sigma)$ is defined by (2.3) in Lemma 2.1.

Proof Using Lemma 2.3 again, we have

$$\begin{aligned}
&\left|\mathcal{T}_{f,g}(k,\alpha;a,b)\right| \\
&\quad\le \frac{1}{k\Gamma_k(\alpha)}\frac{\ln b-\ln a}{2} \\
&\qquad\times \int_0^1\left[\int_a^{L(t)}\left(\ln\frac{s}{a}\right)^{\frac{\alpha}{k}-1}g(s)\frac{\mathrm{d}s}{s}\right]\left[L(t)\left|f'\big(L(t)\big)\right|+U(t)\left|f'\big(U(t)\big)\right|\right]\mathrm{d}t \\
&\quad\le \frac{(\ln b-\ln a)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\int_0^1\left[t^{\frac{\alpha}{k}}L(t)\left|f'\big(L(t)\big)\right|+t^{\frac{\alpha}{k}}U(t)\left|f'\big(U(t)\big)\right|\right]\mathrm{d}t. \qquad (3.20)
\end{aligned}$$

Now, considering the following weighted version of Hölder's inequality, see [9]:

$$\left|\int_I f(s)g(s)h(s)\,\mathrm{d}s\right|\le\left(\int_I\left|f(s)\right|^p h(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_I\left|g(s)\right|^q h(s)\,\mathrm{d}s\right)^{\frac{1}{q}}$$

for $q>1$, $p^{-1}+q^{-1}=1$, and h is non-negative on I and provided all the other integrals exist and are finite, we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}}L(t)\left|f'\big(L(t)\big)\right|\mathrm{d}t \\
&\quad\le\left(\int_0^1\left[L(t)\right]^p t^{\frac{\alpha}{k}}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\big(L(t)\big)\right|^q t^{\frac{\alpha}{k}}\,\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad=\left(\int_0^1\left(a^{1-\frac{t}{2}}b^{\frac{t}{2}}\right)^p t^{\frac{\alpha}{k}}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a^{1-\frac{t}{2}}b^{\frac{t}{2}}\right)\right|^q t^{\frac{\alpha}{k}}\,\mathrm{d}t\right)^{\frac{1}{q}}. \qquad (3.21)
\end{aligned}$$

Considering GA-s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\int_0^1\left|f'\left(a^{1-\frac{t}{2}}b^{\frac{t}{2}}\right)\right|^q t^{\frac{\alpha}{k}}\,\mathrm{d}t \\
&\quad\le\left|f'(a)\right|^q\int_0^1\left(1-\frac{t}{2}\right)^s t^{\frac{\alpha}{k}}\,\mathrm{d}t+\left|f'(b)\right|^q\int_0^1\left(\frac{t}{2}\right)^s t^{\frac{\alpha}{k}}\,\mathrm{d}t \\
&\quad=\frac{1}{\frac{\alpha}{k}+1}\,{}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{2}\right)\left|f'(a)\right|^q+\frac{1}{2^s(\frac{\alpha}{k}+s+1)}\left|f'(b)\right|^q. \qquad (3.22)
\end{aligned}$$

Using inequality (2.3) in Lemma 2.1, we have

$$\int_0^1\left(a^{1-\frac{t}{2}}b^{\frac{t}{2}}\right)^p t^{\frac{\alpha}{k}}\,\mathrm{d}t\le\Upsilon\left(a^p,b^p,\frac{\alpha}{k}\right). \qquad (3.23)$$

Applying (3.22) and (3.23) to (3.21), we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}}L(t)\left|f'\big(L(t)\big)\right|\mathrm{d}t \\
&\quad\le\left(\Upsilon\left(a^p,b^p,\frac{\alpha}{k}\right)\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{1}{\frac{\alpha}{k}+1}\,{}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{2}\right)\left|f'(a)\right|^q+\frac{1}{2^s(\frac{\alpha}{k}+s+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}. \qquad (3.24)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}} U(t)\left|f'\left(U(t)\right)\right| \,\mathrm{d}t \\
&\quad\le \left(\Upsilon\left(b^p, a^p, \frac{\alpha}{k}\right)\right)^{\frac{1}{p}} \\
&\qquad\times \left(\frac{1}{2^s(\frac{\alpha}{k}+s+1)}\left|f'(a)\right|^q + \frac{1}{\frac{\alpha}{k}+1} {}_2F_1\left(-s, \frac{\alpha}{k}+1; \frac{\alpha}{k}+2; \frac{1}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.25}$$

The inequality (3.19) is proved by using (3.24) and (3.25) in (3.20). This ends the proof. □

Corollary 3.9 *If we take $k = 1$, $s = 1$ and $g(x) = 1$ in Theorem 3.5, then the following inequality holds*:

$$\begin{aligned}
&\left|f(\sqrt{ab}) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ln b - \ln a)^\alpha}\left[\mathcal{H}^\alpha_{\sqrt{ab}^-}\left\{f(a)\right\} + \mathcal{H}^\alpha_{\sqrt{ab}^+}\left\{f(b)\right\}\right]\right| \\
&\quad\le \frac{\ln b - \ln a}{4}\left\{\left(\Upsilon\left(a^p, b^p, \alpha\right)\right)^{\frac{1}{p}}\left[\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q + \frac{1}{2(\alpha+2)}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(\Upsilon\left(b^p, a^p, \alpha\right)\right)^{\frac{1}{p}}\left[\frac{1}{2(\alpha+2)}\left|f'(a)\right|^q + \frac{\alpha+3}{2(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Specially, taking $\alpha = 1$, we have

$$\begin{aligned}
&\left|f(\sqrt{ab}) - \frac{1}{\ln b - \ln a}\int_a^b f(x)\frac{\mathrm{d}x}{x}\right| \\
&\quad\le \frac{\ln b - \ln a}{4}\left\{\left(\Upsilon\left(a^p, b^p, 1\right)\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|f'(a)\right|^q + \frac{1}{6}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(\Upsilon\left(b^p, a^p, 1\right)\right)^{\frac{1}{p}}\left(\frac{1}{6}\left|f'(a)\right|^q + \frac{1}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\Upsilon(\mathcal{U}, \mathcal{V}, \sigma)$ is defined by (2.3) in Lemma 2.1.

4 Inequalities for products of two GA-*s*-convex functions

Theorem 4.1 *Let $f, g, w : [a, b] \to \mathbb{R}^+$, $a, b \in (0, \infty)$, $a < b$, be functions satisfying that w and fgw are in $L^1([a, b])$. If f is GA-s_1-convex on $[a, b]$ for some fixed $s_1 \in (0, 1]$, g is GA-s_2-convex on $[a, b]$ for some fixed $s_2 \in (0, 1]$, and if w is geometrically symmetric about $x = \sqrt{ab}$, then we have*

$$\begin{aligned}
&{}_k\mathcal{H}^\alpha_{\sqrt{ab}^+}\left\{fgw(b)\right\} + {}_k\mathcal{H}^\alpha_{\sqrt{ab}^-}\left\{fgw(a)\right\} \\
&\quad\le \frac{\mathbb{M}(a,b)}{k\Gamma_k(\alpha)(\ln b - \ln a)^{s_1+s_2}} \\
&\qquad\times \int_{\sqrt{ab}}^b \left(\ln\frac{b}{u}\right)^{\frac{\alpha}{k}-1}\left[\left(\ln\frac{b}{u}\right)^{s_1+s_2} + \left(\ln\frac{u}{a}\right)^{s_1+s_2}\right] w(u)\frac{\mathrm{d}u}{u} \\
&\qquad + \frac{\mathbb{N}(a,b)}{k\Gamma_k(\alpha)(\ln b - \ln a)^{s_1+s_2}} \\
&\qquad\times \int_{\sqrt{ab}}^b \left(\ln\frac{b}{u}\right)^{\frac{\alpha}{k}-1}\left[\left(\ln\frac{b}{u}\right)^{s_1}\left(\ln\frac{u}{a}\right)^{s_2} + \left(\ln\frac{b}{u}\right)^{s_2}\left(\ln\frac{u}{a}\right)^{s_1}\right] w(u)\frac{\mathrm{d}u}{u},
\end{aligned} \tag{4.1}$$

where

$$\mathbb{M}(a,b) = f(a)g(a) + f(b)g(b)$$

and

$$\mathbb{N}(a,b) = f(a)g(b) + f(b)g(a).$$

Proof Since f is GA-s_1-convex and g is GA-s_2-convex on $[a,b]$, we have

$$f\left(a^t b^{1-t}\right) \le t^{s_1} f(a) + (1-t)^{s_1} f(b)$$

and

$$g\left(a^t b^{1-t}\right) \le t^{s_2} g(a) + (1-t)^{s_2} g(b)$$

for all $t \in [0,1]$. f and g are non-negative, so

$$\begin{aligned} f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) \le{}& t^{s_1+s_2} f(a)g(a) + (1-t)^{s_1+s_2} f(b)g(b) \\ &+ t^{s_1}(1-t)^{s_2} f(a)g(b) + (1-t)^{s_1} t^{s_2} f(b)g(a). \end{aligned} \tag{4.2}$$

Similarly, we also have

$$\begin{aligned} f\left(a^{1-t} b^{t}\right) g\left(a^{1-t} b^{t}\right) \le{}& (1-t)^{s_1+s_2} f(a)g(a) + t^{s_1+s_2} f(b)g(b) \\ &+ (1-t)^{s_1} t^{s_2} f(a)g(b) + t^{s_1}(1-t)^{s_2} f(b)g(a). \end{aligned} \tag{4.3}$$

The sum of (4.2) and (4.3) yields

$$\begin{aligned} &f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) + f\left(a^{1-t} b^{t}\right) g\left(a^{1-t} b^{t}\right) \\ &\quad \le \left(t^{s_1+s_2} + (1-t)^{s_1+s_2}\right)\left[f(a)g(a) + f(b)g(b)\right] \\ &\qquad + \left(t^{s_1}(1-t)^{s_2} + t^{s_2}(1-t)^{s_1}\right)\left[f(a)g(b) + f(b)g(a)\right]. \end{aligned} \tag{4.4}$$

Multiplying both sides of (4.4) by $t^{\frac{\alpha}{k}-1} w(a^t b^{1-t})$, then integrating the obtained inequality with respect to t from 0 to $\frac{1}{2}$, we have

$$\begin{aligned} &\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1} f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) w\left(a^t b^{1-t}\right) \,\mathrm{d}t + \int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1} f\left(a^{1-t} b^{t}\right) g\left(a^{1-t} b^{t}\right) w\left(a^t b^{1-t}\right) \,\mathrm{d}t \\ &\quad \le \left[f(a)g(a) + f(b)g(b)\right] \int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1} \left(t^{s_1+s_2} + (1-t)^{s_1+s_2}\right) w\left(a^t b^{1-t}\right) \,\mathrm{d}t \\ &\qquad + \left[f(a)g(b) + f(b)g(a)\right] \int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1} \left(t^{s_1}(1-t)^{s_2} + t^{s_2}(1-t)^{s_1}\right) w\left(a^t b^{1-t}\right) \,\mathrm{d}t. \end{aligned} \tag{4.5}$$

By the change of variable $u = a^t b^{1-t}$, we get

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1} f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) w\left(a^t b^{1-t}\right) \,\mathrm{d}t = \frac{k\Gamma_k(\alpha)}{(\ln b - \ln a)^{\frac{\alpha}{k}}}\, {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\left\{fgw(b)\right\},$$

$$\begin{aligned}
&\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}\big(t^{s_1+s_2}+(1-t)^{s_1+s_2}\big)w\big(a^t b^{1-t}\big)\,\mathrm{d}t \\
&\quad= \frac{1}{(\ln b-\ln a)^{\frac{\alpha}{k}+s_1+s_2}} \int_{\sqrt{ab}}^{b}\left[\left(\ln\frac{b}{u}\right)^{\frac{\alpha}{k}+s_1+s_2-1}+\left(\ln\frac{b}{u}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{u}{a}\right)^{s_1+s_2}\right]w(u)\frac{\mathrm{d}u}{u}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}\big(t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\big)w\big(a^t b^{1-t}\big)\,\mathrm{d}t \\
&\quad= \frac{1}{(\ln b-\ln a)^{\frac{\alpha}{k}+s_1+s_2}} \\
&\qquad\times \int_{\sqrt{ab}}^{b}\left[\left(\ln\frac{b}{u}\right)^{\frac{\alpha}{k}+s_1-1}\left(\ln\frac{u}{a}\right)^{s_2}+\left(\ln\frac{b}{u}\right)^{\frac{\alpha}{k}+s_2-1}\left(\ln\frac{u}{a}\right)^{s_1}\right]w(u)\frac{\mathrm{d}u}{u}.
\end{aligned}$$

Using the fact that w is geometrically symmetric and by the change of variable $u = a^{1-t}b^t$, we get

$$\begin{aligned}
\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1} f\big(a^{1-t}b^t\big)g\big(a^{1-t}b^t\big)w\big(a^t b^{1-t}\big)\,\mathrm{d}t &= \int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1} f\big(a^{1-t}b^t\big)g\big(a^{1-t}b^t\big)w\big(a^{1-t}b^t\big)\,\mathrm{d}t \\
&= \frac{k\Gamma_k(\alpha)}{(\ln b-\ln a)^{\frac{\alpha}{k}}}\, {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\big\{fgw(a)\big\}.
\end{aligned}$$

Substituting the four equalities above into (4.5), we have the required inequality in (4.1). □

Corollary 4.1 *In Theorem 4.1, if we take $w(u) = 1$, then we have*

$$\begin{aligned}
&{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\big\{fg(b)\big\} + {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\big\{fg(a)\big\} \\
&\quad\le \frac{(\ln b-\ln a)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left[\frac{1}{(\frac{\alpha}{k}+s_1+s_2)2^{\frac{\alpha}{k}+s_1+s_2}}+\beta\left(\frac{1}{2};\frac{\alpha}{k},s_1+s_2+1\right)\right]\mathbb{M}(a,b) \\
&\qquad+\frac{(\ln b-\ln a)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left[\beta\left(\frac{1}{2};\frac{\alpha}{k}+s_1,s_2+1\right)+\beta\left(\frac{1}{2};\frac{\alpha}{k}+s_2,s_1+1\right)\right]\mathbb{N}(a,b).
\end{aligned}$$

Corollary 4.2 *In Theorem 4.1, if we take $k = 1$, $\alpha = 1$ and $w(u) = 1$ for all $u \in [a,b]$, then we have*

$$\frac{1}{\ln b-\ln a}\int_a^b f(x)g(x)\frac{\mathrm{d}x}{x} \le \frac{1}{s_1+s_2+1}\mathbb{M}(a,b)+\beta(s_1+1,s_2+1)\mathbb{N}(a,b).$$

Furthermore, if we choose $s_1 = s_2 = 1$, then we have Corollary 3.12 *in* [18].

Remark 4.1 If we choose $w(u) = 1$ for all $u \in [a,b]$, $k = 1$ and $s_1 = s_2 = 1$ in Theorem 4.1, then we have Theorem 3.9 in [18].

Theorem 4.2 *Suppose that conditions of Theorem 4.1 hold, then we have the following inequality*:

$$\begin{aligned}
&2^{s_1+s_2-1}f(\sqrt{ab})g(\sqrt{ab})\bigl[{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\{w(a)\}+{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\{w(b)\}\bigr]\\
&\quad\le\bigl[{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\{fgw(a)\}+{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\{fgw(b)\}\bigr]\\
&\qquad+\frac{1}{k\Gamma_k(\alpha)}\bigl[\mathbb{M}(a,b)\cdot\Delta_1+\mathbb{N}(a,b)\cdot\Delta_2\bigr],
\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}
\Delta_1&=\frac{1}{(\ln b-\ln a)^{s_1+s_2}}\\
&\quad\times\int_a^{\sqrt{ab}}\left(\ln\frac{\nu}{a}\right)^{\frac{\alpha}{k}-1}\left[\left(\ln\frac{\nu}{a}\right)^{s_1}\left(\ln\frac{b}{\nu}\right)^{s_2}+\left(\ln\frac{\nu}{a}\right)^{s_2}\left(\ln\frac{b}{\nu}\right)^{s_1}\right]w(\nu)\frac{\mathrm{d}\nu}{\nu},\\
\Delta_2&=\frac{1}{(\ln b-\ln a)^{s_1+s_2}}\int_a^{\sqrt{ab}}\left(\ln\frac{\nu}{a}\right)^{\frac{\alpha}{k}-1}\left[\left(\ln\frac{\nu}{a}\right)^{s_1+s_2}+\left(\ln\frac{b}{\nu}\right)^{s_1+s_2}\right]w(\nu)\frac{\mathrm{d}\nu}{\nu},
\end{aligned}$$

and $\mathbb{M}(a,b)$, $\mathbb{N}(a,b)$ *are defined in Theorem 4.1.*

Proof Using the GA-s_1-convexity of f and GA-s_2-convexity of g, we have

$$\begin{aligned}
f(\sqrt{ab})g(\sqrt{ab})&=f\bigl(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\bigr)g\bigl(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\bigr)\\
&\le\left(\frac{1}{2}\right)^{s_1+s_2}\bigl[f\bigl(a^tb^{1-t}\bigr)g\bigl(a^tb^{1-t}\bigr)+f\bigl(a^{1-t}b^t\bigr)g\bigl(a^{1-t}b^t\bigr)\bigr]\\
&\quad+\left(\frac{1}{2}\right)^{s_1+s_2}\bigl[f\bigl(a^tb^{1-t}\bigr)g\bigl(a^{1-t}b^t\bigr)+f\bigl(a^{1-t}b^t\bigr)g\bigl(a^tb^{1-t}\bigr)\bigr].
\end{aligned}$$

Considering the GA-s_1-convexity of f and GA-s_2-convexity of g again, we have

$$\begin{aligned}
f(\sqrt{ab})g(\sqrt{ab})&=f\bigl(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\bigr)g\bigl(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\bigr)\\
&\le\left(\frac{1}{2}\right)^{s_1+s_2}\bigl[f\bigl(a^tb^{1-t}\bigr)g\bigl(a^tb^{1-t}\bigr)+f\bigl(a^{1-t}b^t\bigr)g\bigl(a^{1-t}b^t\bigr)\bigr]\\
&\quad+\left(\frac{1}{2}\right)^{s_1+s_2}\bigl\{\bigl[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\bigr]\bigl[f(a)g(a)+f(b)g(b)\bigr]\\
&\quad+\bigl[t^{s_1+s_2}+(1-t)^{s_1+s_2}\bigr]\bigl[f(a)g(b)+f(b)g(a)\bigr]\bigr\}.
\end{aligned}\tag{4.7}$$

Multiplying both sides of (4.7) by $t^{\frac{\alpha}{k}-1}w(a^{1-t}b^t)$, and integrating the obtained inequality with respect to t from 0 to $\frac{1}{2}$, we obtain

$$\begin{aligned}
&f(\sqrt{ab})g(\sqrt{ab})\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}-1}w\bigl(a^{1-t}b^t\bigr)\,\mathrm{d}t\\
&\le\left(\frac{1}{2}\right)^{s_1+s_2}\left\{\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}-1}f\bigl(a^tb^{1-t}\bigr)g\bigl(a^tb^{1-t}\bigr)w\bigl(a^{1-t}b^t\bigr)\,\mathrm{d}t\right.\\
&\quad\left.+\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}-1}f\bigl(a^{1-t}b^t\bigr)g\bigl(a^{1-t}b^t\bigr)w\bigl(a^{1-t}b^t\bigr)\,\mathrm{d}t\right\}
\end{aligned}$$

$$+\left(\frac{1}{2}\right)^{s_1+s_2}\Bigg\{\big[f(a)g(a)+f(b)g(b)\big]\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}\big[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\big]w\big(a^{1-t}b^t\big)\,\mathrm{d}t$$

$$+\big[f(a)g(b)+f(b)g(a)\big]\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}\big[t^{s_1+s_2}+(1-t)^{s_1+s_2}\big]w\big(a^{1-t}b^t\big)\,\mathrm{d}t\Bigg\}. \tag{4.8}$$

Using the change of variable and Lemma 2.2, we have

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}w\big(a^{1-t}b^t\big)\,\mathrm{d}t=\frac{k\Gamma_k(\alpha)}{(\ln b-\ln a)^{\frac{\alpha}{k}}}{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\big\{w(a)\big\}$$
$$=\frac{k\Gamma_k(\alpha)}{2(\ln b-\ln a)^{\frac{\alpha}{k}}}\big[{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\big\{w(a)\big\}+{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\big\{w(b)\big\}\big],$$

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}f\big(a^{1-t}b^t\big)g\big(a^{1-t}b^t\big)w\big(a^{1-t}b^t\big)\,\mathrm{d}t=\frac{k\Gamma_k(\alpha)}{(\ln b-\ln a)^{\frac{\alpha}{k}}}{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-}\big\{fgw(a)\big\},$$

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}t^{s_1}(1-t)^{s_2}w\big(a^{1-t}b^t\big)\,\mathrm{d}t$$
$$=\frac{1}{(\ln b-\ln a)^{\frac{\alpha}{k}+s_1+s_2}}\int_a^{\sqrt{ab}}\left(\ln\frac{v}{a}\right)^{\frac{\alpha}{k}+s_1-1}\left(\ln\frac{b}{v}\right)^{s_2}w(v)\frac{\mathrm{d}v}{v},$$

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}t^{s_2}(1-t)^{s_1}w\big(a^{1-t}b^t\big)\,\mathrm{d}t$$
$$=\frac{1}{(\ln b-\ln a)^{\frac{\alpha}{k}+s_1+s_2}}\int_a^{\sqrt{ab}}\left(\ln\frac{v}{a}\right)^{\frac{\alpha}{k}+s_2-1}\left(\ln\frac{b}{v}\right)^{s_1}w(v)\frac{\mathrm{d}v}{v},$$

and

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}\big[t^{s_1+s_2}+(1-t)^{s_1+s_2}\big]w\big(a^{1-t}b^t\big)\,\mathrm{d}t$$
$$=\frac{1}{(\ln b-\ln a)^{\frac{\alpha}{k}+s_1+s_2}}\int_a^{\sqrt{ab}}\left(\ln\frac{v}{a}\right)^{\frac{\alpha}{k}-1}\left[\left(\ln\frac{v}{a}\right)^{s_1+s_2}+\left(\ln\frac{b}{v}\right)^{s_1+s_2}\right]w(v)\frac{\mathrm{d}v}{v}.$$

Note that w is geometrically symmetric about $\sqrt{ab}$, we also have

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}f\big(a^tb^{1-t}\big)g\big(a^tb^{1-t}\big)w\big(a^{1-t}b^t\big)\,\mathrm{d}t$$
$$=\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-1}f\big(a^tb^{1-t}\big)g\big(a^tb^{1-t}\big)w\big(a^tb^{1-t}\big)\,\mathrm{d}t$$
$$=\frac{k\Gamma_k(\alpha)}{(\ln b-\ln a)^{\frac{\alpha}{k}}}{}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+}\big\{fgw(b)\big\}.$$

Substituting the six equalities above into (4.8), we have the required inequality in (4.6). This ends the proof. □

Corollary 4.3 *In Theorem* 4.2, *if we take* $w(\nu) = 1$, *then we have*

$$\begin{aligned}
&\frac{2^{s_1+s_2-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} f(\sqrt{ab})g(\sqrt{ab}) \\
&\quad \le \frac{1}{(\ln b - \ln a)^{\frac{\alpha}{k}}} \left\{ {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^-} \{fg(a)\} + {}_k\mathcal{H}^{\alpha}_{\sqrt{ab}^+} \{fg(b)\} \right\} \\
&\quad + \frac{1}{k\Gamma_k(\alpha)} \left\{ \left[\beta\left(\frac{1}{2}; \frac{\alpha}{k} + s_1, s_2 + 1\right) + \beta\left(\frac{1}{2}; \frac{\alpha}{k} + s_2, s_1 + 1\right) \right] \mathbb{M}(a,b) \right. \\
&\quad \left. + \left[\frac{1}{(\frac{\alpha}{k} + s_1 + s_2) 2^{\frac{\alpha}{k} + s_1 + s_2}} + \beta\left(\frac{1}{2}; \frac{\alpha}{k}, s_1 + s_2 + 1\right) \right] \mathbb{N}(a,b) \right\}.
\end{aligned}$$

Corollary 4.4 *In Theorem* 4.2, *if we take* $k = 1$, $\alpha = 1$ *and* $w(\nu) = 1$, *then we have*

$$\begin{aligned}
&2^{s_1+s_2-1} f(\sqrt{ab})g(\sqrt{ab}) \\
&\quad \le \frac{1}{\ln b - \ln a} \int_a^b f(x)g(x)\frac{\mathrm{d}x}{x} + \beta(s_1+1, s_2+1)\mathbb{M}(a,b) + \frac{1}{s_1+s_2+1}\mathbb{N}(a,b).
\end{aligned}$$

Furthermore, if we choose $s_1 = s_2 = 1$, *then we have Corollary* 3.16 *in* [18].

Remark 4.2 If we choose $w(\nu) = 1$ for all $\nu \in [a,b]$, $k = 1$ and $s_1 = s_2 = 1$ in Theorem 4.2, then we have Theorem 3.13 in [18].

5 Applications to special means

For positive numbers $\mu > 0$ and $\nu > 0$ with $\mu \neq \nu$, let us define

$$A(\mu,\nu) = \frac{\mu+\nu}{2}, \qquad L(\mu,\nu) = \frac{\nu-\mu}{\ln\nu - \ln\mu}, \qquad G(\mu,\nu) = \sqrt{\mu\nu}$$

and

$$L_p(\mu,\nu) = \begin{cases} \left[\frac{\nu^{p+1}-\mu^{p+1}}{(p+1)(\nu-\mu)}\right]^{\frac{1}{p}}, & p \neq -1, 0, \\ L(\mu,\nu), & p = -1, \\ \frac{1}{e}\left(\frac{\nu^\nu}{\mu^\mu}\right)^{\frac{1}{\nu-\mu}}, & p = 0, \end{cases}$$

with $p \in \mathbb{R}$, respectively.

Now let $f(x) = x^r$ for $x > 0$, $r \in \mathbb{R}$ with $r \neq 0$. It is easy to check that $|f'(x)|^q = |r|^q x^{q(r-1)}$ is GA-convex on $[a,b]$ for $q \geq 1$ and $r \neq 1$, where $a, b > 0$.

Consider the function

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2, \quad x \in [a,b], 0 < a < b.$$

Clearly, $g(x)$ is geometrically symmetric about $x = \sqrt{ab}$.

Theorem 5.1 *Let $0 < a < b$, $r \in \mathbb{R} \setminus \{-2,0,1,2\}$ and $q \geq 1$. Then the following inequality holds*:

$$\begin{aligned} &\biggl| 2\bigl[G(a,b)\bigr]^{r-2}L\bigl(a^2,b^2\bigr) - 2\bigl[G(a,b)\bigr]^{r} \\ &\quad - \frac{L(a^{r+2},b^{r+2})}{[G(a,b)]^2} - \bigl[G(a,b)\bigr]^2 L\bigl(a^{r-2},b^{r-2}\bigr) + 2L\bigl(a^r,b^r\bigr)\biggr| \\ &\quad \leq \frac{(\ln b - \ln a)|r|}{2^{2+\frac{2}{q}}\cdot 3}\frac{(b-a)^2}{[G(a,b)]^2} \\ &\quad\quad \times \bigl\{a^{\frac{1}{2}}\bigl[A\bigl(a^{\frac{1}{2}},2b^{\frac{1}{2}}\bigr)\bigr]^{1-\frac{1}{q}}\bigl[3A\bigl(a^{\frac{1}{2}+q(r-1)},b^{\frac{1}{2}+q(r-1)}\bigr) + A\bigl(5a^{q(r-1)}b^{\frac{1}{2}},a^{\frac{1}{2}}b^{q(r-1)}\bigr)\bigr]^{\frac{1}{q}} \\ &\quad\quad + b^{\frac{1}{2}}\bigl[A\bigl(2a^{\frac{1}{2}},b^{\frac{1}{2}}\bigr)\bigr]^{1-\frac{1}{q}}\bigl[3A\bigl(a^{\frac{1}{2}+q(r-1)},b^{\frac{1}{2}+q(r-1)}\bigr) + A\bigl(a^{q(r-1)}b^{\frac{1}{2}},5a^{\frac{1}{2}}b^{q(r-1)}\bigr)\bigr]^{\frac{1}{q}}\bigr\}. \end{aligned} \tag{5.1}$$

Proof Applying Corollary 3.3 to the functions

$$f(x) = x^r, \quad \forall x > 0, r \in \mathbb{R} \setminus \{-2,0,1,2\}$$

and

$$g(x) = \biggl(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\biggr)^2, \quad x \in [a,b], 0 < a < b,$$

we derive the required result. □

Corollary 5.1 *Suppose all assumptions of Theorem 5.1 are satisfied and if $r = -1$, then the following inequality holds*:

$$\begin{aligned} &\bigl|2\bigl[G(a,b)\bigr]^{-3}L\bigl(a^2,b^2\bigr) - 2\bigl[G(a,b)\bigr]^{-1} \\ &\quad - \bigl[G(a,b)\bigr]^2 L\bigl(a^{-3},b^{-3}\bigr) + L\bigl(a^{-1},b^{-1}\bigr)\bigr| \\ &\quad \leq \frac{(\ln b - \ln a)}{2^{2+\frac{2}{q}}\cdot 3}\frac{(b-a)^2}{[G(a,b)]^2} \\ &\quad\quad \times \bigl\{a^{\frac{1}{2}}\bigl[A\bigl(a^{\frac{1}{2}},2b^{\frac{1}{2}}\bigr)\bigr]^{1-\frac{1}{q}}\bigl[3A\bigl(a^{\frac{1}{2}-2q},b^{\frac{1}{2}-2q}\bigr) + A\bigl(5a^{-2q}b^{\frac{1}{2}},a^{\frac{1}{2}}b^{-2q}\bigr)\bigr]^{\frac{1}{q}} \\ &\quad\quad + b^{\frac{1}{2}}\bigl[A\bigl(2a^{\frac{1}{2}},b^{\frac{1}{2}}\bigr)\bigr]^{1-\frac{1}{q}}\bigl[3A\bigl(a^{\frac{1}{2}-2q},b^{\frac{1}{2}-2q}\bigr) + A\bigl(a^{-2q}b^{\frac{1}{2}},5a^{\frac{1}{2}}b^{-2q}\bigr)\bigr]^{\frac{1}{q}}\bigr\}. \end{aligned}$$

Corollary 5.2 *Under the assumptions of Theorem 5.1 with $q = 1$, the following inequality holds*:

$$\begin{aligned} &\biggl| 2\bigl[G(a,b)\bigr]^{r-2}L\bigl(a^2,b^2\bigr) - 2\bigl[G(a,b)\bigr]^{r} \\ &\quad - \frac{L(a^{r+2},b^{r+2})}{[G(a,b)]^2} - \bigl[G(a,b)\bigr]^2 L\bigl(a^{r-2},b^{r-2}\bigr) + 2L\bigl(a^r,b^r\bigr)\biggr| \\ &\quad \leq \frac{(\ln b - \ln a)|r|}{48}\frac{(b-a)^2}{[G(a,b)]^2} \\ &\quad\quad \times \bigl\{3A\bigl(a^r,b^r\bigr) + 8A\bigl(a^{\frac{1}{2}}b^{r-\frac{1}{2}},a^{r-\frac{1}{2}}b^{\frac{1}{2}}\bigr) + A\bigl(ab^{r-1},a^{r-1}b\bigr)\bigr\}. \end{aligned}$$

Especially for $r = -1$, we get

$$\begin{aligned}&\big|2\big[G(a,b)\big]^{-3}L\big(a^2,b^2\big) - 2\big[G(a,b)\big]^{-1}\\&\quad - \big[G(a,b)\big]^2 L\big(a^{-3},b^{-3}\big) + L\big(a^{-1},b^{-1}\big)\big|\\&\quad\le \frac{(\ln b - \ln a)}{48}\frac{(b-a)^2}{[G(a,b)]^2}\\&\qquad\times\big\{3A\big(a^{-1},b^{-1}\big) + 8A\big(a^{\frac{1}{2}}b^{-\frac{3}{2}},a^{-\frac{3}{2}}b^{\frac{1}{2}}\big) + A\big(ab^{-2},a^{-2}b\big)\big\}.\end{aligned}$$

Theorem 5.2 *Let $0 < a < b$, $r \in \mathbb{R} \setminus \{-2,0,1,2\}$ and $q > 1$. Then the following inequality holds*:

$$\begin{aligned}&\bigg|2\big[G(a,b)\big]^{r-2}L\big(a^2,b^2\big) - 2\big[G(a,b)\big]^{r}\\&\quad - \frac{L(a^{r+2},b^{r+2})}{[G(a,b)]^2} - \big[G(a,b)\big]^2 L\big(a^{r-2},b^{r-2}\big) + 2L\big(a^r,b^r\big)\bigg|\\&\quad\le \frac{(\ln b - \ln a)|r|}{4}\frac{(b-a)^2}{[G(a,b)]^2}\bigg(\frac{q-1}{2q-1}\bigg)^{1-\frac{1}{q}}\\&\qquad\times\bigg\{\bigg(2a^{q(r-\frac{1}{2})}A\bigg(\frac{5}{12}a^{\frac{q}{2}},\frac{1}{3}b^{\frac{q}{2}}\bigg) + b^{q(r-1)}\bigg[\frac{1}{6}G\big(a^q,b^q\big) + \frac{1}{12}a^q\bigg]\bigg)^{\frac{1}{q}}\\&\qquad + \bigg(a^{q(r-1)}\bigg[\frac{1}{6}G\big(a^q,b^q\big) + \frac{1}{12}b^q\bigg] + 2b^{q(r-\frac{1}{2})}A\bigg(\frac{1}{3}a^{\frac{q}{2}},\frac{5}{12}b^{\frac{q}{2}}\bigg)\bigg)^{\frac{1}{q}}\bigg\}.\end{aligned}\tag{5.2}$$

Proof Using Corollary 3.5 for the functions $f(x) = x^r$, $x > 0$, $r \in \mathbb{R} \setminus \{-2,0,1,2\}$ and $g(x) = (\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x})^2$, $x \in [a,b]$ with $0 < a < b$, we obtain the required result. □

Corollary 5.3 *Suppose the assumptions of Theorem 5.2 are fulfilled and if $r = -1$, then the following inequality holds*:

$$\begin{aligned}&\big|2\big[G(a,b)\big]^{-3}L\big(a^2,b^2\big) - 2\big[G(a,b)\big]^{-1}\\&\quad - \big[G(a,b)\big]^2 L\big(a^{-3},b^{-3}\big) + L\big(a^{-1},b^{-1}\big)\big|\\&\quad\le \frac{(\ln b - \ln a)}{4}\frac{(b-a)^2}{[G(a,b)]^2}\bigg(\frac{q-1}{2q-1}\bigg)^{1-\frac{1}{q}}\\&\qquad\times\bigg\{\bigg(2a^{-\frac{3}{2}q}A\bigg(\frac{5}{12}a^{\frac{q}{2}},\frac{1}{3}b^{\frac{q}{2}}\bigg) + b^{-2q}\bigg[\frac{1}{6}G\big(a^q,b^q\big) + \frac{1}{12}a^q\bigg]\bigg)^{\frac{1}{q}}\\&\qquad + \bigg(a^{-2q}\bigg[\frac{1}{6}G\big(a^q,b^q\big) + \frac{1}{12}b^q\bigg] + 2b^{-\frac{3}{2}q}A\bigg(\frac{1}{3}a^{\frac{q}{2}},\frac{5}{12}b^{\frac{q}{2}}\bigg)\bigg)^{\frac{1}{q}}\bigg\}.\end{aligned}$$

Theorem 5.3 *Let $0 < a < b$, $r \in \mathbb{R} \setminus \{-2,0,1,2\}$ and $q > 1$. Then the following inequality holds*:

$$\begin{aligned}&\bigg|2\big[G(a,b)\big]^{r-2}L\big(a^2,b^2\big) - 2\big[G(a,b)\big]^{r}\\&\quad - \frac{L(a^{r+2},b^{r+2})}{[G(a,b)]^2} - \big[G(a,b)\big]^2 L\big(a^{r-2},b^{r-2}\big) + 2L\big(a^r,b^r\big)\bigg|\end{aligned}$$

$$\leq \frac{(\ln b - \ln a)|r|}{2^{(1+\frac{1}{q})}} \frac{(b-a)^2}{[G(a,b)]^2} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \times \left[\frac{5}{6} A\left(a^{qr}, b^{qr}\right) + A\left(a^{q(r-\frac{1}{2})} b^{\frac{q}{2}}, a^{\frac{q}{2}} b^{q(r-\frac{1}{2})}\right) + \frac{1}{6} A\left(a^{q(r-1)} b^q, a^q b^{q(r-1)}\right) \right]^{1/q}. \tag{5.3}$$

Proof Using Corollary 3.7 for the functions $f(x) = x^r$, $x > 0$, $r \in \mathbb{R} \setminus \{-2, 0, 1, 2\}$ and $g(x) = (\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x})^2$, $x \in [a,b]$ with $0 < a < b$, we deduce the required result. □

Corollary 5.4 *Suppose the assumptions of Theorem 5.3 are satisfied and if* $r = -1$, *then the following inequality holds*:

$$\begin{aligned} &\left|2\left[G(a,b)\right]^{-3} L\left(a^2, b^2\right) - 2\left[G(a,b)\right]^{-1} \right. \\ &\quad \left. - \left[G(a,b)\right]^2 L\left(a^{-3}, b^{-3}\right) + L\left(a^{-1}, b^{-1}\right)\right| \\ &\leq \frac{(\ln b - \ln a)}{2^{(1+\frac{1}{q})}} \frac{(b-a)^2}{[G(a,b)]^2} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \\ &\quad \times \left[\frac{5}{6} A\left(a^{-q}, b^{-q}\right) + A\left(a^{-\frac{3}{2}q} b^{\frac{q}{2}}, a^{\frac{q}{2}} b^{-\frac{3}{2}q}\right) + \frac{1}{6} A\left(a^{-2q} b^q, a^q b^{-2q}\right) \right]^{1/q}. \end{aligned}$$

6 Conclusion

Utilizing mappings whose first-order derivatives absolute values are GA-s-convex, we establish some new Hadamard k-fractional inequalities of Fejér type associated with geometrically symmetric mappings. For the weighted inequalities via products of two different mappings, we also present their upper and lower bounds, which generalize parts of the results given by İşcan and Kunt [18]. With these techniques and the ideas developed in this paper, we hope to motivate the interested reader to further explore this fascinating field of fractional integral inequalities.

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