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Characterizations of the dilation of frame generator dual pairs for group

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Abstract

In this paper, we are interested in the dilation problem on frame generator dual pairs for a unitary representation in Hilbert spaces. We show the existence of a Riesz generator dilation dual pair of a frame generator dual pair in Hilbert spaces. Then we reveal the uniqueness of such dilations in the sense of similarity and give a characterization of the dilation of frame generator alternate dual pairs by that of the canonical dual pair in terms of a special operator. We also exhibit that the corresponding operator between two dilations of a frame generator dual pair is in a special structure.

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1 Introduction

In [12], Sun initiated the concept of g-frame as follows. Let $\mathbb{J}$ be a finite or countable index set. Let $A_i \in B(H, H_i)$, where H_i is a separable Hilbert space for any $i \in \mathbb{J}$. If there exist two constants a, b such that

$$a\|f\|^2 \le \sum_{i\in\mathbb{J}} \|A_i f\|^2 \le b\|f\|^2, \quad \forall f \in H,$$

we call $\{A_i\}_{i\in\mathbb{J}}$ a *g-frame* for H. $\{A_i\}_{i\in\mathbb{J}}$ is called a *tight g-frame* for H if $a = b$. Specially, if $a = b = 1$, we say that $\{A_i\}_{i\in\mathbb{J}}$ is a *Parseval g-frame* for H. If only the right-hand inequality holds, then $\{A_i\}_{i\in\mathbb{J}}$ is called a *g-Bessel sequence* for H. If $\overline{\operatorname{span}}\{A_i^* H_i\}_{i\in\mathbb{J}} = H$, we say that $\{A_i\}_{i\in\mathbb{J}}$ is *g-complete* in H. If $\{A_i\}_{i\in\mathbb{J}}$ is g-complete such that

$$a\bigl\|\{g_i\}_{i\in\mathbb{J}}\bigr\|^2 \le \left\|\sum_{i\in\mathbb{J}} A_i^* g_i\right\|^2 \le b\bigl\|\{g_i\}_{i\in\mathbb{J}}\bigr\|^2, \quad \forall \{g_i\}_{i\in\mathbb{J}} \in \bigoplus_{i\in\mathbb{J}} H_i,$$

we call $\{A_i\}_{i\in\mathbb{J}}$ a *g-Riesz basis* for H. As we know, if $\{A_i\}_{i\in\mathbb{J}}$ is a g-frame for H, we define $S_A f = \sum_{i\in\mathbb{J}} A_i^* A_i f$ for any $f \in H$, then S_A is a well-defined, bounded, positive, invertible operator by [12]. We call S_A a *frame operator* of $\{A_i\}_{i\in\mathbb{J}}$. Another basic fact is that $\{\widetilde{A}_i : \widetilde{A}_i = A_i S_A^{-1}\}_{i\in\mathbb{J}}$ is a g-frame for H, we call it a canonical dual g-frame of $\{A_i\}_{i\in\mathbb{J}}$. Extensively, by [10], if $\{B_i\}_{i\in\mathbb{J}}$ is a g-frame for H such that $f = \sum_{i\in\mathbb{J}} B_i^* A_i f$ for every $f \in H$, we say that it is a *dual g-frame* of $\{A_i\}_{i\in\mathbb{J}}$. Actually, g-frames and operator valued frames (see [9, 10]) are equivalent. Recently, g-frames or operator valued frames in Hilbert spaces have been studied intensively; for more details, see [1, 2, 7, 8, 10–12] and the references therein.

In order to understand the structured g-frames more deeply, Guo studied the wandering generators for a unitary system in [6]. In [10] and [9], the authors studied the frame generators for group representations. In [11] the author studied the frame generator for a group-like unitary system. These researchers showed that such an abstract way to study the structured g-frames is very feasible and fruitful. We denote a unitary representation π for a countable group $\mathcal{G}$ by $(\mathcal{G},\pi,H)$, which is a mapping $g\mapsto\pi(g)$ from $\mathcal{G}$ into the set of unitary operators on a Hilbert space H such that $\pi(g)\pi(h)=\pi(gh)$ for any $g,h\in\mathcal{G}$. In this paper, we focus on the dilation problem on frame generator dual pairs for a unitary representation in Hilbert spaces. Firstly, in order to establish our techniques, we give a direct proof for the existence of a Riesz generator dilation dual pair of a frame generator dual pair in Hilbert spaces. As there may be more than one dual frame generator of a given frame generator in general, there may be different pairs of frame generator dual pairs. Our main result characterizes the relations between the dilations of all these pairs. We illustrate that all the dilations can be mutually transformed by a type of special structured lower triangular operator matrices.

Throughout this paper, H, H_0 denote separable Hilbert spaces. Let $B(H,H_0)$ denote all the bounded linear operators from H to H_0 and $B(H):=B(H,H)$. If M, N are closed subspaces of H, $H=M\dotplus N$ denotes that $M+N=H$ and $M\cap N=\{0\}$. We use $M^{\perp}$ to denote the orthogonal complement of a closed subspace M contained in H. For an operator $T\in B(H,H_0)$, we let $\ker T$ denote the null space of T and $\operatorname{ran} T$ denote the range space of T. For a subset $\mathcal{S}\subset B(H)$, $\mathcal{S}'$ denotes the commutant of $\mathcal{S}$. We denote by $A\pi(\mathcal{G}):=\{A\pi(g)\}_{g\in\mathcal{G}}$ for $A\in B(H,H_0)$.

Definition 1.1 ([10]) Let $(\mathcal{G},\pi,H)$ be a unitary representation of the countable group $\mathcal{G}$ on a Hilbert space H. Suppose $A\in B(H,H_0)$. Then

(1) A is called *a Bessel generator* of $(\mathcal{G},\pi,H)$ if $A\pi(\mathcal{G})$ is a g-Bessel sequence for H.

(2) A is called *a* (resp. *Parseval, tight*) *frame generator* of $(\mathcal{G},\pi,H)$ if $A\pi(\mathcal{G})$ is a (resp. Parseval, tight) g-frame for H.

(3) A is called *a Riesz* (resp. *orthonormal*) *generator* of $(\mathcal{G},\pi,H)$ if $A\pi(\mathcal{G})$ is a g-Riesz (resp. g-orthonormal) basis for H.

Let $A\in B(H,H_0)$ be a Bessel generator of $(\mathcal{G},\pi,H)$. For any $f\in H$, the *analysis operator* of A is defined as

$$\theta_A: H\to l^2(\mathcal{G})\otimes H_0,\qquad \theta_A f=\sum_{g\in\mathcal{G}}\chi_g\otimes A\pi(g)^*f,$$

where $\{\chi_g\}_{g\in\mathcal{G}}$ is the orthonormal basis for $l^2(\mathcal{G})$. And the *frame operator* of A is defined as

$$S_A: H\to H,\qquad S_A f=\sum_{g\in\mathcal{G}}\pi(g)A^*A\pi(g)^*f.$$

By [10], for any $k \in H_0$, *the left regular representation* of $\mathcal{G}$ on $l^2(\mathcal{G}) \otimes H_0$ is defined as

$$\Lambda(g)(\chi_e \otimes k) = (\lambda_g \otimes I_{H_0})(\chi_e \otimes k) = \chi_g \otimes k, \quad \forall g \in \mathcal{G}, k \in H_0,$$

where λ denotes the left regular representation of $\mathcal{G}$ on $l^2(\mathcal{G})$, I_{H_0} is the identity operator on H_0, $e \in \mathcal{G}$ is the unit element.

For every $g \in \mathcal{G}$, define $Q_g : l^2(\mathcal{G}) \otimes H_0 \to H_0$ as

$$Q_g(\chi_h \otimes k) = \delta_{g,h} k, \quad \forall h \in \mathcal{G}, k \in H_0,$$

where $\delta_{g,h} = 1$ for $h = g$ and $\delta_{g,h} = 0$ for $h \neq g$.

Lemma 1.2 ([11]) *Q_h is an orthonormal generator of $(\mathcal{G}, \Lambda, l^2(\mathcal{G}) \otimes H_0)$ for any $h \in \mathcal{G}$.*

Definition 1.3 Let $(\mathcal{G}, \pi, H)$ be a unitary representation of $\mathcal{G}$ on H. If $A, B \in B(H, H_0)$ such that $B\pi(\mathcal{G})$ is a (canonical) dual g-frame (resp. g-Riesz basis) of the g-frame (resp. g-Riesz basis) $A\pi(\mathcal{G})$, we call B *a (canonical) dual frame* (resp. *Riesz*) *generator* of A (with the same structure). In this case, (A, B) is called a pair of dual frame (resp. Riesz) generators of $(\mathcal{G}, \pi, H)$ or a frame (resp. Riesz) generator dual pair.

Lemma 1.4 *Let $(\mathcal{G}, \pi, H)$ be a unitary representation of $\mathcal{G}$ on H, A, B be any Bessel generators of $(\mathcal{G}, \pi, H)$. Then $\theta_A \theta_B^* \in \Lambda(\mathcal{G})'$.*

Proof For any $g, h \in \mathcal{G}$, $k \in H_0$, we have

$$\begin{aligned} \theta_A \theta_B^* \bigl(\Lambda(g)(\chi_h \otimes k)\bigr) &= \theta_A \pi(gh) B^* k = \sum_{g' \in \mathcal{G}} \chi_{g'} \otimes A\pi\left(g'\right)^* \pi(gh) B^* k \\ &= \sum_{g' \in \mathcal{G}} \Lambda(g)\bigl(\chi_{g'} \otimes A\pi\left(g'\right)^* \pi(h) B^* k\bigr) \\ &= \Lambda(g) \theta_A \theta_B^* (\chi_h \otimes k). \end{aligned}$$

□

Theorem 1.5 *Let $(\mathcal{G}, \pi, H)$ be a unitary representation of $\mathcal{G}$ on H. Then the following are equivalent*:

(1) *(A, B) is a frame generator dual pair of $(\mathcal{G}, \pi, H)$, where $A, B \in B(H, H_0)$.*

(2) *There exist a Hilbert space $K \supset H$, a unitary representation $(\mathcal{G}, \sigma, K)$ of $\mathcal{G}$ on K, and a Riesz generator dual pair (C, D) of $(\mathcal{G}, \sigma, K)$ such that $CP = A$, $DP = B$, H is σ-invariant and $\pi = \sigma|_H$, where $C, D \in B(K, H_0)$, P is the orthogonal projection from K onto H.*

Proof Suppose that (A, B) is a frame generator dual pair of $(\mathcal{G}, \pi, H)$. Let $M = \operatorname{ran} \theta_A$, $N = \operatorname{ran} \theta_B$, P_A, P_B be the orthogonal projection from $l^2(\mathcal{G}) \otimes H_0$ onto M, N respectively. We can easily get that

$$P_A = \theta_{AS_A^{-\frac{1}{2}}} \theta^*_{AS_A^{-\frac{1}{2}}} = \theta_A S_A^{-1} \theta_A^*.$$

And then $P_A\theta_B = \theta_A S_A^{-1}$. It follows that $P_A|_N : N \to M$ is invertible. Let I_{l^2} be the identity operator on $l^2(\mathcal{G}) \otimes H_0$. Since

$$I_{l^2} = \begin{pmatrix} P_A & 0 \\ 0 & P_A^{\perp} \end{pmatrix} : \qquad \begin{pmatrix} N \\ N^{\perp} \end{pmatrix} \to \begin{pmatrix} M \\ M^{\perp} \end{pmatrix},$$

$P_A^{\perp} : N^{\perp} \to M^{\perp}$ is invertible. Similarly, $P_B^{\perp} : M^{\perp} \to N^{\perp}$ is invertible.

Let $K = H \oplus N^{\perp}$, $\sigma(g) = \pi(g) \oplus \Lambda(g)$ for any $g \in \mathcal{G}$, $C = A \oplus Q_e P_B^{\perp} \in B(K, H_0)$. Thus $\sigma(g)C^* = \pi(g)A^* \oplus \Lambda(g)P_B^{\perp}Q_e^*$ for any $g \in \mathcal{G}$. Hence, by Lemma 1.4, we have

$$\sigma(g)C^* = \pi(g)A^* \oplus P_B^{\perp}\Lambda(g)Q_e^*.$$

Therefore, for every $x \in H$, $y \in N^{\perp}$, we get

$$\begin{aligned} \theta_C(x \oplus y) &= \sum_{g\in\mathcal{G}} \chi_g \otimes C\sigma(g)^*(x \oplus y) \\ &= \sum_{g\in\mathcal{G}} \chi_g \otimes \bigl(A\pi(g)^*x + Q_e\Lambda(g)^*P_B^{\perp}y\bigr) \\ &= \theta_A x + P_B^{\perp}y. \end{aligned}$$

Let $T = \theta_A\theta_B^*$. Then $T^2 = T$. We get $l^2(\mathcal{G}) \otimes H_0 = M \dotplus N^{\perp}$. It follows that θ_C is invertible. Hence C is a Riesz generator of $(\mathcal{G}, \sigma, K)$. Obviously, $A = CP$, where P is the orthogonal projection from K onto H, and H is σ-invariant.

Since $\rho := P_A^{\perp}P_B^{\perp} : N^{\perp} \to M^{\perp}$ is invertible, there exists $\tau \in B(M^{\perp}, N^{\perp})$ such that $\tau\rho = P_B^{\perp}$. Let $D = B \oplus Q_e P_A^{\perp}\tau^* \in B(K, H_0)$.

Let $\Lambda_1(g) = P_A^{\perp}\Lambda(g)P_A^{\perp}$, $\Lambda_2(g) = P_B^{\perp}\Lambda(g)P_B^{\perp}$ for any $g \in \mathcal{G}$. Then, for arbitrary $u \in N^{\perp}$,

$$\begin{aligned} \rho\Lambda_2(g)u = \rho P_B^{\perp}\Lambda(g)P_B^{\perp}u &= P_A^{\perp}P_B^{\perp}P_B^{\perp}\Lambda(g)P_B^{\perp}u \\ &= \Lambda(g)P_A^{\perp}P_B^{\perp}u = P_A^{\perp}\Lambda(g)P_A^{\perp}\rho u \\ &= \Lambda_1(g)\rho u. \end{aligned}$$

Hence, $\tau\Lambda_1(g)v = \Lambda_2(g)\tau v$ for every $v \in M^{\perp}$.

Therefore, for any $g \in \mathcal{G}$,

$$\sigma(g)D^* = \pi(g)B^* \oplus \Lambda(g)\tau P_A^{\perp}Q_e^* = \pi(g)B^* \oplus \tau P_A^{\perp}\Lambda(g)Q_e^*.$$

Obviously, $Q_e\Lambda(\mathcal{G})P_A^{\perp}\tau^* = \{Q_e\Lambda(g)P_A^{\perp}\tau^*\}_{g\in\mathcal{G}}$ is a g-frame for $N^{\perp}$. It follows that $Q_eP_A^{\perp}\tau^*$ is a frame generator of $(\mathcal{G}, \Lambda, N^{\perp})$.

Hence, for arbitrary $x \in H$, $y \in N^{\perp}$, we have

$$\begin{aligned} \theta_D(x \oplus y) &= \sum_{g\in\mathcal{G}} \chi_g \otimes D\sigma(g)^*(x \oplus y) \\ &= \sum_{g\in\mathcal{G}} \chi_g \otimes \bigl(B\pi(g)^*x + Q_eP_A^{\perp}\tau^*\Lambda(g)^*y\bigr) \\ &= \theta_B x + P_A^{\perp}\tau^*y. \end{aligned}$$

Similarly, since $l^2(\mathcal{G}) \otimes H_0 = M^\perp \dotplus N$, θ_D is invertible. Then D is a Riesz generator of $(\mathcal{G}, \sigma, K)$ and $B = DP$.

For any $x, x_1 \in H$, $y, y_1 \in N^\perp$,

$$\begin{aligned}
&\sum_{g\in\mathcal{G}} \bigl\langle \sigma(g) C^* D \sigma(g)^* x \oplus y, x_1 \oplus y_1 \bigr\rangle \\
&\quad = \sum_{g\in\mathcal{G}} \bigl\langle \bigl(\pi(g) A^* \oplus P_B^\perp \Lambda(g) Q_e^*\bigr) \bigl(B\pi(g)^* \oplus Q_e \Lambda(g)^* P_A^\perp \tau^*\bigr)(x \oplus y), x_1 \oplus y_1 \bigr\rangle \\
&\quad = \sum_{g\in\mathcal{G}} \bigl\langle \bigl(B\pi(g)^* \oplus Q_e \Lambda(g)^* P_A^\perp \tau^*\bigr)(x \oplus y), \bigl(A\pi(g)^* \oplus Q_e \Lambda(g)^* P_B^\perp\bigr)(x_1 \oplus y_1) \bigr\rangle \\
&\quad = \sum_{g\in\mathcal{G}} \bigl\langle B\pi(g)^* x + Q_e \Lambda(g)^* P_A^\perp \tau^* y, A\pi(g)^* x_1 + Q_e \Lambda(g)^* P_B^\perp y_1 \bigr\rangle \\
&\quad = \sum_{g\in\mathcal{G}} \bigl\langle B\pi(g)^* x, A\pi(g)^* x_1 \bigr\rangle + \sum_{g\in\mathcal{G}} \bigl\langle B\pi(g)^* x, Q_e \Lambda(g)^* P_B^\perp y_1 \bigr\rangle \\
&\qquad + \sum_{g\in\mathcal{G}} \bigl\langle Q_e \Lambda_g^* P_A^\perp \tau^* y, A\pi(g)^* x_1 \bigr\rangle + \sum_{g\in\mathcal{G}} \bigl\langle Q_e \Lambda(g)^* P_A^\perp \tau^* y, Q_e \Lambda(g)^* P_B^\perp y_1 \bigr\rangle. \\
&\quad = \langle \theta_B x, \theta_A x_1 \rangle + \bigl\langle \theta_B x, P_B^\perp y_1 \bigr\rangle + \bigl\langle P_A^\perp \tau^* y, \theta_A x_1 \bigr\rangle + \bigl\langle P_A^\perp \tau^* y, P_B^\perp y_1 \bigr\rangle \\
&\quad = \langle x, x_1 \rangle + \langle y, y_1 \rangle = \langle x \oplus y, x_1 \oplus y_1 \rangle,
\end{aligned}$$

which implies (C, D) is a Riesz generator dual pair of $(\sigma, \mathcal{G}, K)$.

The converse is obvious. □

Remark 1.6 (1) From the proof above, the isomorphism of N, M is critical. In the following we provide an easier way of the proof.

In fact, as above, $C \in B(K, H_0)$ is a Riesz generator of $(\mathcal{G}, \sigma, K)$. So $S_C \in \sigma(\mathcal{G})'$. And then, $CS_C^{-1} \in B(K, H_0)$ is a Riesz generator of $(\mathcal{G}, \sigma, K)$ evidently.

Let $CS_C^{-1} = D_1 \oplus D_2$, where $D_1 \in B(H, H_0)$, $D_2 \in B(N^\perp, H_0)$. Then $D_1 = CS_C^{-1}P$. We need to show $D_1 = B$.

Identify H with $H \oplus \{0\}$. For arbitrary $x, x_1 \in H$,

$$\begin{aligned}
\langle x, x_1 \rangle &= \langle x \oplus 0, x_1 \oplus 0 \rangle \\
&= \sum_{g\in\mathcal{G}} \bigl\langle \sigma(g) C^* C S_C^{-1} \sigma(g)^* x \oplus 0, x_1 \oplus 0 \bigr\rangle \\
&= \sum_{g\in\mathcal{G}} \bigl\langle \bigl(\pi(g) A^* \oplus P_B^\perp \Lambda(g) Q_e^*\bigr) \bigl(D_1 \pi(g)^* \oplus D_2 \Lambda(g)^*\bigr)(x \oplus 0), x_1 \oplus 0 \bigr\rangle \\
&= \sum_{g\in\mathcal{G}} \bigl\langle \bigl(D_1 \pi(g)^* \oplus D_2 \Lambda(g)^*\bigr)(x \oplus 0), \bigl(A\pi(g)^* \oplus Q_e \Lambda(g)^* P_B^\perp\bigr)(x_1 \oplus 0) \bigr\rangle \\
&= \sum_{g\in\mathcal{G}} \bigl\langle D_1 \pi(g)^* x, A\pi(g)^* x_1 \bigr\rangle,
\end{aligned}$$

which means $D_1 \in B(H, H_0)$ is a dual frame generator of A.

Moreover, for any $x, x_1 \in H$, $y_1 \in N^\perp$.

$$\langle x, x_1 \rangle = \langle x \oplus 0, x_1 \oplus y_1 \rangle$$

$$\begin{aligned}
&= \sum_{g\in\mathcal{G}} \bigl\langle \sigma(g)C^*CS_C^{-1}\sigma(g)^*x \oplus 0, x_1 \oplus y_1\bigr\rangle \\
&= \sum_{g\in\mathcal{G}} \bigl\langle D_1\pi(g)^*x, A\pi(g)^*x_1 + Q_eP_B^{\perp}\Lambda(g)^*y_1\bigr\rangle \\
&= \langle x, x_1\rangle + \sum_{g\in\mathcal{G}} \bigl\langle D_1\pi(g)^*x, Q_eP_B^{\perp}\Lambda(g)^*y_1\bigr\rangle.
\end{aligned}$$

Then

$$\sum_{g\in\mathcal{G}} \bigl\langle D_1\pi(g)^*x, Q_eP_B^{\perp}\Lambda(g)^*y_1\bigr\rangle = \langle \theta_{D_1}x, y_1\rangle = 0.$$

We get $\operatorname{ran}\theta_{D_1} \subseteq N = \operatorname{ran}\theta_B$. Hence, for every $x \in H$, there exists $x_1 \in H$ such that $\theta_{D_1}x = \theta_B x_1$. Then

$$x = \theta_A^*\theta_{D_1}x = \theta_A^*\theta_B x_1 = x_1.$$

Therefore, for any $x \in H$, we obtain $\theta_{D_1}x = \theta_B x$, which implies $B = D_1$.

(2) We can also have another way to prove $B = D_1$ as follows.

For any $x, x_1 \in H$, $y_1 \in N^{\perp}$, on the one hand,

$$\begin{aligned}
\langle x, x_1\rangle &= \langle x \oplus 0, x_1 \oplus y_1\rangle \\
&= \sum_{g\in\mathcal{G}} \bigl\langle \sigma(g)C^*CS_C^{-1}\sigma(g)^*x \oplus 0, x_1 \oplus y_1\bigr\rangle \\
&= \sum_{g\in\mathcal{G}} \bigl\langle D_1\pi(g)^*x, C\sigma(g)^*(x_1 \oplus y_1)\bigr\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{g\in\mathcal{G}} \bigl\langle B\pi(g)^*x, C\sigma(g)^*(x_1 \oplus y_1)\bigr\rangle \\
&\quad= \sum_{g\in\mathcal{G}} \bigl\langle B\pi(g)^*x, A\pi(g)^*x_1 + Q_1P_B^{\perp}\Lambda(g)^*y_1\bigr\rangle \\
&\quad= \langle x, x_1\rangle.
\end{aligned}$$

Hence, $\theta_C^*\theta_B x = \theta_C^*\theta_{D_1}x$. Since θ_C is invertible, it follows that $\theta_B x = \theta_{D_1}x$, which means $B = D_1$.

Definition 1.7 Let $(\mathcal{G}, \pi, H)$, $(\mathcal{G}, \pi_1, N_1)$ be unitary representations of $\mathcal{G}$ on H, N_1 respectively, (A, B) be a pair of dual frame generators of $(\mathcal{G}, \pi, H)$, where $A, B \in B(H, H_0)$. If there exist $C_1, D_1 \in B(N_1, H_0)$ such that $(E, \widetilde{E})$ is a Riesz generator dual pair of $(\mathcal{G}, \sigma_1, H \oplus N_1)$, where $\sigma_1 = \pi \oplus \pi_1$, $E = A \oplus C_1$, $\widetilde{E} = B \oplus D_1$. We call $(E, \widetilde{E})$ *a dilation* of (A, B). (C_1, D_1) is called *a complementary generator pair* of (A, B). Particularly, C_1 is called *a complementary generator* of A.

Theorem 1.5 illustrates that there exists a dilation for any pair of dual frame generators.

In the next we show that the dilation of a dual frame generator pair is unique in the sense of "similarity".

Theorem 1.8 *Let $(\mathcal{G},\pi,H)$, $(\mathcal{G},\pi_1,N_1)$, $(\mathcal{G},\pi_2,N_2)$ be unitary representations of $\mathcal{G}$ on H, N_1, N_2 respectively, (A,B) be a pair of dual frame generators of $(\mathcal{G},\pi,H)$, where $A,B\in B(H,H_0)$. Suppose that $(E,\widetilde{E})$ is a Riesz generator dual pair of $(\mathcal{G},\sigma_1,H\oplus N_1)$ which is also a dilation of (A,B), where $\sigma_1=\pi\oplus\pi_1$, $E=A\oplus C_1$, $C_1\in B(N_1,H_0)$, $\widetilde{E}$ is the canonical dual Riesz generator of E. If there exist $C_2\in B(N_2,H_0)$ and an invertible operator $T\in B(N_1,N_2)$ such that $C_1\pi_1(g)=C_2\pi_2(g)T$ for every $g\in\mathcal{G}$, then F is a Riesz generator of $(\mathcal{G},\sigma_2,H\oplus N_2)$, where $\sigma_2=\pi\oplus\pi_2$, $F=A\oplus C_2\in B(H\oplus N_2,H_0)$. Moreover, $(F,\widetilde{F})$ is a Riesz generator dual pair of $(\mathcal{G},\sigma_2,H\oplus N_2)$ which is also a dilation of (A,B), where $\widetilde{F}$ is the canonical dual Riesz generator of F.*

Proof Since $C_1\pi_1(g)=C_2\pi_2(g)T$ for any $g\in\mathcal{G}$, we have

$$F\sigma_2(g)=A\pi(g)\oplus C_2\pi_2(g)=A\pi(g)\oplus C_1\pi_1(g)T^{-1}=E\sigma_1(g)\bigl(I\oplus T^{-1}\bigr),$$

which implies F is a Riesz generator of $(\mathcal{G},\sigma_2,H\oplus N_2)$ and $A=FP_2$, where P_2 is the orthogonal projection from $H\oplus N_2$ onto H.

Denote $\widehat{T}=I\oplus T$. As $\widetilde{F}=FS_F^{-1}$, it follows that, for every $g\in\mathcal{G}$,

$$\widetilde{F}\sigma_2(g)=F\sigma_2(g)S_F^{-1}=E\sigma_1(g)\widehat{T}^{-1}S_F^{-1}=\widetilde{E}\sigma_1(g)S_E\widehat{T}^{-1}S_F^{-1}.$$

Besides, we can get $S_F=\theta_F^*\theta_F=(\widehat{T}^*)^{-1}S_E\widehat{T}^{-1}$. Then

$$\widetilde{F}\sigma_2(g)=\widetilde{E}\sigma_1(g)\widehat{T}^*=\widetilde{E}\sigma_1(g)\bigl(I\oplus T^*\bigr).$$

Evidently, $\widetilde{F}\sigma_2(g)P_2=B\pi(g)$. Specially, we have $\widetilde{F}P_2=B$. □

In the following we exhibit that two different complementary generators of a given frame generator are "similar".

Theorem 1.9 *Let $(\mathcal{G},\pi,H)$, $(\mathcal{G},\pi_1,N_1)$, $(\mathcal{G},\pi_2,N_2)$ be unitary representations of $\mathcal{G}$ on H, N_1, N_2 respectively, (A,B) be a pair of dual frame generators of $(\mathcal{G},\pi,H)$, where $A,B\in B(H,H_0)$. Suppose that $(E,\widetilde{E})$ is a Riesz generator dual pair of $(\mathcal{G},\sigma_1,H\oplus N_1)$ which is also a dilation of (A,B), where $\sigma_1=\pi\oplus\pi_1$, $E=A\oplus C_1$, $C_1\in B(N_1,H_0)$, $\widetilde{E}$ is the canonical dual Riesz generator of E. If $(F,\widetilde{F})$ is a Riesz generator dual pair of $(\mathcal{G},\sigma_2,H\oplus N_2)$ which is also a dilation of (A,B), where $\sigma_2=\pi\oplus\pi_2$, $F=A\oplus C_2$, $C_2\in B(N_2,H_0)$, $\widetilde{F}$ is the canonical dual Riesz generator of F, then there exists an invertible operator $T\in B(N_1,N_2)$ such that $C_1\pi_1(g)=C_2\pi_2(g)T$ for every $g\in\mathcal{G}$. In particular, $T\pi_1(g)=\pi_2(g)T$.*

Proof Let $\widetilde{E}=B\oplus D_1$, $\widetilde{F}=B\oplus D_2$, where $D_1\in B(N_1,H_0)$, $D_2\in B(N_2,H_0)$. Then, for every $x,y\in H$, $x_1\in N_1$, we get

$$\begin{aligned}\langle x,y\rangle &=\langle x\oplus 0,y\oplus x_1\rangle=\bigl\langle\theta_E^*\theta_{\widetilde{E}}(x\oplus 0),y\oplus x_1\bigr\rangle\\ &=\sum_{g\in\mathcal{G}}\bigl\langle B\pi(g)^*x,A\pi(g)^*y+C_1\pi_1(g)^*x_1\bigr\rangle\\ &=\langle x,y\rangle+\bigl\langle x,\theta_B^*\theta_{C_1}x_1\bigr\rangle.\end{aligned}$$

Thus $\theta_B^*\theta_{C_1} = 0$, which implies $\operatorname{ran}\theta_{C_1} \subseteq (\operatorname{ran}\theta_B)^\perp$. Since E is a Riesz generator of $(\mathcal{G}, \sigma_1, H \oplus N_1)$, by [2, Proposition 2.3] we have

$$l^2(\mathcal{G}) \otimes H_0 = \operatorname{ran}\theta_A \dotplus \operatorname{ran}\theta_{C_1}.$$

Hence, for any $u \in (\operatorname{ran}\theta_B)^\perp$, there exist $x \in H$, $x_1 \in N_1$ such that $u = \theta_A x + \theta_{C_1} x_1$. Then, $0 = \theta_B^* u = \theta_B^*\theta_A x + \theta_B^*\theta_{C_1} x_1 = x$, therefore, $u = \theta_{C_1} x_1 \in \operatorname{ran}\theta_{C_1}$, equivalently, $(\operatorname{ran}\theta_B)^\perp \subseteq \operatorname{ran}\theta_{C_1}$. Therefore, we obtain $\operatorname{ran}\theta_{C_1} = (\operatorname{ran}\theta_B)^\perp$.

Similarly, we obtain $\operatorname{ran}\theta_{C_2} = (\operatorname{ran}\theta_B)^\perp$. Then $\operatorname{ran}\theta_{C_1} = \operatorname{ran}\theta_{C_2}$. Hence, by [10, Proposition 4.3], there is an invertible operator $T \in B(N_1, N_2)$ such that $C_1\pi_1(g) = C_2\pi_2(g)T$ for arbitrary $g \in \mathcal{G}$.

Specially, for every $x_1 \in N_1$, we have $x = \sum_{g\in\mathcal{G}} \pi_1(g)C_1^*\widetilde{C_1}\pi_1(g)^* x$, where $\widetilde{C_1} \in B(N_1, H_0)$ is the canonical dual frame generator of C_1. Besides, for any $h \in \mathcal{G}$,

$$\begin{aligned}
\pi_2(h)\bigl(T^*\bigr)^{-1}x &= \sum_{g\in\mathcal{G}} \pi_2(h)\bigl(T^*\bigr)^{-1}\pi_1(g)C_1^*\widetilde{C_1}\pi_1(g)^* x \\
&= \sum_{g\in\mathcal{G}} \pi_2(hg)C_2^*\widetilde{C_1}\pi_1(hg)^*\pi_1(h)x \\
&= \sum_{g\in\mathcal{G}} \bigl(T^*\bigr)^{-1}\pi_1(hg)C_1^*\widetilde{C_1}\pi_1(hg)^*\pi_1(h)x \\
&= \bigl(T^*\bigr)^{-1}\pi_1(h)x,
\end{aligned}$$

which implies $T\pi_1(g) = \pi_2(g)T$ for every $g \in \mathcal{G}$. □

By the proof of Theorem 1.8 and Theorem 1.9, two complementary generator pairs of a frame generator dual pair are "similar" by the operator pair (T, T^*), which implies that complementary generator pairs are unique in the sense of "similarity".

2 Characterization of all the dilations of a frame generator dual pair

The following is a result which describes the dilation of the frame generator canonical dual pair.

Theorem 2.1 *Let $(\mathcal{G}, \pi, H)$ be a unitary representation of $\mathcal{G}$ on H, $(A, \widetilde{A})$ be a pair of dual frame generators of $(\mathcal{G}, \pi, H)$, where $A \in B(H, H_0)$, $\widetilde{A}$ is the canonical dual frame generator of A. Denote $E = A \oplus Q_e P_A^\perp$, $\widetilde{E} = \widetilde{A} \oplus Q_e P_A^\perp \in B(H \oplus M^\perp, H_0)$, where $M = \operatorname{ran}\theta_A$, P_A is the orthogonal projection from $l^2(\mathcal{G}) \otimes H_0$ onto M. Then $(E, \widetilde{E})$ is a pair of dual Riesz generators of $(\mathcal{G}, \sigma, H \oplus M^\perp)$ which is a dilation of $(A, \widetilde{A})$, where $\sigma := \pi \oplus \Lambda$, $(\mathcal{G}, \Lambda, l^2(\mathcal{G}) \otimes H_0)$ is the left regular representation of $\mathcal{G}$.*

Proof Let $C = Q_e P_A^\perp \in B(M^\perp, H_0)$. By Lemma 1.4, $C\Lambda(\mathcal{G})$ is a Parseval g-frame for $M^\perp$ and

$$\operatorname{ran}\theta_A = \operatorname{ran}\theta_{\widetilde{A}} = M, \qquad \operatorname{ran}\theta_C = M^\perp.$$

Since $\theta_E(x \oplus y) = \theta_A x + \theta_C y$ for any $x \in H$, $y \in M^\perp$, we have θ_E is invertible, which means E is a Riesz generator of $(\mathcal{G}, \sigma, H \oplus M^\perp)$. Similarly, $\widetilde{E}$ is also a Riesz generator of $(\mathcal{G}, \sigma, H \oplus M^\perp)$.

For arbitrary $x, y \in H$, $x_1, y_1 \in M^{\perp}$,

$$\begin{aligned}
&\Big\langle \sum_{g\in\mathcal{G}} \sigma(g)E^*\widetilde{E}\sigma(g)^*(x\oplus x_1), y\oplus y_1\Big\rangle \\
&\quad= \Big\langle \sum_{g\in\mathcal{G}} \big(\pi(g)A^* \oplus \Lambda(g)P_A^{\perp}Q_e^*\big)\big(\widetilde{A}\pi(g)^* \oplus Q_eP_A^{\perp}\Lambda(g)^*\big)(x\oplus x_1), y\oplus y_1\Big\rangle \\
&\quad= \sum_{g\in\mathcal{G}} \big\langle \widetilde{A}\pi(g)^*x + Q_eP_A^{\perp}\Lambda(g)^*x_1, A\pi(g)^*y + Q_eP_A^{\perp}\Lambda(g)^*y_1\big\rangle \\
&\quad= \sum_{g\in\mathcal{G}} \big\langle \widetilde{A}\pi(g)^*x, A\pi(g)^*y\big\rangle + \sum_{g\in\mathcal{G}} \big\langle \widetilde{A}\pi(g)^*x, Q_eP_A^{\perp}\Lambda(g)^*y_1\big\rangle \\
&\qquad+ \sum_{g\in\mathcal{G}} \big\langle Q_eP_A^{\perp}\Lambda(g)^*x_1, A\pi(g)^*y\big\rangle + \sum_{g\in\mathcal{G}} \big\langle Q_eP_A^{\perp}\Lambda(g)^*x_1, Q_eP_A^{\perp}\Lambda(g)^*y_1\big\rangle \\
&\quad= \sum_{g\in\mathcal{G}} \big\langle \widetilde{A}\pi(g)^*x, A\pi(g)^*y\big\rangle + \sum_{g\in\mathcal{G}} \big\langle Q_eP_A^{\perp}\Lambda(g)^*x_1, Q_eP_A^{\perp}\Lambda(g)^*y_1\big\rangle \\
&\quad= \langle x, y\rangle + \langle x_1, y_1\rangle = \langle x\oplus x_1, y\oplus y_1\rangle,
\end{aligned}$$

which means $(E, \widetilde{E})$ is a pair of dual Riesz generators of $(\mathcal{G}, \sigma, H\oplus M^{\perp})$. □

We call the dilation $(E, \widetilde{E})$ in Theorem 2.1 the *natural dilation* of $(A, \widetilde{A})$. If $A \in B(H, H_0)$ is a frame generator of $(\mathcal{G}, \pi, H)$, let $M = \operatorname{ran}\theta_A$, $\sigma = \pi \oplus \Lambda$, P_A be the orthogonal projection from $l^2(\mathcal{G})\otimes H_0$ onto M. We can characterize all the dilations of frame generator dual pairs in terms of the natural dilation by a type of special lower triangular operator matrices.

Theorem 2.2 *Let $(\mathcal{G}, \pi, H)$ be a unitary representation of $\mathcal{G}$ on H, (A, B) be a frame generator alternate dual pair of $(\mathcal{G}, \pi, H)$, where $A, B \in B(H, H_0)$. Then there exists a pair of dual g-Riesz bases $(E', \widetilde{E}')$ which is a dilation of the g-frame dual pair $(A\pi(\mathcal{G}), B\pi(\mathcal{G}))$ such that $E' := \{E'_g = E\sigma(g)T^*\}_{g\in\mathcal{G}}$, where $\widetilde{E}' := \{\widetilde{E}'_g\}_{g\in\mathcal{G}}$ is the canonical dual g-frame of E',*

$$T = \begin{pmatrix} I_H & 0 \\ T' & I_{M^{\perp}} \end{pmatrix} \in B\big(H\oplus M^{\perp}\big),$$

$T' \in B(H, M^{\perp})$. (Note that $(E, \widetilde{E})$ is the natural dilation of $(A, \widetilde{A})$ in Theorem 2.1.)

Proof Let $\Gamma = \widetilde{A} - B \in B(H, H_0)$. Obviously, Γ is a Bessel generator of $(\mathcal{G}, \pi, H)$. Then

$$\theta_\Gamma^*\theta_A = -\theta_B^*\theta_A + \theta_{\widetilde{A}}^*\theta_A = 0,$$

which implies $\operatorname{ran}\theta_\Gamma \perp \operatorname{ran}\theta_A$. So $T' := \theta_\Gamma \in B(H, M^{\perp})$. Besides, $T = \left(\begin{smallmatrix} I_H & 0 \\ T' & I_{M^{\perp}} \end{smallmatrix}\right) \in B(H\oplus M^{\perp})$ is invertible. Hence, for any $g \in \mathcal{G}$,

$$\begin{aligned}
E\sigma(g)T^* &= \big(A\pi(g) \oplus Q_e\Lambda(g)P_A^{\perp}\big)\begin{pmatrix} I_H & (T')^* \\ 0 & I_{M^{\perp}} \end{pmatrix} \\
&= A\pi(g) \oplus \big(A\pi(g)\big(T'\big)^* + Q_e\Lambda(g)P_A^{\perp}\big).
\end{aligned}$$

Evidently, $E' := \{E'_g = E\sigma(g)T^*\}_{g\in\mathcal{G}}$ is a g-Riesz basis for $H \oplus M^\perp$ and $A\pi(g) = E\sigma(g)T^*P$ for every $g \in \mathcal{G}$, where P is the orthogonal projection from $H \oplus M^\perp$ onto H. We can directly get $\widetilde{E}' = \{\widetilde{E}'_g := \widetilde{E}\sigma(g)T^{-1}\}_{g\in\mathcal{G}}$ is the canonical dual g-Riesz basis of E'. Moreover, for any $g \in \mathcal{G}$,

$$\begin{aligned}\widetilde{E}\sigma(g)T^{-1} &= \bigl(\widetilde{A}\pi(g) \oplus Q_e\Lambda(g)P_A^\perp\bigr)\begin{pmatrix} I_H & 0 \\ -T' & I_{M^\perp}\end{pmatrix}\\ &= \bigl(\widetilde{A}\pi(g) - Q_e\Lambda(g)P_A^\perp T'\bigr) \oplus Q_e\Lambda(g)P_A^\perp.\end{aligned}$$

Obviously, $(T')^*P_A^\perp = (T')^*$, where P_A is the orthogonal projection from $l^2(\mathcal{G}) \otimes H_0$ onto M. And then, for every $g \in \mathcal{G}$, $k \in H_0$, we obtain

$$\bigl(T'\bigr)^* P_A^\perp \Lambda(g)Q_e^* k = -\pi(g)B^*k + \pi(g)\widetilde{A}^*k.$$

Hence, $B\pi(g) = \widetilde{A}\pi(g) - Q_e\Lambda(g)P_A^\perp T'$ which means $\widetilde{E}'_g = B\pi(g) \oplus Q_e\Lambda(g)P_A^\perp$ and $B\pi(g) = \widetilde{E}'_g P$. □

The next conclusion is more general than Theorem 2.2.

Theorem 2.3 *Let $(\mathcal{G},\pi,H)$, $(\mathcal{G},\pi_2,N)$ be unitary representations of $\mathcal{G}$ on H, N respectively, (A,B) be a frame generator alternate dual pair of $(\mathcal{G},\pi,H)$, where $A,B \in B(H,H_0)$. If there is an invertible operator $\widetilde{T} \in B(M^\perp,N)$ such that $\widetilde{T}\Lambda(g) = \pi_2(g)\widetilde{T}$ for every $g \in \mathcal{G}$, then there exists a pair of dual Riesz generators $(E',\widetilde{E}')$ of $(\mathcal{G},\sigma_2,H\oplus N)$ which is also a dilation of (A,B) such that $E'\sigma_2(g) = E\sigma(g)T^*$ for any $g \in \mathcal{G}$, where $\sigma_2 = \pi \oplus \pi_2$, $\widetilde{E}'$ is the canonical dual frame generator of E', $T = \bigl(\begin{smallmatrix} I_H & 0 \\ T' & \widetilde{T}\end{smallmatrix}\bigr) \in B(H \oplus M^\perp, H \oplus N)$, $T' \in B(H,N)$. (Note that $(E,\widetilde{E})$ is the natural dilation of $(A,\widetilde{A})$ in Theorem 2.1.)*

Proof Let $\Gamma = -B + \widetilde{A} \in B(H,H_0)$. It is easy to verify that Γ is a Bessel generator of $(\mathcal{G},\pi,H)$. Denote $T' := \widetilde{T}\theta_\Gamma \in B(H,N)$. Then T is invertible and $T^{-1} = \bigl(\begin{smallmatrix} I_H & 0 \\ -\widetilde{T}^{-1}T' & \widetilde{T}^{-1}\end{smallmatrix}\bigr)$.

Let $E'_g = E\sigma(g)T^*$ for every $g \in \mathcal{G}$, thus

$$\begin{aligned}E'_g = E\sigma(g)T^* &= \bigl(A\pi(g) \oplus Q_e\Lambda(g)P_A^\perp\bigr)\begin{pmatrix} I_H & (T')^* \\ 0 & \widetilde{T}^*\end{pmatrix}\\ &= A\pi(g) \oplus \bigl(A\pi(g)\bigl(T'\bigr)^* + Q_e\Lambda(g)P_A^\perp\widetilde{T}^*\bigr)\\ &= A\pi(g) \oplus \bigl(A\pi(g)\theta_\Gamma^*\widetilde{T}^* + Q_e\widetilde{T}^*\pi_2(g)\bigr)\\ &= A\pi(g) \oplus \bigl(\bigl(A\theta_\Gamma^*\widetilde{T}^* + Q_e\widetilde{T}^*\bigr)\pi_2(g)\bigr)\\ &= \bigl(A \oplus \bigl(A\theta_\Gamma^*\widetilde{T}^* + Q_e\widetilde{T}^*\bigr)\bigr)\bigl(\pi(g) \oplus \pi_2(g)\bigr).\end{aligned}$$

Denote $E' := A \oplus (A\theta_\Gamma^*\widetilde{T}^* + Q_e\widetilde{T}^*) \in B(H \oplus N, H_0)$, $\sigma_2 = \pi \oplus \pi_2$. Therefore, E' is a Riesz generator of $(\mathcal{G},\sigma_2,H\oplus N)$ and $A = E'P$, where P is the orthogonal projection from $H \oplus N$ onto H. Denote $\widetilde{E}'_g := \widetilde{E}\sigma(g)T^{-1}$ for any $g \in \mathcal{G}$. It is easy to examine $\{\widetilde{E}'_g\}_{g\in\mathcal{G}}$ is the canonical dual g-frame of $\{E'_g\}_{g\in\mathcal{G}}$. Then it is also a g-Riesz basis for $H \oplus N$. Moreover, for any $g \in \mathcal{G}$,

$$\widetilde{E}\sigma(g)T^{-1} = \bigl(\widetilde{A}\pi(g) \oplus Q_e\Lambda(g)P_A^\perp\bigr)T^{-1}$$

$$\begin{aligned}
&= \left(\widetilde{A}\pi(g) \oplus Q_e \Lambda(g) P_A^{\perp}\right) \begin{pmatrix} I_H & 0 \\ -\widetilde{T}^{-1}T' & \widetilde{T}^{-1} \end{pmatrix} \\
&= \left(\widetilde{A}\pi(g) - Q_e \Lambda(g) P_A^{\perp} \widetilde{T}^{-1} T'\right) \oplus Q_e \Lambda(g) P_A^{\perp} \widetilde{T}^{-1} \\
&= \left(\widetilde{A}\pi(g) - Q_e \Lambda(g) P_A^{\perp} \theta_{\Gamma}\right) \oplus Q_e \Lambda(g) P_A^{\perp} \widetilde{T}^{-1} \\
&= \left(\widetilde{A}\pi(g) - \left(\widetilde{A}\pi(g) - B\pi(g)\right)\right) \oplus Q_e \Lambda(g) P_A^{\perp} \widetilde{T}^{-1} \\
&= B\pi(g) \oplus Q_e \Lambda(g) \widetilde{T}^{-1} \\
&= B\pi(g) \oplus Q_e \widetilde{T}^{-1} \pi_2(g).
\end{aligned}$$

Denote $\widetilde{E}' = B \oplus Q_e \widetilde{T}^{-1}$. Hence, $\widetilde{E}'$ is a Riesz generator of $(\mathcal{G}, \sigma_2, H \oplus N)$ which is a dual frame generator of E' and such that $B = \widetilde{E}'P$. □

The following result illustrates that if a dilation g-Riesz basis pair of a frame generator alternate dual pair and that of the frame generator canonical dual pair are "similar" by a pair of operators, then one of the corresponding operators is in the form of T in Theorem 2.3.

Theorem 2.4 *Let $(\mathcal{G}, \pi, H)$ be a unitary representation of $\mathcal{G}$ on H, (A, B) be a frame generator alternate dual pair of $(\mathcal{G}, \pi, H)$, where $A, B \in B(H, H_0)$. If there exists a pair of dual g-Riesz basis $(E', \widetilde{E}')$ for $H \oplus N$ which is also a dilation of the dual g-frame pair $(A\pi(\mathcal{G}), B\pi(\mathcal{G}))$ such that $E' := \{E'_g = E\sigma(g)T^*\}_{g \in \mathcal{G}}$ for any $g \in \mathcal{G}$, where N is a Hilbert space, $\widetilde{E}'$ is the canonical dual g-Riesz basis of E', $T \in B(H \oplus M^{\perp}, H \oplus N)$ is invertible. Then T has the following form:*

$$T = \begin{pmatrix} I_H & 0 \\ T' & \widetilde{T} \end{pmatrix},$$

where $T' \in B(H, N)$, $\widetilde{T} \in B(M^{\perp}, N)$. (Note that $(E, \widetilde{E})$ is the natural dilation of $(A, \widetilde{A})$ in Theorem 2.1.)

Proof As in the proofs of Theorem 2.2 and Theorem 2.3, let $\Gamma = -B + \widetilde{A} \in B(H, H_0)$. Then Γ is a Bessel generator of $(\mathcal{G}, \pi, H)$. Let $T_0 = \left(\begin{smallmatrix} I_H & 0 \\ \theta_{\Gamma} & I_{M^{\perp}} \end{smallmatrix}\right) \in B(H \oplus M^{\perp})$. Denote $F := \{F_g = E\sigma(g)T_0^*\}_{g \in \mathcal{G}}$. By Theorem 2.2, $(F, \widetilde{F})$ is a dual g-Riesz basis pair for $H \oplus M^{\perp}$, which is a dilation of $(A\pi(\mathcal{G}), B\pi(\mathcal{G}))$, where $\widetilde{F} = \{\widetilde{F}_g := \widetilde{E}\sigma(g)T_0^{-1}\}_{g \in \mathcal{G}}$ is the canonical dual g-Riesz basis of F. Since $(E', \widetilde{E}')$ is a g-Riesz basis dual pair for $H \oplus N$, which is a dilation of $(A\pi(\mathcal{G}), B\pi(\mathcal{G}))$, for arbitrary $g \in \mathcal{G}$, denote $E'_g = A\pi(g) \oplus C_g$, $F_g = A\pi(g) \oplus C'_g$, where $C_g \in B(N, H_0)$, $C'_g \in B(M^{\perp}, H_0)$. By the proof of Theorem 1.9, there is an invertible operator $\widetilde{T} \in B(M^{\perp}, N)$ such that $C_g = C'_g \widetilde{T}^*$. Therefore,

$$F_g \begin{pmatrix} I_H & 0 \\ 0 & \widetilde{T}^* \end{pmatrix} = \left(A\pi(g) \oplus C'_g\right) \begin{pmatrix} I_H & 0 \\ 0 & \widetilde{T}^* \end{pmatrix} = E'_g = E\sigma(g)T^*.$$

And then,

$$F_g \begin{pmatrix} I_H & 0 \\ 0 & \widetilde{T}^* \end{pmatrix} = E\sigma(g) \begin{pmatrix} I_H & \theta_{\Gamma}^* \\ 0 & I_{M^{\perp}} \end{pmatrix} \begin{pmatrix} I_H & 0 \\ 0 & \widetilde{T}^* \end{pmatrix} = E\sigma(g)T^*.$$

Since $E\sigma(\mathcal{G})$ is a g-Riesz basis for $H \oplus M^{\perp}$, it is g-complete. Then

$$\begin{pmatrix} I_H & \theta_\Gamma^* \\ 0 & I_{M^\perp} \end{pmatrix} \begin{pmatrix} I_H & 0 \\ 0 & \widetilde{T}^* \end{pmatrix} = \begin{pmatrix} I_H & \theta_\Gamma^* \widetilde{T}^* \\ 0 & \widetilde{T}^* \end{pmatrix} = T^*.$$

Hence,

$$T = \begin{pmatrix} I_H & 0 \\ \widetilde{T}\theta_\Gamma & \widetilde{T} \end{pmatrix},$$

where $T' = \widetilde{T}\theta_\Gamma \in B(H,N)$. □

The following is our main result which shows that if the dilation Riesz generator dual pair of a frame generator alternate dual pair has a relationship with that of another frame generator alternate dual pair by a pair of operators, then one of the corresponding operators is in the form of a lower triangular matrix.

Proposition 2.5 *Let $(\mathcal{G},\pi,H)$, $(\mathcal{G},\pi_1,N_1)$, $(\pi_2,\mathcal{G},N_2)$ be unitary representations of $\mathcal{G}$ on H, N_1, N_2 respectively, (A,B), (A,B') be frame generator alternate dual pairs of $(\mathcal{G},\pi,H)$, where $A,B,B' \in B(H,H_0)$. Suppose that there exist pairs of dual Riesz generators $(E',\widetilde{E}')$, $(F,\widetilde{F})$ of $(\mathcal{G},\sigma_1,H\oplus N_1)$, $(\mathcal{G},\sigma_2,H\oplus N_2)$ which are also dilations of (A,B), (A,B') respectively, where $\sigma_1 = \pi \oplus \pi_1$, $\sigma_2 = \pi \oplus \pi_2$, $E',\widetilde{E}' \in B(H\oplus N_1,H_0)$, $F,\widetilde{F} \in B(H\oplus N_2,H_0)$. If there is an invertible operator $T \in B(H\oplus N_1, H\oplus N_2)$ such that $F\sigma_2(g) = E'\sigma_1(g)T^*$ for any $g \in \mathcal{G}$. Then T is in the form of the following*:

$$T = \begin{pmatrix} I_H & 0 \\ T' & \widetilde{T} \end{pmatrix},$$

where $T' \in B(H,N_2)$, $\widetilde{T} \in B(N_1,N_2)$.

Proof Let $\Gamma = \widetilde{A} - B \in B(H,H_0)$, $\Gamma' = \widetilde{A} - B' \in B(H,H_0)$. We can easily examine Γ, Γ' are Bessel generators of $(\mathcal{G},\pi,H)$ and $\operatorname{ran}\theta_\Gamma$, $\operatorname{ran}\theta_{\Gamma'} \subset (\operatorname{ran}\theta_A)^\perp$.

Similar to the proof of Theorem 2.3, we can construct

$$T_1 = \begin{pmatrix} I_H & 0 \\ 0 & \widetilde{T}_1 \end{pmatrix} \begin{pmatrix} I_H & 0 \\ \theta_\Gamma & I_{M^\perp} \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ \widetilde{T}_1\theta_\Gamma & \widetilde{T}_1 \end{pmatrix} \in B\bigl(H\oplus M^\perp, H\oplus N_1\bigr)$$

and

$$T_2 = \begin{pmatrix} I_H & 0 \\ 0 & \widetilde{T}_2 \end{pmatrix} \begin{pmatrix} I_H & 0 \\ \theta_{\Gamma'} & I_{M^\perp} \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ \widetilde{T}_2\theta_{\Gamma'} & \widetilde{T}_2 \end{pmatrix} \in B\bigl(H\oplus M^\perp, H\oplus N_2\bigr),$$

where $\widetilde{T}_1 \in B(M^\perp,N_1)$, $\widetilde{T}_2 \in B(M^\perp,N_2)$ are invertible. (In fact, we can let $E' = A \oplus C_1$, $F = A \oplus C_2$, where $C_1 \in B(N_1,H_0)$, $C_2 \in B(N_2,H_0)$. By Theorem 1.9, there are invertible operators $\widetilde{T}_1 \in B(M^\perp,N_1)$, $\widetilde{T}_2 \in B(M^\perp,N_2)$ such that $C_g^1\widetilde{T}_1^* = C_1\pi_1(g)$ and $C_g^2\widetilde{T}_2^* = C_2\pi_2(g)$, where $E_g^1 = E\sigma(g)(T_0^1)^*$, $E_g^2 = E\sigma(g)(T_0^2)^*$ for every $g \in \mathcal{G}$, $T_0^1 = \left(\begin{smallmatrix} I_H & 0 \\ \theta_\Gamma & I_{M^\perp} \end{smallmatrix}\right)$, $T_0^2 = \left(\begin{smallmatrix} I_H & 0 \\ \theta'_\Gamma & I_{M^\perp} \end{smallmatrix}\right) \in B(H\oplus M^\perp)$.)

Evidently, T_1, T_2 are invertible, $E\sigma(g)T_1^* = E'\sigma_1(g)$, $E\sigma(g)T_2^* = F\sigma_2(g)$ for any $g \in \mathcal{G}$.

Denote $T_1' = \widetilde{T}_1\theta_\Gamma \in B(H,N_1)$, $T_2' = \widetilde{T}_2\theta_{\Gamma'} \in B(H,N_2)$.

Then $F\sigma_2(g) = E\sigma(g)T_2^* = E'\sigma_1(g)(T_1^*)^{-1}T_2^*$ for any $g \in \mathcal{G}$. Because $T_2(T_1)^{-1} \in B(H \oplus N_1, H \oplus N_2)$ and

$$T_2(T_1)^{-1} = \begin{pmatrix} I_H & 0 \\ T_2' & \widetilde{T}_2 \end{pmatrix} \begin{pmatrix} I_H & 0 \\ -\widetilde{T}_1^{-1}T_1' & \widetilde{T}_1^{-1} \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ \widetilde{T}_2(\theta_{\Gamma'} - \theta_\Gamma) & \widetilde{T}_2\widetilde{T}_1^{-1} \end{pmatrix}.$$

Denote $T' := \widetilde{T}_2(\theta_{\Gamma'} - \theta_\Gamma) \in B(H,N_2)$, $\widetilde{T} = \widetilde{T}_2\widetilde{T}_1^{-1} \in B(N_1,N_2)$.

Since $F\sigma_2(g) = E'\sigma_1(g)T^*$ for any $g \in \mathcal{G}$ and $E'\sigma_1(\mathcal{G})$ is g-complete, it follows that $T = T_2(T_1)^{-1}$. □

From the above result, we have another illustration that there is an operator which is in the form of a lower triangular matrix to construct a relationship between the dilation Riesz generator pair of a frame generator alternate dual pair and that of another frame generator alternate dual pair.

Corollary 2.6 *Let $(\mathcal{G},\pi,H)$, $(\mathcal{G},\pi_1,N_1)$, $(\pi_2,\mathcal{G},N_2)$ be unitary representations of $\mathcal{G}$ on H, N_1, N_2 respectively, (A,B), (A,B') be frame generator alternate dual pairs of $(\mathcal{G},\pi,H)$, where $A,B,B' \in B(H,H_0)$. Suppose that there exist pairs of dual Riesz generators $(E',\widetilde{E}')$, $(F,\widetilde{F})$ of $(\mathcal{G},\sigma_1,H\oplus N_1)$, $(\mathcal{G},\sigma_2,H\oplus N_2)$ which are also dilations of (A,B), (A,B') respectively, where $\sigma_1 = \pi \oplus \pi_1$, $\sigma_2 = \pi \oplus \pi_2$, $E',\widetilde{E}' \in B(H\oplus N_1,H_0)$, $F,\widetilde{F} \in B(H\oplus N_2,H_0)$. Then there is an invertible operator $T \in B(H\oplus N_1, H\oplus N_2)$ such that $F\sigma_2(g) = E'\sigma_1(g)T^*$ for any $g \in \mathcal{G}$, where*

$$T = \begin{pmatrix} I_H & 0 \\ T' & \widetilde{T} \end{pmatrix},$$

$T' \in B(H,N_2)$, $\widetilde{T} \in B(N_1,N_2)$.

Proof It is direct by the proof of Theorem 2.5. □

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