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Strong singularities of attractive and repulsive type to 2n-order neutral differential equation

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Abstract

This paper is devoted to the existence of a positive periodic solution for a kind of 2n-order neutral differential equation with a singularity, where nonlinear term $g(t,x)$ has strong singularities of attractive and repulsive type at the origin. Our proof is based on coincidence degree theory.

MSC: 34B16; 34B18; 34C25

Keywords: Positive periodic solution; p-Laplacian; 2n-order; Neutral operator; Strong singularities of attractive and repulsive type

1 Introduction

In this paper, we consider the following 2n-order p-Laplacian neutral differential equation with a singularity:

$$\bigl(\phi_p\bigl(x(t)-cx(t-\tau)\bigr)^{(n)}\bigr)^{(n)}+f\bigl(t,x'(t)\bigr)+g\bigl(t,x\bigl(t-\sigma(t)\bigr)\bigr)=e(t), \tag{1.1}$$

where $\phi_p:\mathbb{R}\to\mathbb{R}$ is given by $\phi_p(s)=|s|^{p-2}s$, and $p>1$ is a constant, c, τ are constants and $|c|\neq 1$, $\tau\in[0,T)$, $\sigma\in C^1(\mathbb{R},\mathbb{R})$ is a T-periodic function, $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous T-periodic function about t and $f(t,0)=0$, $e\in C(\mathbb{R},\mathbb{R})$ is a T-periodic function, n is a positive integer, $g:\mathbb{R}\times(0,+\infty)\to\mathbb{R}$ is a L^2-Carathéodory function, and $g(t,\cdot)=g(t+T,\cdot)$. It is said that equation (1.1) is singularity of attractive type (resp. repulsive type) if $g(t,x)\to+\infty$ (resp. $g(t,x)\to-\infty$) as $x\to 0^+$ for $t\in\mathbb{R}$.

Zhang [21] in 1995 first introduced the property of neutral operator $(Ax)(t):=x(t)-cx(t-\tau)$ and discussed a kind of neutral differential equation

$$\bigl(x(t)-cx(t-\tau)\bigr)'=-ax\bigl(t-r+\gamma\bigl(t,x(t+\cdot)\bigr)\bigr)+e(t). \tag{1.2}$$

The author has given some properties of the neutral operator A, i.e., if $|c|\neq 1$, then A has continuous inverse on $C_T:=\{x\mid x\in C(\mathbb{R},\mathbb{R}),x(t+T)\equiv x(t),\forall t\in\mathbb{R}\}$,

(i) $\|A^{-1}x\|\leq\frac{\|x\|}{|1-|c||}$, $\forall x\in C_T$, here $\|x\|:=\max_{x\in\mathbb{R}}|x(t)|$;

(ii) $\int_0^T|(A^{-1}x)(t)|\,dt\leq\frac{1}{|1-|c||}\int_0^T|x(t)|\,dt$, $\forall x\in C_T$.

Afterwards, using the above properties of the neutral operator A, *a priori estimation* and Leray–Schauder degree theory, Zhang proved that equation (1.2) has at least one periodic solution. Zhu and Lu [23] in 2007 discussed the existence of periodic solution for the following p-Laplacian neutral differential equation:

$$\big(\phi_p\big(x(t)-cx(t-\tau)\big)'\big)'+g\big(t,x\big(t-\delta(t)\big)\big)=p(t).$$

Since $(\phi_p(x'(t)))'$ is a nonlinear term (i.e., quasilinear), coincidence degree theory [5] does not apply directly. In order to get around this difficulty, Zhu and Lu translated the p-Laplacian neutral differential equation into a two-dimensional system

$$\begin{cases}(Ax_1)'(t)=\phi_q(x_2(t))=|x_2(t)|^{q-2}x_2(t),\\ x_2'(t)=-g(t,x_1(t-\delta(t)))+p(t),\end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$, for which coincidence degree theory can be applied. Based on the works of Zhang and Lu, the Krasnoselskii fixed point theorem [1–3, 6], topological degree theory [4, 14, 15, 20], and the fixed point in a cone [12, 13, 17, 18], fixed point theorem of Leray–Schauder type [10] have been employed to discuss the existence of a periodic solution of neutral differential equations.

Nowadays, the existence of periodic solutions for neutral differential equations with singularity has been researched (see [7–9, 19]). Among these, a good deal of work has been performed on the existence of a positive periodic solution of fourth-order neutral Liénard equation with a singularity of repulsive type. Kong and Lu [7] in 2017 studied the following singular Liénard equation:

$$\big(\phi_p\big(x(t)-cx(t-\tau)\big)''\big)''+f\big(x(t)\big)x'(t)+g\big(t,x\big(t-\sigma(t)\big)\big)=e(t), \tag{1.3}$$

where c is a constant with $|c|<1$, $g(t,x(t-\delta(t)))=g_0(x(t))+g_1(t,x(t-\delta(t)))$, $g_0\in C((0,+\infty),\mathbb{R})$ has a strong singularity of repulsive type at $x=0$, and $\int_0^T e(t)\,dt=0$. By applying coincidence degree theory, they proved that equation (1.3) has at least one positive T-periodic solution.

Inspired by the above paper [7], in this paper, we further consider the existence of a positive T-periodic solution for equation (1.1) with strong singularities of attractive and repulsive type. Applying coincidence degree theory, we obtain the following conclusions.

Theorem 1.1 *Assume that the following conditions hold*:

(H_1) *There exists a positive constant N such that*

$$\big|f(t,u)\big|\le N, \quad \textit{for } (t,u)\in[0,T]\times\mathbb{R}.$$

(H_2) *There exist two positive constants D_1, D_2 with $D_1<D_2$ such that $g(t,x)-e(t)<-N$ for all $(t,x)\in[0,T]\times(0,D_1)$, and $g(t,x)-e(t)>N$ for all $(t,x)\in[0,T]\times(D_2,+\infty)$.*

(H_3) *There exist positive constants a, b, p and $1\le p<+\infty$ such that*

$$g(t,x)\le ax^{p-1}+b, \quad \textit{for all } (t,x)\in[0,T]\times(0,+\infty).$$

(H_4) *$g(t,x) = g_0(x) + g_1(t,x)$, where $g_0 \in C((0,\infty);\mathbb{R})$ and $g_1 : [0,T] \times [0,\infty) \to \mathbb{R}$ is an L^2-Carathéodory function.*

(H_5) (*Strong singularity of repulsive type*)

$$\lim_{x\to 0^+} g_0(x) = -\infty, \quad \textit{and} \quad \lim_{x\to 0^+} \int_x^1 g_0(s)\,ds = +\infty.$$

Then (1.1) *has at least one positive T-periodic solution if*

$$0 < \frac{a(1+|c|)T}{(1-\sigma')|1-|c||^p}\left(\frac{T}{\pi_p}\right)^{np-1} < 1,$$

where $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$, $\sigma' := \max_{t\in[0,T]} |\sigma(t)|$.

Remark 1.2 It is worth mentioning that the friction term $f(x)x'(t)$ in equation (1.3) satisfies $\int_0^T f(x(t))x'(t)\,dt = 0$, which is crucial to estimating *a priori bounds* of a positive T-periodic solution for equation (1.3). However, in this paper, the friction term $f(t,x')$ may not satisfy $\int_0^T f(t,x'(t))\,dt = 0$. For example, let

$$f(t,x') = \left(\sin^2(2t) + 5\right)\cos x'(t).$$

Obviously, $\int_0^T (\sin^2(2t)+5)\cos x'(t)\,dt \neq 0$. This implies that our methods to estimate *a priori bounds* of a positive T-periodic solution for equation (1.1) are more difficult than equation (1.3).

Remark 1.3 From [7], the condition composed on $e(t)$ is $\int_0^T e(t)\,dt = 0$. However, the paper is unnecessary. For example, let $e(t) = e^{\sin 4\pi t}$. Obviously, $\int_0^T \frac{1}{4}e^{\sin 2\pi t} \neq 0$. Moreover, coefficient c of neutral operator A satisfies $|c| < 1$ in [7]; in this paper, coefficient c satisfies $|c| < 1$ and $|c| > 1$. At last, the singular term g_0 of equation (1.3) has not a deviating argument (i.e., $\sigma \equiv 0$). The singular term g_0 of this paper satisfies time-dependent deviating argument (see condition (H_4)). It is easy to verify that the work on estimating *lower bounds* of a positive periodic solution for equation (1.1) is more complex than equation (1.3). Therefore, our result can be more general.

Remark 1.4 If equation (1.1) satisfies singularity of attractive type, i.e., $\lim_{x\to 0^+} g_0(x) = +\infty$ and $\lim_{x\to 0^+} \int_x^1 g_0(s)\,ds = -\infty$. Obviously, attractive condition and (H_2), (H_3), (H_5) are contradictions. Therefore, the above method and conditions are no longer applicable to prove the existence of a positive periodic solution for equation (1.1) with singularity of attractive type. Next, we have to find another way and conditions to get over these problems.

Theorem 1.5 *Assume that conditions* (H_1) *and* (H_4) *hold. Suppose the following conditions are satisfied*:

(H_6) *There exist two positive constants D_3, D_4 with $D_3 < D_4$ such that $g(t,x) - e(t) > N$ for all $(t,x) \in [0,T] \times (0,D_3)$, and $g(t,x) - e(t) < -N$ for all $(t,x) \in [0,T] \times (D_4,+\infty)$.*

(H_7) *There exist positive constants a', b' such that*

$$-g(t,x) \le a'x^{p-1} + b', \quad \textit{for all } (t,x) \in [0,T] \times (0,+\infty).$$

(H_8) (*Strong singularity of attractive type*)

$$\lim_{x\to 0^+} g_0(x) = +\infty, \quad \textit{and} \quad \lim_{x\to 0^+}\int_x^1 g_0(s)\,ds = -\infty.$$

Then (1.1) *has at least one positive* T*-periodic solution if*

$$0 < \frac{a'(1+|c|)T}{(1-\sigma')|1-|c||^p}\left(\frac{T}{\pi_p}\right)^{np-1} < 1.$$

2 Preparation

We first recall the coincidence degree theory.

Lemma 2.1 (Gaines and Mawhin [5]) *Suppose that* X *and* Y *are two Banach spaces, and* $L : D(L) \subset X \to Y$ *is a Fredholm operator with index zero. Let* $\Omega \subset X$ *be an open bounded set and* $N : \overline{\Omega} \to Y$ *be* L*-compact on* $\overline{\Omega}$. *Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ *has a solution in* $\overline{\Omega} \cap D(L)$.

Lemma 2.2 (see [11]) *If* $|c| \neq 1$, *then* $(Ax)(t) := x(t) - cx(t-\tau)$ *has continuous bounded inverse on* $C_T := \{x \in C(\mathbb{R},\mathbb{R}) \mid x(t+T) - x(t) \equiv 0\}$ *and*

$$\int_0^T \left|(A^{-1}x)(t)\right|^p dt \le \frac{1}{|1-|c||^p}\int_0^T \left|x(t)\right|^p dt, \quad \forall x \in C_T.$$

Lemma 2.3 (see [22]) *If* $v \in C^1(\mathbb{R},\mathbb{R})$ *and* $v(0) = v(T) = 0$, *then*

$$\left(\int_0^T \left|v(t)\right|^p dt\right)^{\frac{1}{p}} \le \left(\frac{T}{\pi_p}\right)\left(\int_0^T \left|v'(t)\right|^p dt\right)^{\frac{1}{p}}.$$

Similar to Zhu and Lu [23], we rewrite (1.1) in the form:

$$\begin{cases} x_1^{(n)}(t) = A^{-1}(\phi_q(x_2(t))) \\ x_2^{(n)}(t) = -f(t, x_1'(t)) - g(t, x_1(t-\sigma(t))) + e(t). \end{cases} \tag{2.1}$$

Let

$$X := \left\{x = \left(x_1(t), x_2(t)\right) \in C^n\left(\mathbb{R},\mathbb{R}^2\right) : x(t+T) - x(t) \equiv 0\right\}$$

with the norm $\|x\| := \max\{\|x_1\|, \|x_2\|\}$;

$$Y := \left\{x = \left(x_1(t), x_2(t)\right) \in C^n\left(\mathbb{R},\mathbb{R}^2\right) : x(t+T) - x(t) \equiv 0\right\}$$

with the norm $\|x\|_\infty := \max\{\|x\|, \|x'\|\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) \subset X \to Y, \quad \text{by } (Lx)(t) = \begin{pmatrix} x_1^{(n)}(t) \\ x_2^{(n)}(t) \end{pmatrix}, \tag{2.2}$$

where $D(L) = \{x = (x_1, x_2)^\top \in C^n(\mathbb{R}, \mathbb{R}^2) : x(t+T) - x(t) \equiv 0, t \in \mathbb{R}\}$. Define a nonlinear operator $N : X \to Y$ as follows:

$$(Nx)(t) = \begin{pmatrix} A^{-1}(\phi_q(x_2(t))) \\ -f(t, x_1'(t)) - g(t, x_1(t-\sigma(t))) + e(t) \end{pmatrix}. \tag{2.3}$$

Then (2.1) can be converted to the abstract equation $Lx = Nx$.

From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^n, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px := \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}; \qquad Qy := \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Let K denote the inverse of $L|_{\operatorname{Ker} p \cap D(L)}$. It is easy to see that $\operatorname{Ker} L = \operatorname{Im} Q = \mathbb{R}^n$ and

$$[Ky](t) = \operatorname{col}\big((Gy_1)(t), (Gy_2)(t)\big),$$

where

$$[Gy_k](t) = \sum_{i=1}^{n-1} \frac{1}{i!} x_k^{(i)}(0) t^i + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y_k(s)\, ds, \quad k = 1, 2. \tag{2.4}$$

3 Proofs of Theorems 1.1 and 1.5

Proof of Theorem 1.1 Consider the following operator equation:

$$Lx = \lambda Nx, \quad \lambda \in (0, 1),$$

where L and N are defined by equations (2.2) and (2.3). Set

$$\Omega_1 = \big\{x : Lx = \lambda Nx, \lambda \in (0, 1)\big\}.$$

If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} x_1^{(n)}(t) = \lambda A^{-1}(\phi_q(x_2(t))) \\ x_2^{(n)}(t) = -\lambda f(t, x_1'(t)) - \lambda g(t, x_1(t-\sigma(t))) + \lambda e(t), \end{cases} \tag{3.1}$$

since $(Ax_1^{(n)})(t) = (Ax_1)^{(n)}(t)$. Substituting $x_2(t) = \frac{1}{\lambda^{p-1}}(\phi_p(Ax_1)^{(n)}(t))$ into the second equation of equation (3.1), we get

$$\big(\phi_p(Ax_1)^{(n)}(t)\big)^{(n)} + \lambda^p f\big(t, x_1'(t)\big) + \lambda^p g\big(t, x_1\big(t-\sigma(t)\big)\big) = \lambda^p e(t). \tag{3.2}$$

Integrating both sides of equation (3.2) over $[0, T]$, we have

$$\int_0^T \big(f\big(t, x_1'(t)\big) + g\big(t, x_1\big(t - \sigma(t)\big)\big) - e(t)\big)\, dt = 0, \tag{3.3}$$

since $\int_0^T (\phi_p(Ax_1)''(t))'' = 0$. From equation (3.3) and condition (H_1), we deduce

$$-NT \le \int_0^T \big(g\big(t, x_1\big(t - \sigma(t)\big)\big) - e(t)\big)\, dt \le NT.$$

Then, by condition (H_2), we know that there are two points $\xi, \eta \in (0, T)$ such that

$$x_1\big(\xi - \sigma(\xi)\big) \ge D_1, \qquad x_1(\eta) \le D_2. \tag{3.4}$$

Then, from (3.4), we have

$$\begin{aligned} x_1(t) &= \frac{1}{2}\big(x_1(t) + x_1(t - T)\big) \\ &= \frac{1}{2}\left(x_1(\eta) + \int_\eta^t x_1'(s)\, ds + x(\eta) - \int_{t-T}^\eta x_1'(s)\, ds\right) \\ &\le D_2 + \frac{1}{2}\int_0^T \big|x_1'(t)\big|\, dt. \end{aligned} \tag{3.5}$$

Multiplying both sides of equation (3.2) by $(Ax_1)(t)$ and integrating over the interval $[0, T]$, we get

$$\begin{aligned} &\int_0^T \phi_p\big((Ax_1)^{(n)}(t)\big)^{(n)}(Ax_1)(t)\, dt + \lambda^p \int_0^T f\big(t, x_1'(t)\big)(Ax_1)(t)\, dt \\ &\quad + \lambda^p \int_0^T g\big(t, x_1\big(t - \sigma(t)\big)\big)(Ax_1)(t)\, dt \\ &= \lambda^p \int_0^T e(t)(Ax_1)(t)\, dt. \end{aligned} \tag{3.6}$$

Substituting $\int_0^T \phi_p((Ax_1)^{(n)}(t))^{(n)}(Ax_1)(t)\, dt = (-1)^n \int_0^T |(Ax_1)^{(n)}(t)|^p\, dt$ into equation (3.6), we see that

$$\begin{aligned} (-1)^n \int_0^T \big|(Ax_1)^{(n)}(t)\big|^p\, dt = {} & -\lambda^p \int_0^T f\big(t, x_1'(t)\big)(Ax_1)(t)\, dt \\ & - \lambda^p \int_0^T g\big(t, x_1\big(t - \sigma(t)\big)\big)(Ax_1)(t)\, dt \\ & + \lambda^p \int_0^T e(t)(Ax_1)(t)\, dt. \end{aligned}$$

Furthermore, we obtain

$$\left|(-1)^n \int_0^T \big|(Ax_1)^{(n)}(t)\big|^p\, dt\right| = \left|-\lambda^p \int_0^T f\big(t, x_1'(t)\big)(Ax_1)(t)\, dt\right.$$

$$
\begin{aligned}
&- \lambda^p \int_0^T g\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr)(Ax_1)(t)\,dt \\
&+ \lambda^p \int_0^T e(t)(Ax_1)(t)\,dt \biggr|.
\end{aligned}
$$

Therefore, from condition (H_1), it is clear that

$$
\begin{aligned}
&\int_0^T \bigl|(Ax_1)^{(n)}(t)\bigr|^p\,dt \\
&\quad\le \int_0^T \bigl|f\bigl(t, x_1'(t)\bigr)\bigr|\bigl|(Ax_1)(t)\bigr|\,dt + \int_0^T \bigl|g\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr)\bigr|\bigl|(Ax_1)(t)\bigr|\,dt \\
&\qquad + \int_0^T \bigl|e(t)\bigr|\bigl|(Ax_1)(t)\bigr|\,dt \\
&\quad\le \bigl(1+|c|\bigr)\|x_1\|NT + \bigl(1+|c|\bigr)\|x_1\| \int_0^T \bigl|g\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr|\,dt \\
&\qquad + \bigl(1+|c|\bigr)\|x_1\|\|e\|T, \qquad (3.7)
\end{aligned}
$$

where $\|e\| := \max_{t\in[0,T]} |e(t)|$. From conditions (H_1), (H_2) and equation (3.3), we obtain

$$
\begin{aligned}
&\int_0^T \bigl|g\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr)\bigr|\,dt \\
&\quad= \int_{g(t,x_1)\ge 0} g\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr)\,dt - \int_{g(t,x_1)\le 0} g\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr)\,dt \\
&\quad= 2\int_{g(t,x_1)\ge 0} g\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr)\,dt + \int_0^T f\bigl(t, x_1'(t)\bigr)\,dt - \int_0^T e(t)\,dt \\
&\quad\le 2a\int_0^T \bigl|x_1\bigl(t-\sigma(t)\bigr)\bigr|^{p-1}\,dt + 2bT + \int_0^T \bigl|f\bigl(t, x_1'(t)\bigr)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt \\
&\quad= \frac{2a}{1-\sigma'}\int_0^T \bigl|x_1(t)\bigr|^{p-1}\,dt + 2bT + NT + \|e\|T, \qquad (3.8)
\end{aligned}
$$

from $\frac{a(1+|c|)T}{(1-\sigma')|1-|c||^p}(\frac{T}{\pi_p})^{2p-1} > 0$, we obtain $\sigma' < 1$. Substituting equations (3.5) and (3.8) into (3.7), and we have

$$
\begin{aligned}
&\int_0^T \bigl|(Ax_1)^{(n)}(t)\bigr|^p\,dt \\
&\quad\le \bigl(1+|c|\bigr)NT\|x_1\| + \bigl(1+|c|\bigr)\|x_1\|\biggl(\frac{2a}{1-\sigma'}\int_0^T \bigl|x_1(t)\bigr|^{p-1}\,dt + 2bT + NT + \|e\|T\biggr) \\
&\qquad + \bigl(1+|c|\bigr)\|e\|T\|x_1\| \\
&\quad\le 2\bigl(1+|c|\bigr)NT\biggl(D_2 + \frac{1}{2}\int_0^T \bigl|x_1'(t)\bigr|\,dt\biggr) \\
&\qquad + \frac{2a(1+|c|)}{1-\sigma'}\biggl(D_2 + \frac{1}{2}\int_0^T \bigl|x_1'(t)\bigr|\,dt\biggr)\int_0^T \bigl|x_1(t)\bigr|^{p-1}\,dt \\
&\qquad + 2b\bigl(1+|c|\bigr)T\biggl(D_2 + \frac{1}{2}\int_0^T \bigl|x_1'(t)\bigr|\,dt\biggr) + 2\bigl(1+|c|\bigr)\|e\|T\biggl(D_2 + \frac{1}{2}\int_0^T \bigl|x_1'(t)\bigr|\,dt\biggr)
\end{aligned}
$$

$$= \frac{a(1+|c|)}{1-\sigma'} \int_0^T \left|x_1'(t)\right| dt \int_0^T \left|x_1(t)\right|^{p-1} dt + \frac{2a(1+|c|)D_2}{1-\sigma'} \int_0^T \left|x_1(t)\right|^{p-1} dt$$
$$+ \left(1+|c|\right)\left(N+b+\|e\|\right)T \int_0^T \left|x_1'(t)\right| dt + 2\left(1+|c|\right)\left(N+b+\|e\|\right)TD_2.$$

Using the Hölder inequality, we deduce

$$\begin{aligned}\int_0^T \left|(Ax_1)^{(n)}(t)\right|^p dt &\le \frac{a(1+|c|)T}{1-\sigma'} \left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^T \left|x_1(t)\right|^p dt\right)^{\frac{p-1}{p}} \\ &\quad + \frac{2a(1+|c|)D_2 T^{\frac{1}{p}}}{1-\sigma'} \left(\int_0^T \left|x_1(t)\right|^p dt\right)^{\frac{p-1}{p}} \\ &\quad + P_1 T^{\frac{1}{q}} \left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}} + 2P_1D_2, \end{aligned} \tag{3.9}$$

where $P_1 := (1+|c|)(N+b+\|e\|)T$. Let $\nu(t) = x_1(t+\eta) - x_1(\eta)$, here $x_1(\eta) \le D_2$ is as in equation (3.4), and then $\nu(0) = \nu(T) = 0$. By Lemma 2.3 and Minkowski's inequality [16], we have

$$\begin{aligned}\left(\int_0^T \left|x_1(t)\right|^p dt\right)^{\frac{1}{p}} &= \left(\int_0^T \left|\nu(t) + x_1(\eta)\right|^p dt\right)^{\frac{1}{p}} \\ &\le \left(\int_0^T \left|\nu(t)\right|^p dt\right)^{\frac{1}{p}} + \left(\int_0^T \left|x_1(\eta)\right|^p dt\right)^{\frac{1}{p}} \\ &\le \left(\frac{T}{\pi_p}\right)\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}, \end{aligned} \tag{3.10}$$

since $\int_0^T |\nu'(t)|^p dt = \int_0^T |x_1'(t)|^p dt$. On the other hand, in view of $x_1(0) = x_1(T)$, there exists a point $t_1 \in (0,T)$ such that $x_1'(t_1) = 0$. Let $\nu_1(t) = x_1'(t+t_1)$, it is easy to see that $\nu_1(0) = \nu_1(T) = 0$. Applying inductive method, from $x_1^{(n-2)}(0) = x^{(n-2)}(T)$, there exists a point $t_{n-1} \in (0,T)$ such that $x_1^{(n-1)}(t_{n-1}) = 0$. Let $\nu_{n-1}(t) = x_1^{(n-1)}(t+t_{n-1})$, then we get $\nu_{n-1}(0) = \nu_{n-1}(T) = 0$. By Lemma 2.3,

$$\begin{aligned}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{\frac{1}{p}} &= \left(\int_0^T \left|\nu_1(t)\right|^p dt\right)^{\frac{1}{p}} \\ &\le \left(\frac{T}{\pi_p}\right)\left(\int_0^T \left|x_1''(t)\right|^p dt\right)^{\frac{1}{p}} \\ &\le \cdots \\ &\le \left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}. \end{aligned} \tag{3.11}$$

Substituting equations (3.11) into (3.10), we arrive at

$$\left(\int_0^T \left|x_1(t)\right|^p dt\right)^{\frac{1}{p}} \le \left(\frac{T}{\pi_p}\right)^{n}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}. \tag{3.12}$$

Furthermore, substituting equations (3.11) and (3.12) into (3.9), it is easy to verify that

$$\begin{aligned}
&\int_0^T \left|(Ax_1)^{(n)}(t)\right|^p dt \\
&\le \frac{a(1+|c|)T}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\left(\frac{T}{\pi_p}\right)^n\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} + D_2T^{\frac{1}{p}}\right)^{p-1} \\
&\quad+\frac{2a(1+|c|)D_2T^{\frac{1}{p}}}{1-\sigma'}\left(\left(\frac{T}{\pi_p}\right)^n\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} + D_2T^{\frac{1}{p}}\right)^{p-1} \\
&\quad+P_1T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} + 2P_1D_2 \\
&= \frac{a(1+|c|)T}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{np-1}\left(1+\frac{D_2T^{\frac{1}{p}}}{(\frac{T}{\pi_p})^n(\int_0^T |x_1^{(n)}(t)|^p dt)^{\frac{1}{p}}}\right)^{p-1}\int_0^T \left|x_1^{(n)}(t)\right|^p dt \\
&\quad+\frac{2a(1+|c|)D_2T^{\frac{1}{p}}}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{np-2}\left(1+\frac{D_2T^{\frac{1}{p}}}{(\frac{T}{\pi_p})^n(\int_0^T |x_1^{(n)}(t)|^p dt)^{\frac{1}{p}}}\right)^{p-1} \\
&\quad\times\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{p-1}{p}} \\
&\quad+P_1T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} + 2P_1D_2.
\end{aligned} \tag{3.13}$$

Next, we introduce a classical inequality, there exists a positive constant $k(p) > 0$ (dependent on p),

$$(1+x)^p \le 1 + (1+p)x \quad \text{for } x \in \left[0, k(p)\right]. \tag{3.14}$$

Then, we consider the following two cases.

Case 1. If $\frac{D_2T^{\frac{1}{p}}}{(\frac{T}{\pi_p})^n(\int_0^T |x_1^{(n)}(t)|^p dt)^{\frac{1}{p}}} > k(p)$, then it is obvious that

$$\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} < \frac{D_2T^{\frac{1}{p}}}{k(p)}\left(\frac{T}{\pi_p}\right)^{-n}.$$

From equations (3.5) and (3.11), by using the Hölder inequality, we deduce

$$\begin{aligned}
x(t) &\le D_2 + \frac{1}{2}T^{\frac{1}{q}}\left(\int_0^T \left|x'(t)\right|^p dt\right)^{\frac{1}{p}} \\
&\le D_2 + \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \\
&\le D_2 + \frac{TD_2}{2k(p)}\left(\frac{T}{\pi_p}\right)^{-1} := M_1.
\end{aligned} \tag{3.15}$$

Case 2. If $\frac{D_2 T^{\frac{1}{p}}}{(\frac{T}{\pi_p})^n(\int_0^T |x_1^{(n)}(t)|^p\,dt)^{\frac{1}{p}}} < k(p)$, from equations (3.13) and (3.14), we obtain

$$\begin{aligned}
&\int_0^T \left|(Ax_1)^{(n)}(t)\right|^p dt \\
&\quad\le \frac{a(1+|c|)T}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{np-1}\left(1+\frac{D_2 T^{\frac{1}{p}} p}{(\frac{T}{\pi_p})^n(\int_0^T |x_1^{(n)}(t)|^p\,dt)^{\frac{1}{p}}}\right)\int_0^T \left|x_1^{(n)}(t)\right|^p dt \\
&\qquad+\frac{2a(1+|c|)D_2 T^{\frac{1}{p}}}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{np-2}\left(1+\frac{D_2 T^{\frac{1}{p}} p}{(\frac{T}{\pi_p})^n(\int_0^T |x_1^{(n)}(t)|^p\,dt)^{\frac{1}{p}}}\right) \\
&\qquad\cdot\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{p-1}{p}} \\
&\qquad+P_1 T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}+2P_1 D_2 \\
&\quad=\frac{a(1+|c|)T}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{np-1}\int_0^T \left|x_1^{(n)}(t)\right|^p dt+\frac{a(1+|c|)T^{\frac{1}{p}}D_2}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{np-3}\left(Tp+2\left(\frac{T}{\pi_p}\right)\right) \\
&\qquad\cdot\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{p-1}{p}}+\frac{2a(1+|c|)T^{\frac{2}{p}}D_2^2 p}{1-\sigma'}\left(\frac{T}{\pi_p}\right)^{np-4} \\
&\qquad\cdot\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{p-2}{p}}+P_1 T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}+2P_1 D_2. \qquad (3.16)
\end{aligned}$$

Since $(Ax_1)^{(n)}(t) = (Ax_1^{(n)})(t)$, from Lemma 2.2 and (3.16), we see that

$$\begin{aligned}
&\int_0^T \left|x_1^{(n)}(t)\right|^p dt \\
&\quad=\int_0^T \left|(A^{-1}Ax_1^{(n)})(t)\right|^p dt \\
&\quad\le \frac{1}{|1-|c||^p}\int_0^T \left|(Ax_1)^{(n)}(t)\right|^p dt \\
&\quad\le \frac{a(1+|c|)T}{(1-\sigma')|1-|c||^p}\left(\frac{T}{\pi_p}\right)^{np-1}\int_0^T \left|x_1^{(n)}(t)\right|^p dt+\frac{a(1+|c|)T^{\frac{1}{p}}D_2(\frac{T}{\pi_p})^{np-3}(Tp+2(\frac{T}{\pi_p}))}{(1-\sigma')|1-|c||^p} \\
&\qquad\cdot\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{p-1}{p}}+\frac{2a(1+|c|)T^{\frac{2}{p}}D_2^2 p(\frac{T}{\pi_p})^{np-4}}{(1-\sigma')|1-|c||^p}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{p-2}{p}} \\
&\qquad+\frac{P_1 T^{\frac{1}{q}}}{|1-|c||^p}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}+\frac{2P_1 D_2}{|1-|c||^p}.
\end{aligned}$$

Since

$$\frac{a(1+|c|)T}{(1-\sigma')|1-|c||^p}\left(\frac{T}{\pi_p}\right)^{np-1} < 1,$$

obviously, there exists a positive constant M_1' such that

$$\int_0^T \left|x_1^{(n)}(t)\right|^p dt \le M_1'. \tag{3.17}$$

From equations (3.5), (3.11), and (3.17), applying the Hölder inequality, we deduce

$$\begin{aligned} x_1(t) &\le D_2 + \frac{1}{2}\int_0^T \left|x_1'(t)\right| dt \\ &\le D_2 + \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \\ &\le D_2 + \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-1}\left(M_1'\right)^{\frac{1}{p}} := M_1. \end{aligned} \tag{3.18}$$

From equations (3.5), (3.11), and (3.17), we get

$$\begin{aligned} \|x_1'\| &\le x_1'(t_1) + \frac{1}{2}\int_0^T \left|x_1''(t)\right| dt \\ &\le \frac{1}{2}T^{\frac{1}{q}}\left(\int_0^T \left|x_1''(t)\right|^p dt\right)^{\frac{1}{p}} \\ &\le \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-2}\left(\int_0^T \left|x_1^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \\ &\le \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-2} M_1'^{\frac{1}{p}} := M_2 \end{aligned} \tag{3.19}$$

since $x_1'(t_1) = 0$. From $x_2^{(n-2)}(0) = x_2^{(n-2)}(T)$, there exists a point $t_2^* \in (0,T)$ such that $x_2^{(n-1)}(t_2^*) = 0$. From the second equation of (3.1), equations (3.8), (3.18), (3.19), and condition (H_1), we obtain

$$\begin{aligned} \left\|x_2^{(n-1)}\right\| &\le \frac{1}{2}\int_0^T \left|x_2^{(n)}(t)\right| dt \\ &\le \frac{\lambda}{2}\left(\int_0^T \left|f\left(t,x_1'(t)\right)\right| dt + \int_0^T \left|g\left(t,x_1(t-\sigma)\right)\right| dt + \int_0^T \left|e(t)\right| dt\right) \\ &\le \lambda\left(NT + aTM_1^{p-1} + bT + T\|e\|\right) := \lambda M_3. \end{aligned} \tag{3.20}$$

Integrating the first equation of (3.1) over $[0,T]$, we have $\int_0^T x_2(t)\,dt = \int_0^T \phi_p((Ax_1)''(t))\,dt = 0$, which implies there is a point $t_3^* \in (0,T)$ such that $x_2(t_3^*) = 0$. So, from equations (3.5) and (3.20), applying the Hölder inequality and equation (3.11), it is easy to see that

$$\begin{aligned} \|x_2\| &\le \frac{1}{2}\int_0^T \left|x_2'(t)\right| dt \\ &\le \frac{1}{2}T^{\frac{1}{q}}\left(\int_0^T \left|x_2'(t)\right|^p dt\right)^{\frac{1}{p}} \\ &\le \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)^{n-2}\left(\int_0^T \left|x_2^{(n-1)}(t)\right|^p dt\right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2} T\left(\frac{T}{\pi_p}\right)^{n-2} \|x_2^{(n-1)}\| \\
&\le \frac{\lambda}{2} T\left(\frac{T}{\pi_p}\right)^{n-2} M_3 := \lambda M_4.
\end{aligned} \tag{3.21}$$

On the other hand, from equation (3.2) and condition (H_4), we see that

$$\begin{aligned}
&\bigl(\phi_p\bigl((Ax_1)^{(n)}(t)\bigr)\bigr)^{(n)} + \lambda^p f\bigl(t, x_1'(t)\bigr) + \lambda^p\bigl(g_0\bigl(x_1\bigl(t-\sigma(t)\bigr)\bigr) + g_1\bigl(t, x_1\bigl(t-\sigma(t)\bigr)\bigr)\bigr) \\
&\quad = \lambda^p e(t).
\end{aligned} \tag{3.22}$$

Let $\xi \in [0, T]$ be as in equation (3.4) for any $t \in [\xi, T]$. Multiplying both sides of equation (3.22) by $x_1'(t-\sigma(t))(1-\sigma'(t))$ and integrating over $[\xi, t]$, we get

$$\begin{aligned}
\lambda^p \int_{x_1(\xi-\sigma(\xi))}^{x_1(t-\sigma(t))} g_0(u)\,du &= \lambda^p \int_\xi^t g_0\bigl(x_1\bigl(s-\sigma(s)\bigr)\bigr) x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds \\
&= -\int_\xi^t \bigl(\phi_p\bigl((Ax_1)^{(n)}(s)\bigr)\bigr)^{(n)} x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds \\
&\quad - \lambda^p \int_\xi^t f\bigl(s, x_1'(s)\bigr) x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds \\
&\quad - \lambda^p \int_\xi^t g_1\bigl(s, x_1\bigl(s-\sigma(s)\bigr)\bigr) x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds \\
&\quad + \lambda^p \int_\xi^t e(s) x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds.
\end{aligned} \tag{3.23}$$

Furthermore, by equations (3.16), (3.18), (3.19), and (3.20), we have

$$\begin{aligned}
&\lambda^p \left|\int_{x_1(\xi-\sigma(\xi))}^{x_1(t-\sigma(t))} g_0(u)\,du\right| \\
&\quad = \Bigl|-\int_\xi^t \bigl(\phi_p\bigl((Ax_1)^{(n)}(s)\bigr)\bigr)^{(n)} x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds \\
&\qquad - \lambda^p \int_\xi^t f\bigl(s, x_1'(s)\bigr) x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds \\
&\qquad - \lambda^p \int_\xi^t g_1\bigl(s, x_1\bigl(s-\sigma(s)\bigr)\bigr) x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds \\
&\qquad + \lambda^p \int_\xi^t e(s) x_1'\bigl(s-\sigma(s)\bigr)\bigl(1-\sigma'(s)\bigr)\,ds\Bigr| \\
&\quad \le \bigl(1+\sigma'\bigr)\|x_1'\| \int_0^T \bigl|\phi_p\bigl((Ax_1)^{(n)}(s)\bigr)^{(n)}\bigr|\,ds + \lambda^p\bigl(1+\sigma'\bigr)\|x_1'\| \int_0^T \bigl|f\bigl(s, x_1'(s)\bigr)\bigr|\,ds \\
&\qquad + \lambda^p\bigl(1+\sigma'\bigr)\|x_1'\| \int_0^T \bigl|g_1\bigl(s, x_1\bigl(s-\sigma(s)\bigr)\bigr)\bigr|\,ds + \lambda^p\bigl(1+\sigma'\bigr)\|x_1'\| \int_0^T \bigl|e(s)\bigr|\,ds \\
&\quad \le \lambda^p\bigl(1+\sigma'\bigr)M_2\left(\int_0^T \bigl|f\bigl(s, x'(s)\bigr)\right)\bigr|\,ds + \int_0^T \bigl|g\bigl(s, x_1(s)\bigr)\bigr|\,ds + \int_0^T \bigl|e(s)\bigr|\,ds) \\
&\qquad + \lambda^p\bigl(1+\sigma'\bigr)\bigl(M_2 NT + M_2\|g_{M_1}\|T + M_2\|e\|T\bigr) \\
&\quad \le \lambda^p\bigl(1+\sigma'\bigr)M_2\bigl(M_3 + NT + \|g_{M_1}\|T + \|e\|T\bigr),
\end{aligned}$$

where $g_{M_1} = \max_{0\le x\le M_1}|g_1(t,x)| \in L^2(0,T)$ is as in condition (H_4). According to singular condition (H_5), we know that there exists a positive constant M_5 such that

$$x_1(t) \ge M_5, \quad \forall t\in[\xi,T]. \tag{3.24}$$

The case $t\in[0,\xi]$ can be treated similarly.

From equations (3.16), (3.18), (3.19), (3.20), and (3.24), we let

$$\Omega = \left\{x=(x_1,x_2)^\top : E_1 \le x_1(t) \le E_2, \left\|x_1'\right\| \le E_3, \|x_2\| \le E_4, \text{ and } \left\|x_2'\right\| \le E_5, \forall t\in[0,T]\right\},$$

where $0 < E_1 < \min(M_5, D_1)$, $E_2 > \max(M_1, D_2)$, $E_3 > M_2$, $E_4 > M_4$, and $E_5 > M_3$. $\Omega_2 = \{x : x\in\partial\Omega\cap\operatorname{Ker} L\}$, then $\forall x\in\partial\Omega\cap\operatorname{Ker} L$

$$QNx = \frac{1}{T}\int_0^T \begin{pmatrix} A^{-1}(\phi_q(x_2(t))) \\ -f(t,x_1'(t)) - g(t,x_1(t-\sigma(t))) + e(t)\end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0$, $x_1 = E_2$ or E_1. But if $x_1(t) = E_2$, we know

$$0 = \int_0^T \left\{g(t,E_2) - e(t)\right\} dt.$$

From condition (H_2), we have $x_1(t) \le D_2 \le E_2$, which yields a contradiction. Similarly, if $x_1 = E_1$, we also have $QNx \ne 0$, i.e., $\forall x\in\partial\Omega\cap\operatorname{Ker} L$, $x\notin\operatorname{Im} L$, so assumptions (1) and (2) of Lemma 2.1 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ as follows:

$$J(x_1,x_2)^\top = (x_2,-x_1)^\top.$$

Let $H(\mu,x) = -\mu x + (1-\mu)JQNx$, $(\mu,x)\in[0,1]\times\Omega$, then $\forall(\mu,x)\in(0,1)\times(\partial\Omega\cap\operatorname{Ker} L)$,

$$H(\mu,x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T}\int_0^T (g(t,x_1) - e(t))\,dt \\ -\mu x_2 - (1-\mu)A^{-1}(\phi_q(x_2))\end{pmatrix}.$$

From condition (H_2), we get $x^\top H(\mu,x)\ne 0$, $\forall(\mu,x)\in(0,1)\times(\partial\Omega\cap\operatorname{Ker} L)$. Hence

$$\begin{aligned}\deg\{JQN,\Omega\cap\operatorname{Ker} L,0\} &= \deg\left\{H(0,x),\Omega\cap\operatorname{Ker} L,0\right\}\\ &= \deg\left\{H(1,x),\Omega\cap\operatorname{Ker} L,0\right\}\\ &= \deg\{I,\Omega\cap\operatorname{Ker} L,0\}\ne 0.\end{aligned}$$

So assumption (3) of Lemma 2.1 is satisfied. By applying Lemma 2.1, we conclude that equation $Lx = Nx$ has a solution $x = (x_1,x_2)^\top$ on $\bar{\Omega}\cap D(L)$, i.e., (2.1) has a T-periodic solution $x_1(t)$. □

Next, we study the existence of a positive T-periodic solution for equation (1.1) with singularity of attractive type.

Proof of Theorem 1.5 We follow the same strategy and notation as in the proof of Theorem 1.1. From equation (3.3) and condition (H_6), we know that there are points $\mu,\nu\in(0,T)$

such that

$$x_1\big(\mu - \sigma(\mu)\big) \ge D_3, \qquad x_1(\nu) \le D_4. \tag{3.25}$$

Next, we consider $\int_0^T |g(t, x_1(t - \sigma(t)))|\,dt$. From equation (3.8) and conditions (H_1), (H_7), we obtain

$$\begin{aligned}
&\int_0^T \big|g\big(x_1\big(t - \sigma(t)\big)\big)\big|\,dt \\
&\quad = \int_{g(t,x_1)\ge 0} g\big(t, x_1\big(t - \sigma(t)\big)\big)\,dt - \int_{g(t,x_1)\le 0} g\big(t, x_1\big(t - \sigma(t)\big)\big)\,dt \\
&\quad = -2\int_{g(t,x_1)\le 0} g\big(t, x_1\big(t - \sigma(t)\big)\big)\,dt - \int_0^T f\big(t, x_1'(t)\big)\,dt + \int_0^T e(t)\,dt \\
&\quad \le 2a'\int_0^T \big|x_1\big(t - \sigma(t)\big)\big|^{p-1}\,dt + 2b'T + \int_0^T \big|f\big(t, x_1'(t)\big)\big|\,dt + \int_0^T \big|e(t)\big|\,dt \\
&\quad \le \frac{2a'}{1 - \sigma'}\int_0^T \big|x_1(t)\big|^{p-1}\,dt + 2b'T + NT + \|e\|T.
\end{aligned} \tag{3.26}$$

The proof left is the same as that of Theorem 1.1. □

Finally, we present an example to illustrate our result.

Example 3.1 Consider the fourth-order neutral Rayleigh equation with singularity of repulsive type:

$$\begin{aligned}
&\left(\phi_p\left(x(t) - \frac{1}{2}x(t - \tau)\right)''\right)'' + \big(\sin^2(2t) + 5\big)\cos x'(t) + \frac{1}{4\pi}(\sin 4t + 3)x^3\left(t - \frac{1}{8}\sin 4t\right) \\
&\quad - \frac{1}{x^\kappa(t - \frac{1}{8}\sin 4t)} = e^{\sin 4t},
\end{aligned} \tag{3.27}$$

where $\kappa \ge 1$ and $p = 4$, τ is a constant and $0 \le \tau < T$.

It is clear that $T = \frac{\pi}{2}$, $\sigma(t) = \frac{1}{8}\sin 4t$, $\sigma' = \frac{1}{2} < 1$, $f(t, u) = (\sin^2(2t) + 5)\cos u$, $g(t, x) = \frac{1}{4\pi}(\cos 4t + 3)x^3 - \frac{1}{x^\kappa}$, $a = \frac{1}{5\pi}$, $\pi_4 = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\pi/p)} = \frac{2\pi(4-1)^{\frac{1}{4}}}{4\cdot\frac{\sqrt{2}}{2}} = \pi \times (\frac{3}{4})^{\frac{1}{4}}$. Take $N = 6$, $a = \frac{1}{\pi}$, $b = 1$. It is obvious that conditions (H_1)–(H_5) hold. Now we consider

$$\begin{aligned}
&\frac{a(1 + |c|)T}{(1 - \sigma')|1 - |c||^p}\left(\frac{T}{\pi_p}\right)^{2p-1} \\
&\quad = \frac{\frac{1}{\pi} \times \frac{3}{2} \times \frac{\pi}{2}}{(\frac{1}{2})^5}\left(\frac{\frac{\pi}{2}}{\pi \times (\frac{3}{4})^{\frac{1}{4}}}\right)^7 \\
&\quad \approx 0.3104 < 1.
\end{aligned}$$

Therefore, applying Theorem 1.1, we know that equation (3.27) has at least one positive $\frac{\pi}{2}$-periodic solution.

Example 3.2 Consider the following fourth-order neutral Rayleigh equation with singularity of attractive type:

$$\begin{aligned}&\big(x(t)-101x(t-\tau)\big)^{(4)}-\big(\cos^2 t+100\big)\sin x'(t)-\frac{1}{\pi}(\sin 2t+5)x\left(t-\frac{1}{4}\cos 2t\right)\\&\quad+\frac{5}{x^{\kappa'}(t-\frac{1}{4}\cos 2t)}=e^{\cos 2t},\end{aligned}\tag{3.28}$$

where $\kappa'\geq 1$ and $p=2$, τ is a constant and $0\leq\tau<T$.

It is clear that $T=\pi$, $\sigma(t)=\frac{1}{4}\cos 2t$, $\sigma'=\frac{1}{2}<1$, $f(t,u)=-(\cos^2 t+100)\sin u$, $g(t,x)=-\frac{1}{\pi}(\sin 2t+5)x+\frac{5}{x^{\kappa'}}$, $a=\frac{1}{5\pi}$, $\pi_2=\frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\pi/p)}=\frac{2\pi(2-1)^{\frac{1}{2}}}{4\times\frac{1}{2}}=\pi$. Take $N=6$, $a=\frac{6}{\pi}$, $b=1$. It is easy to verify that conditions (H_1), (H_4), (H_6)–(H_8) hold. Now we consider

$$\begin{aligned}&\frac{a(1+|c|)T}{(1-\sigma')|1-|c||^p}\left(\frac{T}{\pi_p}\right)^{2p-1}\\&\quad=\frac{\frac{6}{\pi}\times 102\times\pi}{\frac{1}{2}\times 100^2}\left(\frac{\pi}{\pi}\right)^3\\&\quad=\frac{153}{1250}<1.\end{aligned}$$

Therefore, applying Theorem 1.5, we know that equation (3.28) has at least one positive π-periodic solution.

4 Conclusions

In this article, we introduce the following existence of a positive T-periodic solution for $2n$-order p-Laplacian neutral differential equation with singularities of attractive and repulsive type. Due to the friction term, $f(t,x')$ may not satisfy $\int_0^T f(t,x'(t))\,dt=0$. This implies that the work on estimating *a priori bounds* of periodic solutions for equation (1.1) is more difficult than the corresponding work on equation (1.3) in [7]. In this paper, by using coincidence degree theory and conditions (H_1)–(H_5), we prove the existence of a positive T-periodic solution for equation (1.1) with singularity of repulsive type; applying conditions (H_1), (H_4), (H_6)–(H_8), we obtain that equation (1.1) with singularity of attractive type has at least one positive T-periodic solution.

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