Approximations and inequalities for the exponential beta function

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Abstract

In this paper, motivated by interest in simulating the expenditure patterns of construction projects, we introduce the mathematical concept of Exponential-Beta function by

$$F(\alpha, \beta) := \int_0^1 \exp\left[x^\alpha (1-x)^\beta\right] dx,$$

where α, β are positive numbers. Taylor's-type approximations, several analytic inequalities, and global convexity properties are established.

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1 Introduction

The ability to accurately predict the client's financial commitment, which forms the basis of contractors' revenue, provides a pre-warning for alternative courses of action, which could prove detrimental to the success or, indeed, survival of both stakeholders. Cash flow mishaps account for considerable number of construction contractors' failures. The capital commitment locked in the expenditure flow of construction projects exerts considerable demand on the contractor to secure finance, particularly when the contractor is undertaking a number of projects simultaneously. To this end, an accurate forecast of project cash flow could be critical to the success and, indeed, failure of the firm. Ultimately, forecasting is a crystal gazing exercise, and the forecast of the expenditure is more challenging than the income side. Since the 1970s there have been several models developed by researchers and the industry. These models have been classified under different categories. One classification by Khosrowshahi and Kaka [8] consists of activity-based, element-based, and mathematics-based. While the elemental approach focuses on the time-related accumulation of cost centres (or elements), the activity-based approach consists of a laborious task of identifying and quantifying activity sequences, costs, and times. On the other hand, the mathematical approach offers a fast, cheap, and easy solution. While comparatively alienating to the user, it has the advantage of generating a forecast in early stages, and unlike other models, this approach requires much less information about the project.

Extensive analysis of construction project expenditure patterns has revealed that the shape of the project periodic expenditure pattern is defined in terms of a number of variables represented by the following expression (see [7]):

$$Y_C := \exp\left[bx^a(1-x)^d\right] - 1,$$

where

$$x_p := R = \frac{a}{a+d} \quad \text{and} \quad y_p := Q = \exp\left[bR^a(1-R)^d\right] - 1,$$

with Q and R representing the positions of the project expenditure peak on both the cost and time access; a and b being parameterised in terms of x_p and y_p as follows:

$$a = \frac{x_p d}{1 - x_p}, \qquad b = \frac{\ln(1 + y_p)}{x_p^a(1 - x_p)^d}.$$

Parameter d is calculated numerically and derived to rapidly converge towards a solution within desired error tolerance.

A relationship is established between the properties of the project and the physical shape of the project expenditure pattern. These are then related and reflected on the mathematical expression through its parameters.

Motivated by the above considerations, one can explore the mathematical behavior of the family of functions

$$f_{\alpha,\beta}(x) := \exp\left[x^\alpha(1-x)^\beta\right], \quad x \in [0,1], \alpha, \beta > 0$$

and the "*exponential Beta function*" defined by the integral

$$F(\alpha,\beta) := \int_0^1 \exp\left[x^\alpha(1-x)^\beta\right] dx, \quad \alpha, \beta > 0.$$

In order to establish the fundamental properties of these functions, we need some well-known facts related to the classical Euler's Beta function summarised below.

2 Some facts on beta function

In mathematics, the *Beta function*, also called the *Euler integral of the first kind*, is a special function defined by

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt, \quad \alpha > 0, \beta > 0. \tag{2.1}$$

The utility of the Beta function is often overshadowed by that of the *Gamma function*, partly perhaps because it can be evaluated in terms of the Gamma function. However, since it occurs so frequently in practice, a special designation for it is widely accepted.

It is obvious that the Beta function has the symmetry property

$$B(\alpha,\beta) = B(\beta,\alpha), \quad \alpha > 0, \beta > 0, \tag{2.2}$$

and the following connection between the Beta and Gamma functions holds:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \alpha > 0, \beta > 0, \tag{2.3}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt, \quad \alpha > 0. \tag{2.4}$$

The following properties of the Beta function also hold (see, for example, [2, pp. 68–70]):

$$B(\alpha + 1, \beta) + B(\alpha, \beta + 1) = B(\alpha, \beta), \quad \alpha, \beta > 0; \tag{2.5}$$

$$B(\alpha, \beta + 1) = \frac{\beta}{\alpha} B(\alpha + 1, \beta) = \frac{\beta}{\alpha + \beta} B(\alpha, \beta), \quad \alpha, \beta > 0; \tag{2.6}$$

$$B(\alpha, \alpha) = 2^{1-2\alpha} B\left(\alpha, \frac{1}{2}\right), \quad \alpha > 0; \tag{2.7}$$

$$B(\alpha, \beta) B(\alpha + \beta, \gamma) B(\alpha + \beta + \gamma, \delta)$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha + \beta + \gamma + \delta)}, \quad \alpha, \beta, \gamma, \delta > 0; \tag{2.8}$$

$$B\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right) = \pi \sec\left(\frac{\alpha\pi}{2}\right), \quad 0 < \alpha < 1; \tag{2.9}$$

and

$$B(\alpha, \beta) = \frac{1}{2} \int_0^1 \frac{t^{\alpha-1} + t^{\beta-1}}{(t+1)^{\alpha+\beta}}\, dt = p^\alpha (1+p)^{\alpha+\beta} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{(t+p)^{\alpha+\beta}}\, dt \tag{2.10}$$

for $\alpha, \beta, p > 0$.

In [5], Dragomir et al. obtained the following basic inequalities for the Beta function:

Theorem 1 *Let m, n, p, q be positive numbers with the property that*

$$(p - m)(q - n) \le (\ge) 0. \tag{2.11}$$

Then

$$B(p, q) B(m, n) \ge (\le) B(p, n) B(m, q). \tag{2.12}$$

In particular, the following is true:

Corollary 1 *For any $p, m > 0$, we have the inequalities*

$$B(m, p) \ge \left[B(p, p) B(m, m) \right]^{\frac{1}{2}}. \tag{2.13}$$

The positive real numbers a and b may be called similarly (oppositely) unitary if

$$(a - 1)(b - 1) \ge (\le) 0. \tag{2.14}$$

Theorem 2 *Let $a, b > 0$ be similarly (oppositely) unitary. Then*

$$B(a, b) \geq (\leq) \frac{1}{ab}. \tag{2.15}$$

We also have

Theorem 3 *Let m, n, p and q be positive numbers. Then*

$$\left| B(m + p + 1, n + q + 1) - B(m + 1, n + 1)B(p + 1, q + 1) \right|$$
$$\leq \frac{1}{4} \frac{p^p q^q}{(p + q)^{p+q}} \frac{m^m n^n}{(m + n)^{m+n}}. \tag{2.16}$$

Another simpler inequality that one can derive via Grüss' inequality is the following [5]:

Theorem 4 *Let $p, q > 0$. Then we have the inequality*

$$\left| B(p + 1, q + 1) - \frac{1}{(p + 1)(q + 1)} \right| \leq \frac{1}{4} \tag{2.17}$$

or, equivalently,

$$\max\left\{ 0, \frac{3 - pq - p - q}{4(p + 1)(q + 1)} \right\} \leq B(p + 1, q + 1) \leq \frac{5 + pq + p + q}{4(p + 1)(q + 1)}. \tag{2.18}$$

We also have the following global convexity property; see, for instance, [5].

Theorem 5 *The mapping B is logarithmically convex on $(0, \infty)^2$ as a function of two variables.*

For other properties of the Beta function, see the recent papers [1, 9, 11] and the references therein.

3 Basic facts on the generating function

We consider the two-parameter generating function $f_{\alpha,\beta} : [0, 1] \to [0, \infty)$,

$$f_{\alpha,\beta}(x) := \exp\left[x^\alpha (1 - x)^\beta \right],$$

where α, β are positive constants. This family can be extended for negative numbers by eliminating either ends of the closed interval $[0, 1]$. However, we do not consider this case here.

Define a simpler two-parameter family that generates the Beta function, $g_{\alpha,\beta} : [0, 1] \to [0, \infty)$, given by

$$g_{\alpha,\beta}(x) = x^\alpha (1 - x)^\beta, \tag{3.1}$$

where α, β are positive constants.

We start with the simple fact incorporated in the following:

Proposition 1 *Let $\alpha, \beta > 0$. The function $f_{\alpha,\beta}$ is increasing on $[0, \frac{\alpha}{\alpha+\beta}]$, decreasing on $[\frac{\alpha}{\alpha+\beta}, 1]$, and*

$$\max_{x \in [0,1]} f_{\alpha,\beta}(x) = f_{\alpha,\beta}\left(\frac{\alpha}{\alpha+\beta}\right) = \exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]. \tag{3.2}$$

Proof We have

$$f_{\alpha,\beta}(x) = \exp\left[g_{\alpha,\beta}(x)\right]$$

and

$$f'_{\alpha,\beta}(x) = g'_{\alpha,\beta}(x)\exp\left[g_{\alpha,\beta}(x)\right], \quad x \in [0,1]$$

showing that the sign of $f'_{\alpha,\beta}$ on $[0,1]$ is the same with that of $g'_{\alpha,\beta}$.

Further, we have

$$\begin{aligned}
g'_{\alpha,\beta}(x) &= \alpha x^{\alpha-1}(1-x)^{\beta} - \beta x^{\alpha}(1-x)^{\beta-1} \\
&= x^{\alpha-1}(1-x)^{\beta-1}\left[\alpha(1-x) - \beta x\right] \\
&= x^{\alpha-1}(1-x)^{\beta-1}\left[\alpha - (\alpha+\beta)x\right], \quad x \in (0,1).
\end{aligned}$$

This shows that $g'_{\alpha,\beta}(x) > 0$ for $x \in (0, \frac{\alpha}{\alpha+\beta})$ and $g'_{\alpha,\beta}(x) < 0$ for $(\frac{\alpha}{\alpha+\beta}, 1)$, which proves the statement. □

We need the following lemma that is of interest in itself:

Lemma 1 *Let $\alpha, \beta > 0$.*

(i) *If $0 < \alpha + \beta \leq 1$, then $g_{\alpha,\beta}$ is strictly concave on $[0,1]$.*

Define

$$x_{1,\alpha,\beta} := \frac{\alpha(\alpha+\beta-1) - \sqrt{\alpha\beta(\alpha+\beta-1)}}{(\alpha+\beta)(\alpha+\beta-1)} < \frac{\alpha}{\alpha+\beta}$$

and

$$x_{2,\alpha,\beta} := \frac{\alpha(\alpha+\beta-1) + \sqrt{\alpha\beta(\alpha+\beta-1)}}{(\alpha+\beta)(\alpha+\beta-1)} > \frac{\alpha}{\alpha+\beta}.$$

(ii) *If $\alpha, \beta \in (0,1)$ with $\alpha + \beta > 1$, then $g_{\alpha,\beta}$ is strictly concave on $[0,1]$.*

(iii) *If $\alpha > 1$ and $\beta \in (0,1)$ then $g_{\alpha,\beta}$ is strictly convex on $(0, x_{1,\alpha,\beta})$ and strictly concave on $(x_{1,\alpha,\beta}, 1)$.*

(iv) *If $\alpha \in (0,1)$ and $\beta > 1$, then $g_{\alpha,\beta}$ is strictly concave on $(0, x_{2,\alpha,\beta})$ and strictly convex on $(x_{2,\alpha,\beta}, 1)$.*

(v) *If $\alpha, \beta > 1$, then $g_{\alpha,\beta}$ is strictly concave on $(x_{1,\alpha,\beta}, x_{2,\alpha,\beta})$ and strictly convex on $(0, x_{1,\alpha,\beta}) \cup (x_{2,\alpha,\beta}, 1)$.*

Proof If we take the second derivative of $g_{\alpha,\beta}$ on $(0, 1)$, then we get

$$
\begin{aligned}
g''_{\alpha,\beta}(x) &= \alpha(\alpha - 1)x^{\alpha-2}(1 - x)^{\beta} - \alpha\beta x^{\alpha-1}(1 - x)^{\beta-1} \\
&\quad - \alpha\beta x^{\alpha-1}(1 - x)^{\beta-1} + \beta(\beta - 1)x^{\alpha}(1 - x)^{\beta-2} \\
&= \alpha(\alpha - 1)x^{\alpha-2}(1 - x)^{\beta} - 2\alpha\beta x^{\alpha-1}(1 - x)^{\beta-1} + \beta(\beta - 1)x^{\alpha}(1 - x)^{\beta-2} \\
&= x^{\alpha-2}(1 - x)^{\beta-2}\big[\alpha(\alpha - 1)(1 - x)^2 - 2\alpha\beta x(1 - x) + \beta(\beta - 1)x^2\big]
\end{aligned}
$$

for all $\alpha, \beta > 0$ and $x \in (0, 1)$.

Now, consider the two-parameter family of parabolas

$$
h_{\alpha,\beta}(x) := \alpha(\alpha - 1)(1 - x)^2 - 2\alpha\beta x(1 - x) + \beta(\beta - 1)x^2, \quad x \in \mathbb{R}.
$$

We have

$$
\begin{aligned}
h_{\alpha,\beta}(x) &= \alpha(\alpha - 1)\big(x^2 - 2x + 1\big) - 2\alpha\beta\big(x - x^2\big) + \beta(\beta - 1)x^2 \\
&= \big[\alpha(\alpha - 1) + 2\alpha\beta + \beta(\beta - 1)\big]x^2 - 2\big(\alpha(\alpha - 1) + \alpha\beta\big)x + \alpha(\alpha - 1) \\
&= \big[\alpha^2 + 2\alpha\beta + \beta^2 - (\alpha + \beta)\big]x^2 - 2\alpha(\alpha + \beta - 1)x + \alpha(\alpha - 1) \\
&= \big[(\alpha + \beta)^2 - (\alpha + \beta)\big]x^2 - 2\alpha(\alpha + \beta - 1)x + \alpha(\alpha - 1) \\
&= (\alpha + \beta)(\alpha + \beta - 1)x^2 - 2\alpha(\alpha + \beta - 1)x + \alpha(\alpha - 1)
\end{aligned}
$$

for $x \in \mathbb{R}$.

The discriminant of this family of parabolas is

$$
\begin{aligned}
\Delta_{\alpha,\beta} &:= 4\alpha^2(\alpha + \beta - 1)^2 - 4(\alpha + \beta)(\alpha + \beta - 1)\alpha(\alpha - 1) \\
&= 4\alpha(\alpha + \beta - 1)\big[\alpha(\alpha + \beta - 1) - (\alpha + \beta)(\alpha - 1)\big] \\
&= 4\alpha(\alpha + \beta - 1)\big(\alpha^2 + \alpha\beta - \alpha - \alpha^2 - \alpha\beta + \alpha + \beta\big) \\
&= 4\alpha\beta(\alpha + \beta - 1)
\end{aligned}
$$

for $\alpha, \beta > 0$.

Now, if $0 < \alpha + \beta < 1$, then $\Delta_{\alpha,\beta} < 0$ which shows that the parabola $h_{\alpha,\beta}(x) < 0$ for all $x \in \mathbb{R}$, implying that $g''_{\alpha,\beta}(x) < 0$ for $x \in (0, 1)$, namely $g_{\alpha,\beta}$ is strictly concave on $[0, 1]$.

If $\alpha + \beta = 1$, then $h_{\alpha,\beta}(x) = \alpha(\alpha - 1) < 0$, namely $g_{\alpha,\beta}$ is strictly concave on $[0, 1]$.

If $\alpha + \beta > 1$ with $\alpha, \beta > 0$ then $\Delta_{\alpha,\beta} > 0$ and the parabola $h_{\alpha,\beta}(\cdot)$ has two distinct intercepts with the axis ox, namely

$$
x_{1,\alpha,\beta} = \frac{\alpha(\alpha + \beta - 1) - \sqrt{\alpha\beta(\alpha + \beta - 1)}}{(\alpha + \beta)(\alpha + \beta - 1)}
$$

and

$$
x_{2,\alpha,\beta} = \frac{\alpha(\alpha + \beta - 1) + \sqrt{\alpha\beta(\alpha + \beta - 1)}}{(\alpha + \beta)(\alpha + \beta - 1)}.
$$

The x coordinate for the vertex is

$$x_{V,\alpha,\beta} = \frac{\alpha}{\alpha + \beta} \in (0,1)$$

for all $\alpha, \beta > 0$.

We also have $h_{\alpha,\beta}(0) = \alpha(\alpha - 1)$ and $h_{\alpha,\beta}(1) = \beta(\beta - 1)$.

Now, if $\alpha, \beta \in (0,1)$ with $\alpha + \beta > 1$ then $x_{1,\alpha,\beta} < 0$ and $x_{2,\alpha,\beta} > 1$, showing that $h_{\alpha,\beta}(x) < 0$, namely $g_{\alpha,\beta}$ is strictly concave on $[0,1]$.

If $\alpha > 1$ and $\beta \in (0,1)$, then $\alpha + \beta > 1$, $x_{1,\alpha,\beta} \in (0, \frac{\alpha}{\alpha+\beta})$, $x_{2,\alpha,\beta} > 1$, which shows that $h_{\alpha,\beta}(x) > 0$ for $x \in (0, x_{1,\alpha,\beta})$ and $h_{\alpha,\beta}(x) < 0$ for $x \in (x_{1,\alpha,\beta}, 1)$, showing that $g_{\alpha,\beta}$ is strictly convex on $(0, x_{1,\alpha,\beta})$ and strictly concave on $(x_{1,\alpha,\beta}, 1)$.

If $\alpha \in (0,1)$ and $\beta > 1$, then $\alpha + \beta > 1$, $x_{1,\alpha,\beta} < 0$, $x_{2,\alpha,\beta} \in (\frac{\alpha}{\alpha+\beta}, 1)$, which shows that $h_{\alpha,\beta}(x) < 0$ for $x \in (0, x_{2,\alpha,\beta})$ and $h_{\alpha,\beta}(x) > 0$ for $x \in (x_{2,\alpha,\beta}, 1)$, showing that $g_{\alpha,\beta}$ is strictly concave on $(0, x_{2,\alpha,\beta})$ and strictly convex on $(x_{2,\alpha,\beta}, 1)$.

If $\alpha, \beta > 1$, then $x_{1,\alpha,\beta} \in (0, \frac{\alpha}{\alpha+\beta})$ and $x_{2,\alpha,\beta} \in (\frac{\alpha}{\alpha+\beta}, 1)$, which shows that $h_{\alpha,\beta}(x) < 0$ for $(x_{1,\alpha,\beta}, x_{2,\alpha,\beta})$ and $h_{\alpha,\beta}(x) > 0$ for $x \in (0, x_{1,\alpha,\beta}) \cup (x_{2,\alpha,\beta}, 1)$ showing that $g_{\alpha,\beta}$ is strictly concave on $(x_{1,\alpha,\beta}, x_{2,\alpha,\beta})$ and strictly convex on $(0, x_{1,\alpha,\beta}) \cup (x_{2,\alpha,\beta}, 1)$. □

We can state the following fact concerning the logarithmic convexity of $f_{\alpha,\beta}$.

Proposition 2 *Let $\alpha, \beta > 0$. Define $x_{1,\alpha,\beta}$ and $x_{2,\alpha,\beta}$ as in Lemma* 1.
  (1) *If $\alpha, \beta \in (0,1)$, then $f_{\alpha,\beta}(x)$ is strictly log-concave on $[0,1]$.*
  (2) *If $\alpha > 1$ and $\beta \in (0,1)$ then $f_{\alpha,\beta}(x)$ is strictly log-convex on $(0, x_{1,\alpha,\beta})$ and strictly log-concave on $(x_{1,\alpha,\beta}, 1)$.*
  (3) *If $\alpha \in (0,1)$ and $\beta > 1$, then $f_{\alpha,\beta}(x)$ is strictly log-concave on $(0, x_{2,\alpha,\beta})$ and strictly log-convex on $(x_{2,\alpha,\beta}, 1)$.*
  (4) *If $\alpha, \beta > 1$, then $f_{\alpha,\beta}(x)$ is strictly log-concave on $(x_{1,\alpha,\beta}, x_{2,\alpha,\beta})$ and strictly log-convex on $(0, x_{1,\alpha,\beta}) \cup (x_{2,\alpha,\beta}, 1)$.*

The proof is obvious by Lemma 1 observing that $\ln[f_{\alpha,\beta}(x)] = g_{\alpha,\beta}(x) = x^\alpha(1 - x)^\beta$, $x \in [0,1]$ and $\alpha, \beta > 0$.

We have also the following simple fact:

Proposition 3 *Let $\alpha, \beta > 0$. Then for all $x \in (0,1)$ we have the double inequality*

$$x^\alpha(1 - x)^\beta < f_{\alpha,\beta}(x) - 1 < \frac{x^\alpha(1 - x)^\beta}{1 - x^\alpha(1 - x)^\beta}. \tag{3.3}$$

Proof We use the following elementary exponential inequality for $y \in (0,1)$, see [10]:

$$y < e^y - 1 < \frac{y}{1 - y}$$

for $y \in (0,1)$. If in this inequality we take $g_{\alpha,\beta}(x) = y \in (0,1)$ for $x \in (0,1)$, we get the desired result (3.3). □

4 Taylor's-type expansion for the generating function

We have the following representation result:

Theorem 6 *Let $\alpha, \beta > 0$, then for all $x \in [0,1]$ and natural number $n \geq 1$, we have*

$$
f_{\alpha,\beta}(x) = 1 + \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k}
$$
$$
+ \frac{1}{n!} x^{\alpha(n+1)}(1-x)^{\beta(n+1)} \int_0^1 \exp\left[sx^\alpha(1-x)^\beta\right](1-s)^n \, ds,
\tag{4.1}
$$

where $x \in [0,1]$.

Proof Let $I \subset \mathbb{R}$ be a closed interval, $c \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{C}$ is such that the nth derivative $f^{(n)}$ is absolutely continuous on I, then for each $y \in I$

$$
f(y) = T_n(f;c,y) + R_n(f;c,y),
\tag{4.2}
$$

where $T_n(f;c,y)$ is *Taylor's polynomial*, i.e.

$$
T_n(f;c,y) := \sum_{k=0}^{n} \frac{(y-c)^k}{k!} f^{(k)}(c).
\tag{4.3}
$$

Note that $f^{(0)} := f$ and $0! := 1$, and the remainder is given by

$$
R_n(f;c,y) := \frac{1}{n!} \int_c^y (y-t)^n f^{(n+1)}(t) \, dt.
\tag{4.4}
$$

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$, that

$$
\int_c^d h(t) \, dt = (d-c) \int_0^1 h\left((1-s)c + sd\right) ds.
$$

Therefore,

$$
\int_c^y f^{(n+1)}(t)(y-t)^n \, dt
$$
$$
= (y-c) \int_0^1 f^{(n+1)}\left((1-s)c + sy\right)\left(x - (1-s)c - sy\right)^n ds
$$
$$
= (y-c)^{n+1} \int_0^1 f^{(n+1)}\left((1-s)c + sy\right)(1-s)^n \, ds,
$$

and from (4.4) we get the representation

$$
f(y) = \sum_{k=0}^{n} \frac{(y-c)^k}{k!} f^{(k)}(c)
$$
$$
+ \frac{1}{n!}(y-c)^{n+1} \int_0^1 f^{(n+1)}\left((1-s)c + sy\right)(1-s)^n \, ds
\tag{4.5}
$$

for all $y, c \in I$.

Now, if we write equality (4.5) for the exponential function $f(y) = e^y$, $y \in \mathbb{R}$, and the point $c = 0$, we get

$$\exp y - 1 = \sum_{k=1}^{n} \frac{y^k}{k!} + \frac{1}{n!} y^{n+1} \int_0^1 \exp(sy)(1-s)^n \, ds \tag{4.6}$$

for any real number $y \in \mathbb{R}$.

If we take in (4.6) $y = g_{\alpha,\beta}(x)$, $x \in [0,1]$, we get

$$\exp\left[g_{\alpha,\beta}(x)\right] - 1 = \sum_{k=1}^{n} \frac{[g_{\alpha,\beta}(x)]^k}{k!}$$
$$+ \frac{1}{n!}\left[g_{\alpha,\beta}(x)\right]^{n+1} \int_0^1 \exp\left[s g_{\alpha,\beta}(x)\right](1-s)^n \, ds,$$

which produces the desired result (4.1).   □

We have some simple upper and lower bounds as follows:

Corollary 2 *Let* $\alpha, \beta > 0$, *then for all* $x \in [0,1]$ *and natural number* $n \geq 1$, *we have*

$$\sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k}$$
$$\leq f_{\alpha,\beta}(x) - 1$$
$$\leq \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k} + \frac{e}{(n+1)!} x^{\alpha(n+1)}(1-x)^{\beta(n+1)}. \tag{4.7}$$

Proof The inequalities in (4.7) follow by (4.1) observing that

$$0 \leq \frac{1}{n!}\left[g_{\alpha,\beta}(x)\right]^{n+1} \int_0^1 \exp\left[s g_{\alpha,\beta}(x)\right](1-s)^n \, ds$$
$$\leq \frac{1}{n!}\left[g_{\alpha,\beta}(x)\right]^{n+1} \max_{s \in [0,1]} \exp\left[s g_{\alpha,\beta}(x)\right] \int_0^1 (1-s)^n \, ds$$
$$\leq \frac{e}{(n+1)!}\left[g_{\alpha,\beta}(x)\right]^{n+1}$$

for all $x \in [0,1]$.   □

Corollary 3 *Let* $\alpha, \beta > 0$, *then we have function series expansion*

$$f_{\alpha,\beta}(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k}$$

uniformly on the interval $[0,1]$.

Proof By (4.1) we have

$$
\left| f_{\alpha,\beta}(x) - 1 - \sum_{k=1}^{n} \frac{[g_{\alpha,\beta}(x)]^k}{k!} \right|
$$

$$
= \left| \frac{1}{n!} [g_{\alpha,\beta}(x)]^{n+1} \int_0^1 \exp[s g_{\alpha,\beta}(x)](1-s)^n \, ds \right|
$$

$$
\le \frac{1}{n!} [g_{\alpha,\beta}(x)]^{n+1} \int_0^1 \left| \exp[s g_{\alpha,\beta}(x)](1-s)^n \right| ds
$$

$$
\le \frac{e}{n!} \int_0^1 (1-s)^n \, ds = \frac{e}{(n+1)!} \to 0
$$

uniformly for $x \in [0,1]$ as $n \to \infty$.　　　　　　　　　　　　　　　□

Now, we can introduce the two-variable function $F : (0,\infty) \times (0,\infty) \to (0,\infty)$, that we can call the *Exponential-Beta function,* defined by the integral

$$
F(\alpha,\beta) := \int_0^1 \exp[x^\alpha (1-x)^\beta] \, dx > 1.
$$

Then we have the following representation result in terms of the beta function:

Theorem 7 *For any natural number $n \ge 1$ and any $\alpha, \beta > 0$, we have the representation*

$$
F(\alpha,\beta) = 1 + \sum_{k=1}^{n} \frac{1}{k!} B(\alpha k + 1, \beta k + 1) + R_n(\alpha,\beta), \tag{4.8}
$$

where the remainder $R_n(\alpha,\beta)$ is given by

$$
R_n(\alpha,\beta)
$$

$$
:= \frac{1}{n!} \int_0^1 \left(\int_0^1 \left\{ x^{\alpha(n+1)} (1-x)^{\beta(n+1)} \exp[s x^\alpha (1-x)^\beta] \right\} dx \right) (1-s)^n \, ds. \tag{4.9}
$$

Proof If we integrate identity (4.1), then we get

$$
F(\alpha,\beta) - 1 = \int_0^1 f_{\alpha,\beta}(x) \, dx - 1
$$

$$
= \sum_{k=1}^{n} \int_0^1 \frac{1}{k!} [g_{\alpha,\beta}(x)]^k \, dx
$$

$$
+ \frac{1}{n!} \int_0^1 [g_{\alpha,\beta}(x)]^{n+1} \left(\int_0^1 \exp[s g_{\alpha,\beta}(x)](1-s)^n \, ds \right) dx
$$

$$
= \sum_{k=1}^{n} \int_0^1 \frac{[x^\alpha (1-x)^\beta]^k}{k!} \, dx
$$

$$
+ \frac{1}{n!} \int_0^1 [x^\alpha (1-x)^\beta]^{n+1} \left(\int_0^1 \exp[s x^\alpha (1-x)^\beta](1-s)^n \, ds \right) dx
$$

$$
= \sum_{k=1}^{n} \frac{1}{k!} \int_0^1 x^{\alpha k} (1-x)^{\beta k} \, dx
$$

$$+ \frac{1}{n!} \int_0^1 x^{\alpha(n+1)}(1-x)^{\beta(n+1)} \left(\int_0^1 \exp\left[sx^{\alpha}(1-x)^{\beta}\right](1-s)^n \, ds \right) dx$$

$$= \sum_{k=1}^n \frac{1}{k!} B(\alpha k + 1, \beta k + 1)$$

$$+ \frac{1}{n!} \int_0^1 \left(\int_0^1 \left\{ x^{\alpha(n+1)}(1-x)^{\beta(n+1)} \exp\left[sx^{\alpha}(1-x)^{\beta}\right] \right\} dx \right)(1-s)^n \, ds,$$

where for the last equality we used Fubini's theorem.   □

Corollary 4 *We have the following beta series expansion*:

$$F(\alpha, \beta) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} B(\alpha k + 1, \beta k + 1) \tag{4.10}$$

uniformly over $\alpha, \beta > 0$.

Proof We observe that for all $\alpha, \beta > 0$ we have

$$\left| R_n(\alpha, \beta) \right| \le \frac{e}{n!} \int_0^1 (1-s)^n \, ds = \frac{e}{(n+1)!} \to 0$$

as $n \to \infty$, which proves the claim.   □

Theorem 8 *For any natural number $n \ge 1$ and any $\alpha, \beta > 0$, we have the representation*

$$\ln\left[f_{\alpha,\beta}(x) \right] = x^{\alpha} + \sum_{k=1}^n (-1)^k \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!} x^{k+\alpha} + E_n(\alpha, \beta), \tag{4.11}$$

where the remainder $E_n(\alpha, \beta)$ is given by

$$E_n(\alpha, \beta) := (-1)^{n+1} \frac{\beta(\beta-1)\cdots(\beta-n)}{n!} x^{n+\alpha+1} \int_0^1 (1-sx)^{\beta-n-1}(1-s)^n \, ds.$$

Proof Consider the function $f : (-1, 1) \to (0, \infty), f(x) = (1-x)^{\beta}$. Then

$$f^{(k)}(0) = (-1)^k \beta(\beta-1)\cdots(\beta-k+1) \quad \text{for } k = 1, \ldots, n,$$

and

$$f^{(n+1)}(x) = (-1)^{n+1} \beta(\beta-1)\cdots(\beta-n)(1-x)^{\beta-n-1}.$$

Using the representation (4.5) for $c = 0$, we get

$$(1-x)^{\beta} = 1 + \sum_{k=1}^n (-1)^k \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!} x^k$$

$$+ (-1)^{n+1} \frac{\beta(\beta-1)\cdots(\beta-n)}{n!} x^{n+1} \int_0^1 (1-sx)^{\beta-n-1}(1-s)^n \, ds$$

for all $x \in (0, 1)$.

If we multiply this equality by x^α, we get

$$x^\alpha (1-x)^\beta = x^\alpha + \sum_{k=1}^{n} (-1)^k \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!} x^{k+\alpha}$$

$$+ (-1)^{n+1} \frac{\beta(\beta-1)\cdots(\beta-n)}{n!} x^{n+\alpha+1} \int_0^1 (1-sx)^{\beta-n-1}(1-s)^n \, ds,$$

which is equivalent to the desired representation (4.11). □

Remark 1 If we take the exponential in (4.11), then we get

$$f_{\alpha,\beta}(x)$$

$$= \exp\left[x^\alpha + \sum_{k=1}^{n} (-1)^k \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!} x^{k+\alpha} + E_n(\alpha,\beta) \right]$$

$$= e^{x^\alpha} \prod_{k=1}^{n} \exp\left[(-1)^k \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!} x^{k+\alpha} \right] e^{E_n(\alpha,\beta)} \tag{4.12}$$

for $x \in (0,1)$ and $\alpha, \beta > 0$.

5 Some analytic inequalities

We start with the following fact:

Theorem 9 *Let $\alpha, \beta > 0$. For any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$0 \le f_{\alpha,\beta}(x) - 1 \le \left[\exp(x^{\alpha p}) - 1 \right]^{1/p} \left[\exp((1-x)^{q\beta}) - 1 \right]^{1/q} \tag{5.1}$$

for all $x \in [0,1]$.

In particular, we have

$$\left[f_{\alpha,\beta}(x) - 1 \right]^2 \le \left[\exp(x^{2\alpha}) - 1 \right] \left[\exp((1-x)^{2\beta}) - 1 \right] \tag{5.2}$$

for all $x \in [0,1]$.

Proof If we make use of Hölder's discrete weighted inequality

$$0 \le \sum_{k=1}^{n} m_k a_k b_k \le \left(\sum_{k=1}^{n} m_k a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} m_k b_k^q \right)^{1/q},$$

where $m_k, a_k, b_k \ge 0$, $k \in \{1,\ldots,n\}$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we can write

$$0 \le \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k} \le \left(\sum_{k=1}^{n} \frac{1}{k!} x^{\alpha p k} \right)^{1/p} \left(\sum_{k=1}^{n} \frac{1}{k!} (1-x)^{q\beta k} \right)^{1/q}$$

$$= \left(\sum_{k=1}^{n} \frac{1}{k!} (x^{\alpha p})^k \right)^{1/p} \left(\sum_{k=1}^{n} \frac{1}{k!} [(1-x)^{q\beta}]^k \right)^{1/q} \tag{5.3}$$

for all $n > 1$ and $x \in [0,1]$.

Since the series $\sum_{k=1}^{\infty} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k}$, $\sum_{k=1}^{\infty} \frac{1}{k!}(x^{\alpha p})^k$ and $\sum_{k=1}^{\infty} \frac{1}{k!}[(1-x)^{q\beta}]^k$ are convergent and

$$\sum_{k=1}^{\infty} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k} = \exp\left[x^{\alpha}(1-x)^{\beta}\right] - 1 = f_{\alpha,\beta}(x) - 1,$$

$$\sum_{k=1}^{\infty} \frac{1}{k!} \left(x^{\alpha p}\right)^k = \exp\left(x^{\alpha p}\right) - 1$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k!}\left[(1-x)^{q\beta}\right]^k = \exp\left[(1-x)^{q\beta}\right] - 1,$$

then, by taking the limit as $n \to \infty$ in (5.3), we get the desired result (5.1).   □

The following result also provides some lower bounds for $f_{\alpha,\beta}(x)$.

Theorem 10 *Let $\alpha, \beta > 0$. Then for all $x \in [0,1]$, we have*

$$\frac{1}{e-1}\left[\exp\left(x^{\alpha}\right) - 1\right]\left[\exp\left((1-x)^{\beta}\right) - 1\right]$$

$$\leq f_{\alpha,\beta}(x) - 1$$

$$\leq \frac{1}{e-1}\left[\exp\left(x^{\alpha}\right) - 1\right]\left[\exp\left((1-x)^{\beta}\right) - 1\right] + \frac{1}{4}(e-1)x^{\alpha}(1-x)^{\beta}. \tag{5.4}$$

Proof We use the weighted *Čebyšev's inequality* for sequences a_k, b_k, $k \in \{1, \ldots, n\}$ that have the same monotonicity

$$\sum_{k=1}^{n} m_k a_k \sum_{k=1}^{n} m_k b_k \leq \sum_{k=1}^{n} m_k \sum_{k=1}^{n} m_k a_k b_k, \tag{5.5}$$

where $m_k \geq 0$, $k \in \{1, \ldots, n\}$.

Consider the sequences $a_k := x^{\alpha k}$, $b_k := (1-x)^{\beta k}$, $k \in \{1, \ldots, n\}$, for $x \in [0,1]$. We observe that both sequences are monotonic nonincreasing and, by applying Čebyšev's inequality for the positive weights $m_k := \frac{1}{k!}$, we get

$$\sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k} \sum_{k=1}^{n} \frac{1}{k!}(1-x)^{\beta k} \leq \sum_{k=1}^{n} \frac{1}{k!} \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k}, \tag{5.6}$$

for all $x \in [0,1]$ and $n \geq 1$.

Since the series $\sum_{k=1}^{\infty} \frac{1}{k!} x^{\alpha k}$, $\sum_{k=1}^{\infty} \frac{1}{k!}(1-x)^{\beta k}$ and $\sum_{k=1}^{\infty} \frac{1}{k!}$ are convergent and

$$\sum_{k=1}^{\infty} \frac{1}{k!} x^{\alpha k} = \sum_{k=1}^{\infty} \frac{1}{k!}\left(x^{\alpha}\right)^k = \exp\left(x^{\alpha}\right) - 1,$$

$$\sum_{k=1}^{\infty} \frac{1}{k!}(1-x)^{\beta k} = \sum_{k=1}^{\infty} \frac{1}{k!}\left((1-x)^{\beta}\right)^k = \exp\left((1-x)^{\beta}\right) - 1$$

and $\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$, then, by taking the limit as $n \to \infty$ in (5.6), we get the first inequality in (5.4).

Further, we use the weighted Grüss inequality for the bounded sequences $a \le a_k \le A$, $b \le b_k \le B$, $k \in \{1, \dots, n\}$ and nonnegative weights $m_k \ge 0$, $k \in \{1, \dots, n\}$,

$$\left| \sum_{k=1}^{n} m_k \sum_{k=1}^{n} m_k a_k b_k - \sum_{k=1}^{n} m_k a_k \sum_{k=1}^{n} m_k b_k \right|$$
$$\le \frac{1}{4} \left(\sum_{k=1}^{n} m_k \right)^2 (A - a)(B - b). \tag{5.7}$$

Now, if we consider the sequences $a_k := x^{\alpha k}$, $b_k := (1-x)^{\beta k}$ $k \in \{1, \dots, n\}$, for $x \in [0, 1]$, then we observe that $0 \le a_k \le x^\alpha$ and $0 \le b_k \le (1-x)^\beta$ for all natural numbers $k \ge 1$. So, by utilising inequality (5.7) for $a = b = 0$, $A = x^\alpha$, $B = (1-x)^\beta$ and $m_k := \frac{1}{k!}$, we get

$$\left| \sum_{k=1}^{n} \frac{1}{k!} \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k} (1-x)^{\beta k} - \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k} \sum_{k=1}^{n} \frac{1}{k!} (1-x)^{\beta k} \right|$$
$$\le \frac{1}{4} \left(\sum_{k=1}^{n} \frac{1}{k!} \right)^2 x^\alpha (1-x)^\beta, \tag{5.8}$$

which holds for all $x \in [0, 1]$ and $n \ge 1$.

Since all the series are involved in (5.8), by taking the limit as $n \to \infty$ in this inequality, we get the second part of (5.4). $\qquad \square$

Theorem 11 *Let $\alpha, \beta > 0$. Then for all $x \in (0, 1)$, we have*

$$(e - 1)\left[x^\alpha (1-x)^\beta \right]^{\frac{e}{e-1}} \le f_{\alpha,\beta}(x) - 1 \tag{5.9}$$

and

$$0 \le \ln \left(\frac{f_{\alpha,\beta}(x) - 1}{e - 1} \right) - \ln \left\{ \left[x^\alpha (1-x)^\beta \right]^{\frac{e}{e-1}} \right\}$$
$$\le \frac{1}{(e-1)^2} \left[f_{-\alpha,-\beta}(x) - 1 \right] \left[f_{\alpha,\beta}(x) - 1 \right] - 1. \tag{5.10}$$

Proof Since ln is a concave function, by *Jensen's discrete inequality* for concave functions g, namely

$$g \left(\frac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k} \right) \ge \frac{\sum_{k=1}^{n} p_k g(x_k)}{\sum_{k=1}^{n} p_k},$$

where $p_k > 0$, $k \in \{1, \dots, n\}$, we have for $x_k = x^{\alpha k}(1-x)^{\beta k}$ and $p_k = \frac{1}{k!}$, $k \in \{1, \dots, n\}$ that

$$\ln \left(\frac{\sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k}}{\sum_{k=1}^{n} \frac{1}{k!}} \right) \ge \frac{\sum_{k=1}^{n} \frac{1}{k!} \ln[x^{\alpha k}(1-x)^{\beta k}]}{\sum_{k=1}^{n} \frac{1}{k!}}$$
$$= \frac{\sum_{k=1}^{n} \frac{k}{k!} \ln[x^\alpha (1-x)^\beta]}{\sum_{k=1}^{n} \frac{1}{k!}}$$

$$= \ln\left[x^{\alpha}(1-x)^{\beta}\right] \frac{\sum_{k=1}^{n} \frac{k}{k!}}{\sum_{k=1}^{n} \frac{1}{k!}}$$

$$= \ln\left[x^{\alpha}(1-x)^{\beta}\right] \frac{\sum_{k=1}^{n} \frac{1}{(k-1)!}}{\sum_{k=1}^{n} \frac{1}{k!}}, \tag{5.11}$$

for all $x \in (0,1)$ and $n \geq 1$.

Since $\sum_{k=1}^{\infty} \frac{1}{(k-1)!} = e$ and $\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$, by taking the limit as $n \to \infty$ in (5.11), we get

$$\ln\left(\frac{f_{\alpha,\beta}(x)-1}{e-1}\right) \geq \frac{e}{e-1} \ln\left[x^{\alpha}(1-x)^{\beta}\right] = \ln\left\{\left[x^{\alpha}(1-x)^{\beta}\right]^{\frac{e}{e-1}}\right\},$$

which is equivalent to the first inequality in (5.9).

Further, we use Dragomir–Ionescu's reverse of Jensen's inequality [6] for concave functions

$$0 \leq g\left(\frac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k}\right) - \frac{\sum_{k=1}^{n} p_k g(x_k)}{\sum_{k=1}^{n} p_k}$$

$$\leq \frac{1}{\sum_{k=1}^{n} p_k} \sum_{k=1}^{n} p_k g'(x_k) \frac{1}{\sum_{k=1}^{n} p_k} \sum_{k=1}^{n} p_k x_k - \frac{1}{\sum_{k=1}^{n} p_k} \sum_{k=1}^{n} p_k x_k g'(x_k), \tag{5.12}$$

which gives for $g(x) = \ln x$, $x_k = x^{\alpha k}(1-x)^{\beta k}$ and $p_k = \frac{1}{k!}$, $k \in \{1, \ldots, n\}$ that

$$0 \leq \ln\left(\frac{\sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k}}{\sum_{k=1}^{n} \frac{1}{k!}}\right) - \frac{\sum_{k=1}^{n} \frac{1}{k!} \ln[x^{\alpha k}(1-x)^{\beta k}]}{\sum_{k=1}^{n} \frac{1}{k!}}$$

$$\leq \frac{1}{\sum_{k=1}^{n} \frac{1}{k!}} \sum_{k=1}^{n} \frac{1}{k!} x^{-\alpha k}(1-x)^{-\beta k} \frac{1}{\sum_{k=1}^{n} \frac{1}{k!}} \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k}(1-x)^{\beta k} - 1 \tag{5.13}$$

for $x \in (0,1)$ and $n \geq 1$.

Since the series $\sum_{k=1}^{\infty} \frac{1}{k!} x^{-\alpha k}(1-x)^{-\beta k}$ is convergent and

$$\sum_{k=1}^{\infty} \frac{1}{k!} x^{-\alpha k}(1-x)^{-\beta k} = \sum_{k=1}^{\infty} \frac{1}{k!}\left[x^{-\alpha}(1-x)^{-\beta}\right]^k = \exp\left[x^{-\alpha}(1-x)^{-\beta}\right] - 1,$$

by letting $n \to \infty$ in (5.13), we get

$$0 \leq \ln\left(\frac{f_{\alpha,\beta}(x)-1}{e-1}\right) - \ln\left\{\left[x^{\alpha}(1-x)^{\beta}\right]^{\frac{e}{e-1}}\right\}$$

$$\leq \frac{1}{(e-1)^2}\left\{\exp\left[x^{-\alpha}(1-x)^{-\beta}\right] - 1\right\}\left\{\exp\left[x^{\alpha}(1-x)^{\beta}\right] - 1\right\} - 1,$$

which is equivalent to (5.10). □

Remark 2 As a simple consequence of inequality (5.10) we note that

$$(e-1)^2 \leq \left[f_{-\alpha,-\beta}(x) - 1\right]\left[f_{\alpha,\beta}(x) - 1\right] \tag{5.14}$$

for all positive $\alpha, \beta > 0$ and $x \in (0,1)$.

In 1984, S.S. Dragomir [3] obtained the following Cauchy–Bunyakovsky–Schwarz related weighted inequality (see also [4, Theorem 2.20]):

$$\sum_{k=1}^{n} m_k a_k^2 \sum_{k=1}^{n} m_k b_k^2 \geq \frac{\sum_{k=1}^{n} m_k a_k \sum_{k=1}^{n} m_k b_k \sum_{k=1}^{n} m_k a_k b_k}{\sum_{k=1}^{n} m_k} \tag{5.15}$$

where a_k, b_k are real numbers, $m_k \geq 0$ for $k \in \{1, \ldots, n\}$, and $\sum_{k=1}^{n} m_k > 0$.

Theorem 12 *Let $\alpha, \beta > 0$. Then for all $x \in (0,1)$ we have*

$$0 \leq f_{\alpha,\beta}(x) - 1 \leq (e-1) \frac{[\exp(x^{2\alpha}) - 1][\exp((1-x)^{2\beta}) - 1]}{[\exp(x^{\alpha}) - 1][\exp((1-x)^{\beta}) - 1]}. \tag{5.16}$$

Proof By taking $a_k := x^{\alpha k}$, $b_k := (1-x)^{\beta k}$ and $m_k := \frac{1}{k!}$ in (5.15), we get

$$\sum_{k=1}^{n} \frac{1}{k!} x^{2\alpha k} \sum_{k=1}^{n} \frac{1}{k!} (1-x)^{2\beta k}$$

$$\geq \frac{\sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k} \sum_{k=1}^{n} \frac{1}{k!} (1-x)^{\beta k} \sum_{k=1}^{n} \frac{1}{k!} x^{\alpha k} (1-x)^{\beta k}}{\sum_{k=1}^{n} \frac{1}{k!}}. \tag{5.17}$$

Since all the series involved in (5.17) are convergent, by taking the limit as $n \to \infty$ in this inequality, we get the desired result (5.16). □

Now, recall the well-known inequality between the *weighted arithmetic mean* and *weighted geometric mean,*

$$a^{1-t} b^t \leq (1-t)a + tb, \tag{GA}$$

which holds for all $a, b > 0$ and $t \in [0,1]$. This inequality is also known in the literature as *Young's inequality.*

We have the following global convexity result for the function $f_{\alpha,\beta}(x)$ as a mapping of the positive parameters (α, β). More precisely, we have

Theorem 13 *The mapping*

$$(0,\infty) \times (0,\infty) \ni (\alpha,\beta) \quad \mapsto \quad f_{\alpha,\beta}(x) \in [0,\infty)$$

is globally convex on $(0,\infty) \times (0,\infty)$ for any $x \in (0,1)$.

Proof Fix $x \in (0,1)$. Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in (0,\infty) \times (0,\infty)$ and $t \in [0,1]$. Then

$$(1-t)(\alpha_1,\beta_1) + t(\alpha_2,\beta_2) = \big((1-t)\alpha_1 + t\alpha_2, (1-t)\beta_1 + t\beta_2\big) \in (0,\infty) \times (0,\infty)$$

and

$$f_{(1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2}(x) - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} x^{[(1-t)\alpha_1+t\alpha_2]k}(1-x)^{[(1-t)\beta_1+t\beta_2]k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} x^{(1-t)\alpha_1 k+t\alpha_2 k}(1-x)^{k(1-t)\beta_1+t\beta_2 k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} x^{(1-t)\alpha_1 k} x^{t\alpha_2 k}(1-x)^{k(1-t)\beta_1}(1-x)^{t\beta_2 k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} x^{(1-t)\alpha_1 k}(1-x)^{k(1-t)\beta_1} x^{t\alpha_2 k}(1-x)^{t\beta_2 k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} \left[x^{\alpha_1 k}(1-x)^{k\beta_1} \right]^{(1-t)} \left[x^{\alpha_2 k}(1-x)^{\beta_2 k} \right]^{t} =: A.$$

By Young's inequality (GA), we have

$$\left[x^{\alpha_1 k}(1-x)^{k\beta_1} \right]^{(1-t)} \left[x^{\alpha_2 k}(1-x)^{\beta_2 k} \right]^{t}$$

$$\leq (1-t)x^{\alpha_1 k}(1-x)^{k\beta_1} + t x^{\alpha_2 k}(1-x)^{\beta_2 k}$$

$$= (1-t)\left[x^{\alpha_1}(1-x)^{\beta_1} \right]^{k} + t\left[x^{\alpha_2}(1-x)^{\beta_2} \right]^{k}$$

for all $k \geq 1$, and, by summing this inequality over k, we get

$$A \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ (1-t)\left[x^{\alpha_1}(1-x)^{\beta_1} \right]^{k} + t\left[x^{\alpha_2}(1-x)^{\beta_2} \right]^{k} \right\}$$

$$= (1-t)\sum_{k=1}^{\infty} \frac{1}{k!} \left[x^{\alpha_1}(1-x)^{\beta_1} \right]^{k} + t\sum_{k=1}^{\infty} \frac{1}{k!} \left[x^{\alpha_2}(1-x)^{\beta_2} \right]^{k}$$

$$= (1-t)\left[f_{(\alpha_1,\beta_1)}(x) - 1 \right] + t\left[f_{(\alpha_2,\beta_2)}(x) - 1 \right]$$

$$= (1-t)f_{(\alpha_1,\beta_1)}(x) + t f_{(\alpha_2,\beta_2)}(x) - 1,$$

which implies that

$$f_{(1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2}(x) \leq (1-t)f_{(\alpha_1,\beta_1)}(x) + t f_{(\alpha_2,\beta_2)}(x)$$

and the claim is thus proved. □

Corollary 5 *The function $F : (0,\infty) \times (0,\infty) \to [0,\infty)$ defined by*

$$F(\alpha,\beta) = \int_0^1 f_{\alpha,\beta}(x)\, dx$$

is convex as a function of two variables on $(0,\infty) \times (0,\infty)$.

Proof Let (α_1, β_1), $(\alpha_2, \beta_2) \in (0, \infty) \times (0, \infty)$ and $t \in [0, 1]$. Then

$$F\big((1 - t)\alpha_1 + t\alpha_2, (1 - t)\beta_1 + t\beta_2\big) = \int_0^1 f_{(1-t)\alpha_1 + t\alpha_2, (1-t)\beta_1 + t\beta_2}(x)\, dx$$

$$\leq \int_0^1 \big[(1 - t)f_{(\alpha_1, \beta_1)}(x) + t f_{(\alpha_2, \beta_2)}(x)\big]\, dx$$

$$= (1 - t)\int_0^1 f_{(\alpha_1, \beta_1)}(x)\, dx + t \int_0^1 f_{(\alpha_2, \beta_2)}(x)\, dx$$

$$= (1 - t)F(\alpha_1, \beta_1) + tF(\alpha_2, \beta_2),$$

which proves the statement.   □

Finally, we have the following logarithmic convexity property:

Theorem 14 *The function $F - 1$ is logarithmically convex on $(0, \infty) \times (0, \infty)$ as a function of two variables.*

Proof Let (α_1, β_1), $(\alpha_2, \beta_2) \in (0, \infty) \times (0, \infty)$ and $t \in [0, 1]$. Then by the representation (4.10) we have

$$F\big((1 - t)\alpha_1 + t\alpha_2, (1 - t)\beta_1 + t\beta_2\big) - 1$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} B\big(\big[(1 - t)\alpha_1 + t\alpha_2\big]k + 1, \big[(1 - t)\beta_1 + t\beta_2\big]k + 1\big)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} B(1 - t)(\alpha_1 k + 1) + t(\alpha_2 k + 1), (1 - t)(\beta_1 k + 1) + t(\beta_2 k + 1)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} B\big[(1 - t)(\alpha_1 k + 1, \beta_1 k + 1) + t(\alpha_2 k + 1, \beta_2 k + 1)\big] =: T.$$

By the logarithmic convexity of the beta function, we have

$$B\big[(1 - t)(\alpha_1 k + 1, \beta_1 k + 1) + t(\alpha_2 k + 1, \beta_2 k + 1)\big]$$

$$\leq \big[B(\alpha_1 k + 1, \beta_1 k + 1)\big]^{1-t}\big[B(\alpha_2 k + 1, \beta_2 k + 1)\big]^{t}$$

for (α_1, β_1), $(\alpha_2, \beta_2) \in (0, \infty) \times (0, \infty)$ and $t \in [0, 1]$.

This implies that

$$T \leq \sum_{k=1}^{\infty} \frac{1}{k!}\big[B(\alpha_1 k + 1, \beta_1 k + 1)\big]^{1-t}\big[B(\alpha_2 k + 1, \beta_2 k + 1)\big]^{t}$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k!}\big[B(\alpha_1 k + 1, \beta_1 k + 1)\big]^{1-t}\big[B(\alpha_2 k + 1, \beta_2 k + 1)\big]^{t}$$

$$\leq \left[\sum_{k=1}^{\infty} \frac{1}{k!}\big([B(\alpha_1 k + 1, \beta_1 k + 1)]^{1-t}\big)^{\frac{1}{1-t}}\right]^{1-t}$$

$$\times \left[\sum_{k=1}^{\infty} \frac{1}{k!} \left(\left[B(\alpha_2 k + 1, \beta_2 k + 1) \right]^t \right)^{\frac{1}{t}} \right]^t$$

$$= \left[\sum_{k=1}^{\infty} \frac{1}{k!} \left[B(\alpha_1 k + 1, \beta_1 k + 1) \right] \right]^{1-t}$$

$$\times \left[\sum_{k=1}^{\infty} \frac{1}{k!} \left[B(\alpha_2 k + 1, \beta_2 k + 1) \right] \right]^t,$$

where for the last inequality we used Hölder's inequality with $p = \frac{1}{1-t}$ and $q = \frac{1}{t}$, for which we have $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$.

Therefore, we have

$$F\big((1-t)\alpha_1 + t\alpha_2, (1-t)\beta_1 + t\beta_2\big) - 1$$

$$\leq \big[F(\alpha_1, \beta_1) - 1 \big]^{1-t} \big[F(\alpha_2, \beta_2) - 1 \big]^t$$

for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in (0, \infty) \times (0, \infty)$ and $t \in [0, 1]$.

This proves the statement.　　　　　　　　　　　　　　　　　　　　　　　□

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