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Generalized Ekeland's variational principle with applications

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Abstract

By using the concept of Γ-distance, we prove EVP (Ekeland's variational principle) on quasi-*F*-metric (q-*F*-m) spaces. We apply EVP to get the existence of the solution to EP (equilibrium problem) in complete q-*F*-m spaces with Γ-distances. Also, we generalize Nadler's fixed point theorem.

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1 Introduction and preliminaries

Ekeland [1] was first to study EVP. EVP is a theorem that shows that for some optimization problems there exist nearly optimal solutions. In this paper, we study the concept of Γ-distances defined on a q-F-m space which generalizes the notion of w-distance. We inaugurate EVP in the setting of q-F-m spaces with Γ-distances but without completeness assumption and then in the setting of complete q-F-m spaces with Γ-distances. The equilibrium version of the EVP in the setting of q-F-m spaces with Γ-distances is also presented. We prove some equivalences of our variational principles with Caristi–Kirk type fixed point theorem for multi-valued maps, Takahashi's minimization theorem, and some other related results. As applications of our results, we derive existence results for solutions of equilibrium problems and fixed point theorems for multi-valued maps. We also extend Nadler's fixed point theorem for multi-valued maps to q-F-m spaces with Γ-distances. The results of this paper extend and generalize many results that have appeared recently in Al-Homidan, Ansari, and Yao [2], Lin, Balaj, and Ye [3], Bianchi, Kassay, and Pini [4, 5], Ha [6], and Lin and Du [7].

Definition 1.1 ([8]) Assume that $T \neq \emptyset$. A function $F : T^3 \to [0,\infty)$ is called quasi-F-metric (q-F-m) if

(i) $F(p,q,r) = 0$ if and only if $p = q = r$,
(ii) $F(p,p,q) > 0$ for all $p,q \in T$, with $p \neq q$,
(iii) $F(p,p,r) \leq F(p,q,r)$ for all $p,q,r \in T$, with $r \neq q$,
(iv) $F(p,q,r) \leq F(p,s,s) + F(s,q,r)$ for all $p,q,r,s \in T$.

The pair (T,F) is called q-F-m space.

Let (T,F) be a q-F-m space.

(1) A sequence $\{u_n\}$ in T is an F-Cauchy sequence if, for every $\varepsilon > 0$, there exists a positive integer n_0 such that $F(u_m, u_n, u_\ell) < \varepsilon$ for all $m, n, \ell \geq n_0$.
(2) A sequence $\{u_n\}$ in T is F-convergent to a point $u \in T$ if, for every $\varepsilon > 0$, there exists a positive integer n_0 such that $F(u_m, u_n, u) < \varepsilon$ for all $m, n \geq n_0$.

In this paper, T is assumed to be a q-F-m space.

Definition 1.2 ([9]) A function $\Gamma : T^3 \to [0,\infty)$ is called a Γ-distance if

(Γ1) $\Gamma(p,q,r) \leq \Gamma(p,s,s) + \Gamma(s,q,r)$ for all $p,q,r \in T$,
(Γ2) for each $p \in T$, the functions $\Gamma(p,\cdot,\cdot) : T \to [0,\infty)$ are lower semicontinuous,
(Γ3) for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\Gamma(p,s,s) \leq \delta$ and $\Gamma(s,q,r) \leq \delta$ imply $F(p,q,r) \leq \varepsilon$.

It is easy to see that if the functions $\Gamma(p,\cdot,\cdot) : T \to [0,\infty)$ are lower semicontinuous, then the functions $\Gamma(p,q,\cdot), \Gamma(p,\cdot,q) : T \to [0,\infty)$ are lower semicontinuous, also we conclude that if $q \in T$ and $\{u_m\}$ is a sequence in T which converges to a point $p \in T$ (with respect to the quasi-F-metric) and $\Gamma(q,u_m,u_m) \leq K$ for some $K = K(q) > 0$, then $\Gamma(q,p,p) \leq K$.

Example 1.3 Let $T = \mathbb{R}$ and $F : T^3 \longrightarrow [0,\infty)$. Define

$$F(p,q,r) = \frac{1}{2}\big(|r-p| + |p-q|\big).$$

Then F is a q-F-m.

Example 1.4 The function $\Gamma := F$, given in the above example, is a Γ-distance.

Proof The proofs of (Γ1) and (Γ2) are obvious. For (Γ3), let $\epsilon > 0$, and put $\delta = \frac{\epsilon}{2}$. If

$$\Gamma(p,s,s) = \frac{1}{2}\big(|r-s| + |s-q|\big) < \frac{\epsilon}{2},$$

then

$$F(p,q,r) = \frac{1}{2}\big(|r-p| + |p-q|\big) \leq \frac{1}{2}\big(|r-s| + |s-p| + |p-s| + |s-q|\big) < \epsilon. \qquad \square$$

Example 1.5 Let $T = \mathbb{R}$ and $F : T^3 \to [0,\infty)$ be a q-F-m defined as

$$F(p,q,r) = \begin{cases} 0, & p = q = r, \\ |r-p|, & \text{otherwise}. \end{cases}$$

Then the function $\Gamma : T^3 \to [0,\infty)$ defined by $\Gamma(p,q,r) = |r-p|$ for each $q,r \in T$ is a Γ-distance. But it is not a q-F-m on T.

Proof The proofs of (Γ1) and (Γ2) are obvious. For (Γ3), let $\epsilon > 0$, and put $\delta = \frac{\epsilon}{2}$. If

$$\Gamma(p,s,s) = |s-p| < \frac{\epsilon}{2}$$

and

$$\Gamma(s,q,r) = |r-s| < \frac{\epsilon}{2},$$

then

$$F(p,q,r) = |r-p| \le |r-s| + |s-p| < \epsilon. \qquad \square$$

Example 1.6 Let $T = \mathbb{R}$ and $F : T^3 \longrightarrow [0,\infty)$ be a q-F-m defined as in Example 1.3. Then the function $\Gamma : T^3 \to [0,\infty)$ defined by $\Gamma(p,q,r) = a$ for each $p,q,r \in T$, in which $a > 0$, is a Γ-distance.

Proof The proofs of ($\Gamma 1$) and ($\Gamma 2$) are obvious. For ($\Gamma 3$), let $\epsilon > 0$, and put $\delta = \frac{a}{2}$. Then we have that

$$\Gamma(p,s,s) < \frac{a}{2}$$

and

$$\Gamma(s,q,r) < \frac{a}{2},$$

which imply that

$$F(p,q,r) \le \epsilon. \qquad \square$$

Remark 1.7 ([10]) Let Γ be a Γ-distance. If ξ from $\mathbb{R}_+$ to $\mathbb{R}_+$ is a decreasing and subadditive function with $\xi(0) = 0$, then $\xi \circ \Gamma$ is a Γ-distance.

Now, we present some properties of Γ-distance.

Lemma 1.8 ([9]) *Let* $\{u_n\}$, $\{v_n\}$ *be two sequences in* T *and* $\{\rho_n\}$, $\{\varphi_n\}$ *be nonnegative sequences converging to* 0, *and let* $p,q,r,s \in T$. *Then we have*

(1) $\Gamma(q,u_n,u_n) \le \rho_n$ *and* $\Gamma(u_n,q,r) \le \varphi_n$ *for all* $n \in \mathbb{N}$ *imply that* $F(q,q,r) < \varepsilon$ *and* $q = r$;
(2) $\Gamma(v_n,u_n,u_n) \le \rho_n$ *and* $\Gamma(u_n,v_m,r) \le \rho_n$ *for any* $m > n \in \mathbb{N}$ *imply that* $F(v_n,v_m,r) \to 0$ *and hence* $v_n \to r$;
(3) *if* $\Gamma(u_n,u_m,u_\ell) \le \rho_n$ *for all* $m,n,\ell \in \mathbb{N}$ *with* $\ell \le n \le m$, *then* $\{u_n\}$ *is an F-Cauchy sequence*;
(4) *if* $\Gamma(u_n,s,s) \le \rho_n$ *for all* $n \in \mathbb{N}$, *then the sequence* $\{u_n\}$ *is an F-Cauchy sequence.*

Definition 1.9 ([2]) Let T have a binary relation $\preccurlyeq$.

(i) If the relation $\preccurlyeq$ on T has transitivity and reflexive properties, then it is quasi-order.
(ii) A sequence $\{u_n\}$ in T is said to be decreasing when $u_{n+1} \preccurlyeq u_n$ for all $n \in \mathbb{N}$.
(iii) The relation $\preccurlyeq$ is called lower closed when, for each p in T, $Q(p) = \{q \in T : q \preccurlyeq p\}$ is lower closed; in other words, if $\{u_n\} \subset Q(p)$ is decreasing and converges to $\tilde{p} \in T$, then $\tilde{p} \in Q(p)$.

Definition 1.10 Suppose that (T,F) is a q-F-m space quasi-ordered by $\preccurlyeq$. Define

$$Q(p) := \{q \in T : q \preccurlyeq p\}.$$

We say that $Q(p)$ is $\preccurlyeq$-complete when every decreasing (with respect to $\preccurlyeq$) F-Cauchy sequence of elements from $Q(p)$ converges in $Q(p)$.

Definition 1.11 A function $g : T \to \mathbb{R} \cup \{+\infty\}$ is *lower semicontinuous from above* (in short, lsca) if, for every sequence $\{u_n\}_{n\in\mathbb{N}} \subset T$ converging to $p \in T$ and satisfying $g(u_{n+1}) \leq g(u_n)$ for all $n \in \mathbb{N}$, we have $g(p) \leq \lim_{n\to\infty} g(u_n)$.

2 Ekeland's variational principle (EVP)

Here, we give two generalizations of EVP by using the concept of Γ-distance, both in the incomplete and the complete q-F-m spaces.

Theorem 2.1 *Assume that $\Gamma : T \times T \times T \longrightarrow \mathbb{R}_+$ is a Γ-distance on a q-F-m space (T, F) (not necessarily complete). Let $\omega : (-\infty, \infty] \to (0, \infty)$ be an increasing function and $g : T \to \mathbb{R} \cup \{\infty\}$ be lsca, bounded from below and proper. The relation $\preccurlyeq$ defined by*

$$q \preccurlyeq p \quad \textit{if and only if} \quad p = q \quad \textit{or} \quad \Gamma(p, q, q) \leq \omega\big(g(p)\big)\big(g(p) - g(q)\big) \tag{2.1}$$

is quasi-order. Further, assume that there exists $\hat{p} \in T$ such that $\inf_{p\in T} g(p) < g(\hat{p})$ and $Q(\hat{p}) = \{q \in T : q \preccurlyeq \hat{p}\}$ are $\preccurlyeq$-complete. Then we can find $\bar{p} \in T$ such that

(a) $\Gamma(\hat{p}, \bar{p}, \bar{p}) \leq \omega(g(\hat{p})(g(\hat{p}) - g(\bar{p}))$,

(b) $\Gamma(\bar{p}, p, p) > \omega(g(\bar{p}))(g(\bar{p}) - g(p))$, $p \in T$, $p \neq \bar{p}$.

Proof Reflexivity is obvious. We prove that $\preccurlyeq$ is transitive. Let $r \preccurlyeq q$ and $q \preccurlyeq p$. Then we have

$$r = q \quad \text{or} \quad \Gamma(q, r, r) \leq \omega\big(g(q)\big)\big(g(p) - g(r)\big), \tag{2.2}$$

$$q = p \quad \text{or} \quad \Gamma(p, q, q) \leq \omega\big(g(p)\big)\big(g(p) - g(q)\big). \tag{2.3}$$

If $r = q$ or $p = q$, then transitivity is confirmed. Let $p \neq q \neq r$. Since $\Gamma(p, q, r) \geq 0$ and $\omega(p) > 0$, from (2.2) and (2.3), we get $g(q) \geq g(r)$ and $g(p) \geq g(q)$, i.e., $g(r) \leq g(q) \leq g(p)$. Since ω is increasing, we get $\omega(g(q)) \leq \omega(g(p))$. By using ($\Gamma$1), (2.2), and (2.3), we obtain

$$\begin{aligned}
\Gamma(p, r, r) &\leq \Gamma(p, q, q) + \Gamma(q, r, r) \\
&\leq \omega\big(g(p)\big)\big(g(p) - g(q)\big) + \omega\big(g(q)\big)\big(g(q) - g(r)\big) \\
&\leq \omega\big(g(p)\big)\big(g(p) - g(q)\big) + \omega\big(g(p)\big)\big(g(q) - g(r)\big) \\
&= \omega\big(g(p)\big)\big(g(p)\big) - g(r)).
\end{aligned}$$

Thus $r \preccurlyeq p$, that is, $\preccurlyeq$ is quasi-order on T.

Now, a sequence $\{u_n\}$ in $Q(\hat{p})$ is constructed as follows. Let

$$\begin{aligned}
Q(u_n) &= \big\{q \in Q(\hat{p}) : q = u_n \text{ or } \Gamma(u_n, q, q) \leq \omega\big(g(u_n)\big)\big(g(u_n) - g(q)\big)\big\} \\
&= \big\{q \in Q(\hat{p}) : q \preccurlyeq u_n\big\}.
\end{aligned}$$

Put $\hat{p} = u_0$ and choose $u_2 \in Q(u_1)$ so that $g(u_2) \leq \inf_{p\in Q(u_1)} g(p) + \frac{1}{2}$. Suppose that $u_{n-1} \in T$ is defined and choose $u_n \in Q(u_{n-1})$ so that

$$g(u_n) \leq \inf_{p\in Q(u_{n-1})} g(p) + \frac{1}{n}. \tag{2.4}$$

Since $u_n \in Q(u_{n-1})$, we have $u_n \preccurlyeq u_{n-1}$, and $\{u_n\}$ is decreasing. Also

$$\Gamma(u_{n-1},u_n,u_n) \leq \omega(g(u_{n-1})\big(g(u_{n-1}) - g(u_n)\big).$$

Hence $g(u_n) \leq g(u_{n-1})$ for all $n \in \mathbb{N}$, that is, $\{g(u_n)\}$ is decreasing. Also, g is bounded from below, so $\{g(u_n)\}$ is convergent. Let $\lim_{n\to\infty} g(u_n) = w$. Also, we prove that the sequence $\{u_n\}$ is F-Cauchy in $Q(\hat{p})$. Assume that $n < m$. Then we have

$$\begin{aligned}
\Gamma(u_n,u_m,u_m) &\leq \Gamma(u_n,u_{n+1},u_{n+1}) + \Gamma(u_{n+1},u_m,u_m) \\
&\leq \Gamma(u_n,u_{n+1},u_{n+1}) + \Gamma(u_n,u_{n+2},u_{n+2}) + \cdots + \Gamma(u_{n+1},u_m,u_m) \\
&\leq \omega\big(g(u_n)\big)\big(g(u_n) - g(u_{n+1})\big) + \omega\big(g(u_{n+1})\big)\big(g(u_{n+1}) - g(u_{n+2})\big) \\
&\quad + \cdots + \omega\big(g(u_{m-1})\big)\big(g(u_{m-1}) - g(u_m)\big) \\
&\leq \omega\big(g(u_n)\big)\big(g(u_n) - g(u_{n+1})\big) + \omega\big(g(u_n)\big)\big(g(u_{n+1}) - g(u_{n+2})\big) \\
&\quad + \cdots + \omega\big(g(u_n)\big)\big(g(u_{m-1}) - g(u_m)\big) \\
&\leq \omega\big(g(u_n)\big)\big(g(u_n) - g(u_m)\big) \leq \omega\big(g(u_n)\big)\big(g(u_n) - w\big).
\end{aligned}$$

Put $\rho_n = \omega(g(u_n))(g(u_n) - w)$. Then $\lim_{n\to\infty} \rho_n = 0$, and according to Lemma 1.8(3), the sequence $\{u_n\}$ is nonincreasing and F-Cauchy in $Q(\hat{p})$. $\preccurlyeq$-completeness $Q(\hat{p})$ implies that $\{u_n\}$ converges to a point $\bar{p} \in P(\hat{p})$. From transitivity of $\preccurlyeq$, we conclude $Q(u_n) \subset Q(u_{n-1})$ for all $n \in \mathbb{N}$.

Now we are ready to show that $\{\bar{p}\} = Q(\bar{p})$. Assume that $p \in Q(\bar{p})$, and $p \neq \bar{p}$. Then $\Gamma(\bar{p},p,p) \leq \omega(g(\bar{p}))(g(\bar{p}) - g(p))$. Since Γ is nonnegative and $\omega \geq 0$, we conclude that $g(p) \leq g(\bar{p})$.

Since $\bar{p} \in Q(\hat{p}) = Q(u_0)$, we have $\bar{p} \in Q(u_{n-1})$ for all $n \in \mathbb{N}$. Thus $p \preccurlyeq \bar{p}$ and $\bar{p} \preccurlyeq u_{n-1}$, and so $p \preccurlyeq u_{n-1}$ (transitivity of $\preccurlyeq$) for $n \in \mathbb{N}$. Also, we have $g(\bar{p}) \leq g(u_n) \leq g(p) + \frac{1}{n}$ and $\lim_{n\to\infty} g(u_n) = w$. Hence $g(\bar{p}) \leq w \leq g(p) \leq g(\bar{p})$ and so $g(\bar{p}) = w = g(p)$. Since $p \preccurlyeq u_n$ for all $n \in \mathbb{N}$, we get

$$\Gamma(u_n,p,p) \leq \omega\big(g(u_n)\big)\big(g(u_n) - g(p)\big) = \omega(g(u_n)\big(g(u_n) - w\big) = \rho_n. \tag{2.5}$$

Also, $\bar{p} \preccurlyeq u_n$ for all $n \in \mathbb{N}$. Thus we have

$$\Gamma(u_n,\bar{p},\bar{p}) \leq \omega(g(u_n)\big(g(u_n) - g(\bar{p})\big) = \omega(g(u_n)\big(g(u_n) - w\big) = \rho_n \tag{2.6}$$

and $\lim_{n\to\infty} \rho_n = 0$. By using (2.5), (2.6), and Lemma 1.8(1), we conclude that $p = \bar{p}$, $\{\bar{p}\} = Q(\bar{p})$, and so we have $\Gamma(\bar{p},p,p) > \omega(g(\bar{p}))(g(\bar{p}) - g(p))$ for all $p \in T$ and $p \neq \bar{p}$. □

Theorem 2.2 *Assume that (T,F) is a complete q-F-m space and that $\Gamma : T \times T \times T \longrightarrow \mathbb{R}_+$ is a Γ-distance on Z. Let $\omega : (-\infty,\infty] \to (0,\infty)$ be an increasing function, and $g : T \to \mathbb{R} \cup \{\infty\}$ be lsca, bounded from below, and proper. Let $\hat{p} \in T$ in which $\inf_{p\in T} g(p) < g(\hat{p})$. Then we can find $\bar{p} \in T$ such that*

(a) $\Gamma(\hat{p},\bar{p},\bar{p}) \leq \omega(g(\hat{p})(g(\hat{p}) - g(\bar{p}))$,

(b) $\Gamma(\bar{p},p,p) > \omega(g(\bar{p}))(g(\bar{p}) - g(p))$, $p \in T$, $p \neq \bar{p}$.

Proof Define a relation $\preccurlyeq$ by

$$q \preccurlyeq p \quad \text{if and only if} \quad p = q \quad \text{or} \quad \Gamma(p,q,q) \le \omega\big(g(p)\big)\big(g(p) - g(q)\big). \tag{2.7}$$

In the proof of the previous theorem, we proved that $\preccurlyeq$ is quasi-order. Now we are ready to show that $\preccurlyeq$ is lower closed. According to Definition 1.9, assume that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is decreasing in T, which converges to p, and $u_{n+1} \preccurlyeq u_n$. We have

$$\Gamma(u_n,u_{n+1},u_{n+1}) \le \omega\big(g(u_n)\big)\big(g(u_n) - g(u_{n+1})\big). \tag{2.8}$$

Since $\Gamma \ge 0$ and $\omega \ge 0$, we have $g(u_{n+1}) \le g(u_n)$, and so $\{g(u_n)\}$ is a decreasing sequence. Since g is bounded from below, we have that $\lim_{n\to\infty} g(u_n)$ is finite. Let $\lim_{n\to\infty} g(u_n) = w$. Then $w \le g(u_n)$ for all $n \in \mathbb{N}$. Since g is lsca, we conclude that $g(p) \le \lim_{n\to\infty} g(u_n)$, and so we get $g(p) \le w \le g(u_n)$.

Assume that $n \in \mathbb{N}$ is fixed. For all $m \in \mathbb{N}$, where $m > n$, similar to the proof of Theorem 2.1, we get

$$\Gamma(u_n,u_m,u_m) \le \omega\big(g(u_n)\big)\big(g(u_n) - g(u_m)\big) \le \omega\big(g(u_n)\big)\big(g(u_n) - g(u)\big).$$

Therefore, we conclude that $g(p) \le g(u_n)$ for all $n \in \mathbb{N}$. Let $K = \omega(g(u_n))(g(u_n) - g(u))$. According to $(\Gamma 2)$, we have $\Gamma(u_n,u_m,u_m) \le K$ and then $\Gamma(u_n,p,p) \le K$ for $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we get $\Gamma(u_n,p,p) \le K = \omega(g(u_n))(g(u_n) - g(u))$. So $p \preccurlyeq u_n$ and we conclude that $\preccurlyeq$ is lower closed for all $r \in T$. Also $Q(r) = \{q \in T : q \preccurlyeq r\}$ is lower closed. The sequence $\{u_n\}$ is constructed as follows:

$$\begin{aligned} Q(u_n) &= \big\{q \in T : q = u_n \text{ or } \Gamma(u_n,q,q) \le \omega\big(g(u_n)\big)\big(g(u_n) - g(q)\big)\big\} \\ &= \{q \in T : q \preccurlyeq u_n\}. \end{aligned}$$

Then, for all $n \in \mathbb{N}$, $Q(u_n)$ is a lower closed subset of a complete q-F-m space and therefore $\preccurlyeq$-complete. The assertion concludes from Theorem 2.1. □

Corollary 2.3 *Assume that g, Γ, T, and ω are the same as in Theorem 2.2. Let $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be increasing and sub-additive with $\xi(0) = 0$. If there is $\hat{p} \in T$, such that $\inf_{p\in Z} g(p) < g(\hat{p})$, then there is $\bar{p} \in T$ such that*

(a) $\xi(\Gamma(\hat{p},\bar{p},\bar{p})) \le \omega(g(\hat{p})(g(\hat{p}) - g(\bar{p}))$,

(b) $\xi(\Gamma(\bar{p},p,p)) > \omega(g(\bar{p})(g(\bar{p}) - g(p))$ *for all* $p \in T$, $p \ne \bar{p}$.

Proof From Remark 1.7, $\xi \circ \Gamma$ is a Γ-distance on T. So, by Theorem 2.2, we obtain the conclusion. □

3 Equivalences

Theorem 3.1 *Assume that (T,F) is a complete q-F-m space. Let $\Gamma : T \times T \times T \to \mathbb{R}_+$ be a Γ-distance on T, $\omega : (-\infty,\infty] \to (0,\infty)$ be an increasing function and g be lsca, proper, and bounded from below. Then the following statements are equivalent to Theorem 2.2:*

(i) (Caristi–Kirk fixed point theorem). *Let $P : T \to 2^T$ be a multi-valued mapping with nonempty values. If the following condition*

$$\text{for each } q \in P(p), \quad \Gamma(p,q,q) \le \omega\big(g(p)\big)\big(g(p) - g(p)\big) \tag{3.1}$$

is satisfied, then we can find $\bar{p} \in T$ such that $\{\bar{p}\} = P(\bar{p})$. If the following condition

$$\text{there is } q \in P(p) \text{ such that } \Gamma(p,q,q) \le \omega\big(g(p)\big)\big(g(p) - g(q)\big) \tag{3.2}$$

is satisfied, then we can find $\bar{p} \in T$ such that $\bar{p} \in P(\bar{p})$.

(ii) (Takahashi's minimization theorem). *Assume that, for all $\hat{p} \in T$ with $\inf_{r \in T} g(r) < g(\hat{p})$, there is $p \in T$ such that*

$$p \ne \hat{p} \quad \text{and} \quad \Gamma(\hat{p},p,p) \le \omega\big(g(\hat{p})\big)\big(g(\hat{p}) - g(p)\big). \tag{3.3}$$

Then we can find $\bar{p} \in T$ such that $g(\bar{p}) = \inf_{q \in T} g(q)$.

(iii) (Equilibrium version of EVP). *Let $G : T \times T \to \mathbb{R} \cup \{\infty\}$ be a function satisfying*:
- (E_1) *for every $p,q,r \in T$, $G(p,r) \le G(p,q) + G(q,r)$;*
- (E_2) *for all fixed $p \in T$, the function $G(p,\cdot) : T \to \mathbb{R} \cup \{\infty\}$ is proper and lsca;*
- (E_3) *there is $p \in T$ such that $\inf_{p \in T} G(\hat{p},p) > -\infty$.*

  Then we can find $\bar{p} \in T$ such that
  - (A) $\omega(g(\hat{p})G(\hat{p},\bar{p}) + \Gamma(\hat{p},\bar{p},\bar{p}) \le 0$,
  - (B) $\omega(g(\bar{p})G(\bar{p},p) + \Gamma(\bar{p},p,p) > 0$ *for all $p \in T$, $p \ne \bar{p}$.*

Proof Assertion (i) follows from Theorem 2.2. By Theorem 2.2(b), there exists $\bar{p} \in T$ such that

$$\Gamma(\bar{p},p,p) > \omega\big(g(\bar{p})\big)\big(g(\bar{p}) - g(p)\big) \quad \text{for all } p \in T, p \ne \bar{p}. \tag{3.4}$$

We prove that $\{\bar{p}\} = T(\bar{p})$ (respectively, $\bar{p} \in T(\bar{p})$). On the contrary, assume that $q \in P(\bar{p})$ and $q \ne \bar{q}$. Then, by (3.1), $\Gamma(\bar{p},q,q) \le \omega(g(\bar{p}))(g(\bar{p}) - g(q))$, and by (3.4), $\Gamma(\bar{p},q,q) > \omega(g(\bar{p}))(g(\bar{p}) - g(q))$. Therefore $\{\bar{p}\} = P(\bar{p})$ (respectively, $\bar{p} \in P(\bar{p})$).

(i) $\Rightarrow$ (ii): Let $P : T \to 2^T$. Then we define $P(p) = \{q \in T : \Gamma(p,q,q) \le \omega(g(p))(g(p) - g(q))\}$ for every $p \in T$. Then P has property (3.1). By (i), there exists $\bar{p} \in T$ such that $\{\bar{p}\} = P(\bar{p})$. Moreover, by assumption, there exists $p \in T$ such that $p \ne \hat{p}$ and $\Gamma(\hat{p},p,p) \le \omega(g(\hat{p}))(g(\hat{p}) - g(p))$ for all $\hat{p} \in T$ when $\inf_{r \in T} g(r) < g(\hat{p})$. Therefore, $p \in T(\hat{p})$ and $P(\hat{p}) \setminus \{\hat{p}\} \ne \emptyset$. Hence $g(\bar{p}) = \inf_{p \in T} g(p)$.

(ii) $\Rightarrow$ (iii): Let $g : T \to \mathbb{R} \cup \{\infty\}$. Then we define $g(p) = G(\hat{p},p)$, where $\hat{p}$ is the same as in (E_3). Then from (E_3) we get $\inf_{p \in T} g(p) > -\infty$, and so g is bounded from below. Assume that (A) is false. So, for all $p \in T$, we can find $q \in T$ such that

$$q \ne p \quad \text{and} \quad \omega\big(g(p)\big)G(x,y) + \Gamma(p,q,q) \le 0. \tag{3.5}$$

By (E_1), we get $G(\hat{p},q) \le G(\hat{p},p) + G(p,q)$, i.e., $G(\hat{p},q) - G(\hat{p},p) \le G(p,q)$.

Then by (3.5) we get

$$\omega\big(g(p)\big)\big(G(\hat{p},q) - G(\hat{p},p)\big) + \Gamma(p,q,q) \le \omega\big(g(p)\big)G(p,q) + \Gamma(p,q,q) \le 0. \tag{3.6}$$

So, for every $p \in T$, we can find $q \in T$ such that $q \neq p$ and $\omega(g(p))(g(q) - g(p)) + \Gamma(p,q,q) \leq 0$. Also, $\Gamma(p,q,q) \leq \omega(g(p))(g(q) - g(p))$.

Now, by (ii), $g(\bar{p}) = \inf_{q\in T} g(q) \leq g(r)$. Replace p by $\bar{p}$ in the last relation. Then there exists $q \in T$ such that $q \neq p$ and $\omega(g(\bar{p}))(G(\hat{p},q) - G(\hat{p},\bar{p})) + \Gamma(\bar{p},q,q) \leq 0$, that is,

$$\omega\bigl(g(\bar{p})\bigr)\bigl(g(q) - g(\bar{p})\bigr) + \Gamma(\bar{p},q,q) \leq 0 \quad \text{or} \quad \Gamma(\bar{p},q,q) \leq \omega\bigl(g(\bar{p})\bigr)\bigl(g(\bar{p}) - g(q)\bigr). \tag{3.7}$$

Since $q \neq p$, by using Lemma 1.8(1), $\Gamma(\bar{p},\bar{p},\bar{p}) \neq 0$, and $\Gamma(\bar{p},q,q) \neq 0$, we get $\Gamma(\bar{p},q,q) > 0$, and by (3.7), we obtain $0 < \omega(g(\bar{p}))(g(\bar{p}) - g(q)) \Rightarrow g(q) < g(\bar{p})$. That is a contradiction.

(iii) $\Rightarrow$ Theorem 2.2: Let $G : T \times T \to \mathbb{R} \cup \{\infty\}$ be a function defined by $G(p,q) = g(q) - h(p)$ for all $p,q \in T$. According to Theorem 2.2, G satisfies all the conditions of (iii). By (A), we get

$$\omega\bigl(g(\hat{p})\bigr)G(\hat{p},\bar{p}) + \Gamma(\hat{p},\bar{p},\bar{p}) \leq 0 \quad \Rightarrow \quad \omega\bigl(g(\hat{p})\bigr)\bigl(g(\bar{p}) - g(\hat{p})\bigr) + \Gamma(\hat{p},\bar{p},\bar{p}) \leq 0.$$

Then

$$\Gamma(\hat{p},\bar{p},\bar{p}) \leq \omega\bigl(g(\hat{p})\bigr)\bigl(g(\hat{p}) - g(\bar{p})\bigr).$$

Also, by (B), we get $\omega(g(\bar{p}))G(\bar{p},p) + \Gamma(\bar{p},p,p) > 0$ for all $p \in T$, $p \neq \bar{p}$. Then

$$\omega\bigl(g(\bar{p})\bigr)\bigl(g(p) - g(\bar{p})\bigr) + \Gamma(\bar{p},p,p) > 0 \quad \Rightarrow \quad \Gamma(\bar{p},p,p) > \omega\bigl(g(\bar{p})\bigr)\bigl(g(\bar{p}) - g(p)\bigr)$$

for all $p \in T$, $p \neq \bar{p}$. □

Corollary 3.2 *Let g, Γ, T, ω be the same as in Theorem 3.1 and suppose that $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a subadditive and increasing function such that $\xi(0) = 0$. Assume that $P : T \to 2^T$ is a multi-valued mapping with nonempty values. If, for all $p \in T$, there is $q \in P(p)$ such that*

$$\xi\bigl(\Gamma(p,q,q)\bigr) \leq \omega\bigl(g(p)\bigr)\bigl(g(p) - g(q)\bigr),$$

then P has a fixed point in T.

Proof Note that $\xi \circ \Gamma$ is a Γ-distance on T by Remark 1.7. Then, by Theorem 3.1(i), P has a fixed point in T. □

Corollary 3.3 *Suppose that (T,F) is a complete q-F-m space. Let $\Gamma : T \times T \times T \to \mathbb{R}_+$ be a Γ-distance on T and $G : T \times T \to \mathbb{R}$ be a function satisfying the conditions:*

(F_1) *$G(p,r) \leq G(p,q) + G(q,r)$ for all $p,q,r \in T$;*

(F_2) *for every constant $p \in T$, the function $G(p,\cdot) : T \to \mathbb{R}$ is lsca and bounded from below. Then, for each $\epsilon > 0$ and every $\hat{p} \in T$, there exists $\bar{p} \in T$ such that*

(C) *$G(\hat{p},\bar{p}) + \epsilon\Gamma(\hat{p},\bar{p},\bar{p}) \leq 0$;*

(D) *$G(\bar{p},p) + \epsilon\Gamma(\bar{p},p,p) > 0$ for all $p \in T$, $p \neq \bar{p}$.*

Proof Let $g : T \to \mathbb{R} \cup \{\infty\}$. Then we define $g(\hat{p}) = G(p,\hat{p})$ for all $\hat{p} \in T$ and fixed $p \in T$. Then, by Theorem 3.1(iii), (C) and (D) are established. □

Corollary 3.4 *Let $G : T \times T \to (-\infty, \infty)$ be proper, lsca, and bounded from below in the first argument and $\omega : (-\infty, \infty) \to (0, \infty)$ be nondecreasing. Assume that, for every $p \in T$ with $\{x \in T : G(p,x) < 0\} \neq \emptyset$, there exists $q = q(p) \in T$ with $q \neq p$ such that*

$$\Gamma(p,q,q)) \leq \omega\bigl(G(p,t)\bigr)\bigl(G(p,t) - G(q,t)\bigr)$$

for all $t \in \{p \in T : G(x,p) > \inf_{a\in T} G(a,p)\}$. Then there exists $y \in T$ such that $G(y,p) \geq 0$ for all $q \in T$.

Proof By Theorem 2.2(b), for all $r \in T$, there exists $y(r) \in T$ such that

$$\Gamma\bigl(y(r),q,q)\bigr) > \omega\bigl(G\bigl(y(r),r\bigr)\bigr)\bigl(G\bigl(y(r),r\bigr) - G(q,r)\bigr)$$

for all $q \in T$ and $p \neq y(r)$. We show that there exists $y \in T$ such that $G(y,q) \geq 0$ for all $q \in T$. Suppose it is false. Then, for all $p \in T$, there exists $q \in T$ such that $G(p,q) < 0$, and thus $\{x \in T : G(p,x) < 0\} \neq \emptyset$. Then, according to the assumption, there exists $q = q(y(r))$, $q \neq y(r)$ such that

$$\Gamma\bigl(y(r),q,q)\bigr) \leq \omega\bigl(G\bigl(y(r),r\bigr)\bigr)\bigl(G\bigl(y(r),r\bigr) - G(q,r)\bigr),$$

which is a contradiction. □

Example 3.5 Let $T = [0,1]$ and $F(p,q,r) = \frac{1}{2}\max\{|p-q|, |p-r|, |q-r|\}$. So (T,F) is a complete q-F-m. Assume that $G : T \times T \to \mathbb{R}$ is defined by $G(p,q) = 3p - 2q$. Then the function $x \to G(p,q)$ is proper, lsca, and bounded from below. Also, for every $q \in T$, $G(1,q) \geq 0$ and for all $p \in [\frac{2}{3}, 1]$, $G(p,q) \geq 0$ for all $q \in T$. On the other hand, when $p \in [0, \frac{2}{3}]$ and $q \in [\frac{3}{2}p, 1]$, we have $G(p,q) = 3p - 2q < 0$. Then $\{x \in T, G(p,x) < 0\} \neq \emptyset$. Let $p, q \in T$ and $p \geq q$. Then we have $p - q = \frac{1}{3}\{(3p - 2x) - (3q - 2x)\}$ for all $x \in T$. Suppose that $\omega : [0,\infty) \to [0,\infty)$ with $\omega(t) = \frac{1}{3}$. Then

$$F(p,q,q)) \leq \omega\bigl(G(p,x)\bigr)\bigl(G(p,x) - G(q,x)\bigr)$$

for all $p \geq q$. By Corollary 3.4, there exists $y \in T$ such that $G(y,p) \geq 0$ for all $p \in T$.

4 Equilibrium problem

The EP (equilibrium problem) is a new research subject in nonlinear science and engineering [11].

Definition 4.1 Suppose that S is a nonempty subset of a metric space T, $G : S \times S \to \mathbb{R}$ is a function on $\mathbb{R}$, and Γ is a Γ-distance on T. Let $\delta > 0$. If there is $\bar{p} \in T$ such that

$$G(\bar{p},q) + \delta\Gamma(\bar{p},q,q) \geq 0 \quad \text{for all } q \in S, \tag{4.1}$$

then $\bar{p}$ is a δ-solution to EP. Moreover, if (4.1) is satisfied as strict, then $\bar{p}$ is called a δ-solution to strict EP.

Theorem 4.2 *Suppose that $S \neq \emptyset$ is a compact subset of a complete metric space T and that Γ is a Γ-distance. If a real-valued function $G : S \times S \to \mathbb{R}$ satisfies the following conditions:*

(E_1) *$G(p,r) \le G(p,q) + G(q,r)$ for all $p,q,r \in S$;*
(E_2) *the function $G(p,\cdot) : S \to \mathbb{R}$ is lsca and bounded from below for each fixed $p \in T$;*
(E_3) *the function $G(\cdot,q) : S \to \mathbb{R}$ is upper semicontinuous for each fixed $q \in S$, then we can find a solution $\bar{p} \in S$ to EP.*

Proof By Corollary 3.3, there is $u_n \in S$ such that

$$G(u_n,q) + \frac{1}{n}\Gamma(u_n,q,q) \ge 0 \quad \text{for each } q \in S.$$

In other words, for $\epsilon = \frac{1}{n}$, $u_n \in S$ is a δ-solution to EP. Since S is compact, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to \bar{p}$. Since $G(\cdot,q)$ is upper semicontinuous, we have

$$G(\bar{p},q) \ge \limsup_{S\to\infty}\left(G(u_{n_k},q) + \frac{1}{n_k}\Gamma(u_{n_k},q,q)\right) \ge 0 \quad \text{for all } q \in S.$$

Hence $\bar{p}$ is a solution to EP. □

Definition 4.3 Assume that (T,F) is a complete q-F-m space and that Γ is a Γ-distance on T. An element $u_0 \in T$ satisfies the condition (Ξ) if every sequence $\{u_n\} \subset T$, satisfying $G(u_0,u_n) \le \frac{1}{n}$ for all $n \in \mathbb{N}$ and $G(u_n,p) + \frac{1}{n}\Gamma(u_n,p,p) \ge 0$ for every $p \in T$ and $n \in \mathbb{N}$, has a convergent subsequence.

Theorem 4.4 *Suppose that (T,F) is a complete q-F-m space and that Γ is a Γ-distance on T. Let $G : T \times T \longrightarrow \mathbb{R}$ satisfy conditions (F_1) and (F_2) of Corollary 3.3 and G be upper semicontinuous in the first variable. If $u_0 \in T$ satisfies the condition (Ξ), then we can find a solution $\bar{p} \in T$ to EP.*

Proof If in Corollary 3.3 we put $\epsilon = \frac{1}{n}$, then for every $n \in \mathbb{N}$ and for each $u_0 \in T$, there is $u_n \in T$ satisfying the following conditions:

$$G(u_0,u_n) + \frac{1}{n}\Gamma(u_0,u_n,u_n) \le 0 \tag{4.2}$$

and

$$G(u_n,p) + \frac{1}{n}\Gamma(u_n,p,p) > 0 \quad \text{for all } p \in T. \tag{4.3}$$

Since $\Gamma(u_0,u_n,u_n) \ge 0$, by (4.2), we conclude that $G(u_0,u_n) \le 0$ for all $n \in \mathbb{N}$. From (Ξ), there is a subsequence $\{u_n\}$ converging to $\bar{p} \in T$. Since $G(\cdot,p)$ is upper semicontinuous and by (4.3), we get that $\bar{p}$ is a solution to *EP*. □

5 A generalization of Nadler's fixed point theorem

In this section, we are ready to prove Nadler's fixed point theorem in q-F-m spaces with Γ-distance.

Definition 5.1 Suppose that (T,F) is a q-F-m space. A mapping $P : T \longrightarrow 2^T$ is called Γ-contractive if there are a Γ-distance Γ on T and w in $[0,1]$ such that, for all $p,q \in T$

and $x \in P(p)$, there is $y \in P(q)$ satisfying

$$\Gamma(x,y,y) \leq w\Gamma(x,q,q).$$

Then $w \in \mathbb{R}$ is called a Γ-contractive constant. In particular, $g : T \to T$ is said to be Γ-contractive if there are a Γ-distance on T and $w \in [0,1]$ such that

$$\Gamma\big(g(p),g(q),g(q)\big) \leq w\Gamma(p,q,q) \quad \text{for all } p,q \in T.$$

Theorem 5.2 *Suppose that (T,F) is a complete q-F-m space, $P : T \to 2^T$ is a Γ-contractive multi-valued mapping, and Γ is a Γ-distance such that, for each p in T, $P(p)$ is a nonempty closed subset. Then there is $\bar{p} \in T$ such that $\bar{p} \in P(\bar{p})$ and $\Gamma(\bar{p},\bar{p},\bar{p}) = 0$.*

Proof Suppose that Γ is a Γ-distance on T and $w \in [0,1)$ is a Γ-contractive constant. Assume that $u_0 \in T$ and $u_1 \in P(u_0)$ is fixed. Then, by the definition of Γ-contractivity, there exists $u_2 \in P(u_1)$ such that

$$\Gamma(u_1,u_2,u_2) \leq w\Gamma(u_0,u_1,u_1).$$

In the same way, we make the sequence $\{u_n\}$ such that $u_{n+1} \in P(u_n)$ and

$$\Gamma(u_n,u_{n+1},u_{n+1}) \leq w\Gamma(u_{n-1},u_n,u_n) \quad \text{for all } n \in \mathbb{N}.$$

We have

$$\begin{aligned}
\Gamma(u_n,u_{n+1},u_{n+1}) &\leq w\Gamma(u_{n-1},u_n,u_n) \\
&\leq w^2\Gamma(u_{n-2},u_{n-1},u_{n-1}) \\
&\ \ \vdots \\
&\leq w^n\Gamma(u_0,u_1,u_1).
\end{aligned}$$

Then, for all $m,n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
\Gamma(u_n,u_m,u_m) &\leq \Gamma(u_n,u_{n+1},u_{n+1}) + \Gamma(u_{n+1},u_m,u_m) \\
&\leq \Gamma(u_n,u_{n+1},u_{n+1}) + \Gamma(u_{n+1},u_{n+2},u_{n+2}) \\
&\quad + \cdots + \Gamma(u_{m-1},u_m,u_m) \\
&\leq w^n\Gamma(u_0,u_1,u_1) + w^{n+1}\Gamma(u_0,u_1,u_1) \\
&\quad + \cdots + w^{m-1}\Gamma(u_{m-1},u_m,u_m) \\
&= w^n\big(1 + w + w^2 + \cdots + w^{m-n-1}\big)\Gamma(u_0,u_1,u_1) \\
&\leq \frac{w^n}{1-w}\Gamma(u_0,u_1,u_1).
\end{aligned}$$

Then the sequence $\{\rho_n\} = \{\frac{w^n}{1-w}\}$ is a nonnegative sequence on $\mathbb{R}$ tending to 0 as $n \to \infty$. By Lemma 1.8(3), $\{u_n\}$ is an F-Cauchy sequence in T. The sequence $\{u_n\}$ is convergent to a $\bar{p} \in T$ since T is complete. Let $n \in \mathbb{N}$. Then we have

$$\Gamma(u_n, u_m, u_m) \leq \frac{w^n}{1-w} \Gamma(u_0, u_1, u_1) \tag{5.1}$$

for all $m > n$.

Let $S = \frac{w^n}{1-w} \Gamma(u_0, u_1, u_1)$. Then $S \geq 0$. Now, by $(\Gamma 2)$ and $\Gamma(u_n, u_m, u_m) \leq S$, we have $\Gamma(u_n, \bar{p}, \bar{p}) \leq S$ for all $n \in \mathbb{N}$. Since n is an arbitrary constant, we have

$$\Gamma(u_n, \bar{p}, \bar{p}) \leq \frac{w^n}{1-w} \Gamma(u_0, u_1, u_1) \quad \text{for all } n \in \mathbb{N}. \tag{5.2}$$

By the assumption, there is $w_n \in P(\bar{p})$ such that

$$\begin{aligned} \Gamma(u_n, w_n, w_n) &\leq w\Gamma(u_{n-1}, \bar{p}, \bar{p}) \\ &\leq \frac{w^n}{1-w} \Gamma(u_0, u_1, u_1). \end{aligned} \tag{5.3}$$

By (5.2), (5.3), and Lemma 1.8(2), we have that the sequence $\{w_n\}$ converges to $\bar{p}$. Since $P(\bar{p})$ is closed, we have $\bar{u} \in P(\bar{u})$.

Now, we prove that $\Gamma(\bar{u}, \bar{u}, \bar{u}) = 0$. Since P is Γ-contractive, there is $v_1 \in P(\bar{p})$ such that

$$\Gamma(\bar{p}, v_1, v_1) \leq w\Gamma(\bar{p}, \bar{p}, \bar{p}).$$

Now, we construct a sequence $\{v_n\}$ as follows: $v_{n+1} \in P(v_n)$ and

$$\Gamma(\bar{p}, v_{n+1}, v_{n+1}) \leq w\Gamma(\bar{p}, v_n, v_n) \quad \text{for all } n \in N.$$

Therefore, for all $n \in \mathbb{N}$, we get

$$\Gamma(\bar{p}, v_n, v_n) \leq \Gamma(\bar{p}, v_{n-1}, v_{n-1}) \leq \cdots \leq w^n \Gamma(\bar{p}, \bar{p}, \bar{p}). \tag{5.4}$$

Since $\Gamma(\bar{p}, \bar{p}, \bar{p}) \geq 0$ and $w^n \geq 0$ for all $n \in \mathbb{N}$ and $w^n \to 0$ as $n \to \infty$, $\{v_n\}$ is an F-Cauchy sequence in T according to Lemma 1.8(4).

On the other hand, since T is complete, $\{v_n\}$ converges to $\bar{q} \in T$. Let $S = \sup_{n\in\mathbb{N}} w^n \Gamma(\bar{p}, \bar{p}, \bar{p})$. Then, from (5.4) and (Γ_2), we have

$$\Gamma(\bar{p}, v_n, v_n) \leq S \quad \Longrightarrow \quad \Gamma(\bar{p}, \bar{q}, \bar{q}) \leq S = \sup_{n\in\mathbb{N}} w^n \Gamma(\bar{p}, \bar{p}, \bar{p}).$$

So $\Gamma(\bar{p}, \bar{q}, \bar{q}) \leq 0$ and $\Gamma(\bar{p}, \bar{q}, \bar{q}) = 0$. Moreover, we have

$$\begin{aligned} \Gamma(u_n, \bar{q}, \bar{q}) &\leq \Gamma(u_n, \bar{p}, \bar{p}) + \Gamma(\bar{p}, \bar{q}, \bar{q}) \\ &\leq \frac{w^n}{1-w} \Gamma(u_0, u_1, u_1) \end{aligned} \tag{5.5}$$

for all $n \in \mathbb{N}$. By (5.2), (5.5), and by Lemma 1.8(1), we obtain $\bar{p} = \bar{q}$ and so $\Gamma(\bar{p}, \bar{p}, \bar{p}) = 0$. □

As a new approach, one can generalize the results presented in [12–21] in q-F-m spaces with Γ-distance.

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