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Generalized k-fractional integral inequalities associated with (α, m)-convex functions

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Abstract

This research investigates bounds of the sum of left-sided and right-sided fractional integrals in a compact form. These bounds are established by using (α, m)-convex functions. The results of this paper also have connection with some known and already published results (Dragomir and Agarwal in Appl. Math. Lett. 11(5):91–95, 1998; Farid in J. Anal. 2018, https://doi.org/10.1007/s41478-0079-4; Farid et al. in Mathematics 2018(6):248, 2018). Moreover, the investigated results are applied to the end points as well as the mid point of the domain and some interesting consequences have been obtained.

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1 Introduction

A function $f : I \to \mathbb{R}$ is said to be convex if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If inequality (1.1) is reversed, then the function f will be the concave on $[a,b]$.

Convex functions have been generalized theoretically extensively; these generalizations include the m-convex function, n-convex function, r-convex function, h-convex function, (h,m)-convex function, s-convex function and many others. Here the authors are interested in the (α, m)-convex functions defined by Mihesan [16]:

Definition 1 A function $f : [0,b] \subset \mathbb{R} \to \mathbb{R}$, $b > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0,1]^2$ if

$$f\big(tx + m(1-t)y\big) \le t^\alpha f(x) + m\big(1-t^\alpha\big)f(y) \tag{1.2}$$

holds for all $x, y \in [0,b]$ and $t \in [0,1]$.

For suitable choices of α and m, class of (α, m)-convex functions reduces to different known classes of functions defined on $[0,b]$ given in the following remark.

Remark 1

(i) If $(\alpha, m) = (1, m)$, then (1.2) produces the definition of m-convex function.
(ii) If $(\alpha, m) = (1, 1)$, then (1.2) produces the definition of a convex function.
(iii) If $(\alpha, m) = (1, 0)$, then (1.2) produces the definition of a star-shaped function: A function $f : [0, b] \to \mathbb{R}$ is called star-shaped if $f(tx) \leq tf(x)$ holds for all $t \in [0, 1]$ and $x \in [0, b]$.

For some recent citations and utilizations of (α, m)-convex functions one can see [2, 10, 19, 20] and the references therein.

Next we give the definition of the well-known Riemann–Liouville fractional integrals.

Definition 2 Let $f \in L_1[a, b]$. Then the Riemann–Liouville fractional integrals of order $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ are defined by

$$ {}^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\,dt, \quad x > a \tag{1.3}$$

and

$$ {}^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)\,dt, \quad x < b. \tag{1.4}$$

The left-sided and right-sided Riemann–Liouville k-fractional integrals are given in [17].

Definition 3 Let $f \in L_1[a, b]$. Then the Riemann–Liouville k-fractional integrals of order $\mu \in \mathbb{C}$, $\Re(\mu) > 0$, $k > 0$, are defined by

$$ {}^{\mu}I^k_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)\,dt, \quad x > a, \tag{1.5}$$

and

$$ {}^{\mu}I^k_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)\,dt, \quad x < b. \tag{1.6}$$

A more general definition of the Riemann–Liouville fractional integrals is given in [14].

Definition 4 Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ are defined by

$$ {}^{\mu}_{g}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \big(g(x)-g(t)\big)^{\mu-1} g'(t) f(t)\,dt, \quad x > a, \tag{1.7}$$

and

$$ {}^{\mu}_{g}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \big(g(t)-g(x)\big)^{\mu-1} g'(t) f(t)\,dt, \quad x < b, \tag{1.8}$$

where $\Gamma(\cdot)$ is the Gamma function.

A k-fractional analogue of the above definition is given in [15] (see also [1]).

Definition 5 Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\mu \in \mathbb{C}$, $\Re(\mu) > 0$, $k > 0$ are defined by

$$ {}^{\mu}_{g}I^{k}_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x \big(g(x) - g(t)\big)^{\frac{\mu}{k}-1} g'(t)f(t)\,dt, \quad x > a, \tag{1.9}$$

and

$$ {}^{\mu}_{g}I^{k}_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b \big(g(t) - g(x)\big)^{\frac{\mu}{k}-1} g'(t)f(t)\,dt, \quad x < b, \tag{1.10}$$

where $\Gamma_k(\cdot)$ is the k-Gamma function.

These are compact forms of fractional integrals; which contain several fractional integrals defined by different authors in recent years.

Remark 2 In the above Definition 5:

(i) By setting $k = 1$, Definition 4 can be obtained.
(ii) By setting $g(x) = x$, Definition 3 can be obtained.
(iii) By setting $g(x) = x$ and $k = 1$, Definition 2 can be obtained.
(iv) By setting $g(x) = \frac{x^\rho}{\rho}$, $\rho > 0$ and $k = 1$, the definition of the Katugampola fractional integrals given in [3] can be obtained.
(v) By setting $g(x) = \frac{(x)^{\tau+s}}{\tau+s}$ and $k = 1$, the definition of the generalized conformable fractional integrals defined by Khan et al. in [13], can be obtained.
(vi) By setting $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10), the definition of the conformable (k,s)-fractional integrals defined by Sidra et al. in [9] can be obtained.
(vii) By setting $g(x) = \frac{x^{1+s}}{1+s}$, the definition of the conformable fractional integrals defined by Sarikaya et al. in [18], can be obtained.
(viii) By setting $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10) with $k = 1$, the definition of the conformable fractional integrals defined by Jarad et al. in [11], can be obtained.

Recently many authors have been working on the applications of fractional integral operators for inequalities point of view see the references [1, 3, 5–8, 12, 15, 21].

The aim of this paper is the study of fractional integral operators which have been analyzed by the researchers of this age, in the form of a compact fractional integral operator. This general and compact form of fractional integrals has been investigated via (α,m)-convex function. In Sect. 2, the bounds of the sum of the left-sided and right-sided generalized k-fractional integrals defined in (1.9) and (1.10) are established by using (α,m)-convex functions. These results provide all analogue results for fractional integrals which have been deduced in Remark 2. Furthermore, some of the results published in [4, 5, 7] have been obtained in particular. In Sect. 3, results of Sect. 2 are applied at end as well as mid points of the interval $[a,b]$ and in consequent some interesting implications have been obtained.

2 Main results

The first result of this section is stated in the following theorem.

Theorem 1 *Let $f,g:[a,b]\subset[0,\infty)\longrightarrow\mathbb{R}$, $a<b$, be two functions such that g be differentiable and $f\in L[a,b]$. Also let f be (α,m)-convex and g be strictly increasing function on $[a,b]$ with $g'\in L[a,b]$. Then, for $(\alpha,m)\in[0,1]\times(0,1]$, the following estimation is valid*:

$$\begin{aligned} &k\Gamma_k(\mu)^{\mu}_{g}I^{k}_{a^+}f(x)+k\Gamma_k(\nu)^{\nu}_{g}I^{k}_{b^-}f(x)\\ &\quad\le\frac{(g(x)-g(a))^{\frac{\mu}{k}-1}}{(x-a)^{\alpha}}\biggl[(x-a)^{\alpha}\biggl(mf\biggl(\frac{x}{m}\biggr)g(x)-f(a)g(a)\biggr)\\ &\qquad-\Gamma(\alpha+1)\biggl(mf\biggl(\frac{x}{m}\biggr)-f(a)\biggr)^{\alpha}I_{a^+}g(x)\biggr]\\ &\qquad+\frac{(g(b)-g(x))^{\frac{\nu}{k}-1}}{(b-x)^{\alpha}}\biggl[(b-x)^{\alpha}\biggl(f(b)g(b)-mf\biggl(\frac{x}{m}\biggr)g(x)\biggr)\\ &\qquad-\Gamma(\alpha+1)\biggl(f(b)-mf\biggl(\frac{x}{m}\biggr)\biggr)^{\alpha}I_{b^-}g(x)\biggr] \end{aligned}\tag{2.1}$$

for all $x\in[a,b]$ and $\mu,\nu\ge k$.

Proof Under given assumptions for the function g: for $x\in[a,b]$; and $t\in[a,x]$, $\mu\ge k$, the following inequality holds true:

$$g'(t)\bigl(g(x)-g(t)\bigr)^{\frac{\mu}{k}-1}\le g'(t)\bigl(g(x)-g(a)\bigr)^{\frac{\mu}{k}-1}.\tag{2.2}$$

Since the identity

$$t=\frac{x-t}{x-a}a+m\frac{t-a}{x-a}\frac{x}{m}$$

exists, by using (α,m)-convexity of f, we have the following inequality:

$$f(t)\le\biggl(\frac{x-t}{x-a}\biggr)^{\alpha}f(a)+m\biggl(1-\biggl(\frac{x-t}{x-a}\biggr)^{\alpha}\biggr)f\biggl(\frac{x}{m}\biggr).\tag{2.3}$$

From the inequalities in (2.2) and (2.3), one can get the following inequality:

$$\begin{aligned} &\int_a^x\bigl(g(x)-g(t)\bigr)^{\frac{\mu}{k}-1}f(t)g'(t)\,dt\\ &\quad\le\bigl(g(x)-g(a)\bigr)^{\frac{\mu}{k}-1}\biggl[f(a)\int_a^x\biggl(\frac{x-t}{x-a}\biggr)^{\alpha}g'(t)\,dt\\ &\qquad+mf\biggl(\frac{x}{m}\biggr)\int_a^x\biggl(1-\biggl(\frac{x-t}{x-a}\biggr)^{\alpha}\biggr)g'(t)\,dt\biggr]. \end{aligned}\tag{2.4}$$

By using (1.9) of Definition 5, and (1.3) of Definition 2, we get the estimation of the left sided fractional integral operator defined in (1.9),

$$\begin{aligned} &k\Gamma_k(\mu)_g^{\mu}I_{a^+}^k f(x) \\ &\quad\le \frac{(g(x)-g(a))^{\frac{\mu}{k}-1}}{(x-a)^{\alpha}}\left[(x-a)^{\alpha}\left(mf\left(\frac{x}{m}\right)g(x)-f(a)g(a)\right)\right. \\ &\quad\quad\left.-\Gamma(\alpha+1)\left(mf\left(\frac{x}{m}\right)-f(a)\right)^{\alpha}I_{a^+}g(x)\right]. \end{aligned} \tag{2.5}$$

Now for $x\in[a,b]$, $t\in[x,b]$ and $\nu\ge 1$, the following inequality holds true:

$$g'(t)\big(g(t)-g(x)\big)^{\frac{\nu}{k}-1}\le g'(t)\big(g(b)-g(x)\big)^{\frac{\nu}{k}-1}. \tag{2.6}$$

By using (α,m)-convexity of f, we have

$$f(t)\le\left(\frac{t-x}{b-x}\right)^{\alpha}f(b)+m\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right)f\left(\frac{x}{m}\right). \tag{2.7}$$

By adopting the same way as we have done for (2.2) and (2.3), one can get from (2.6) and (2.7), the estimation of the right-sided fractional integral operator defined in (1.10),

$$\begin{aligned} &k\Gamma_k(\nu)_g^{\nu}I_{b^-}^k f(x) \\ &\quad\le \frac{((g(b)-g(x))^{\frac{\nu}{k}-1}}{(b-x)^{\alpha}}\left[(b-x)^{\alpha}\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)\right)\right. \\ &\quad\quad\left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{x}{m}\right)\right)^{\alpha}I_{b^-}g(x)\right]. \end{aligned} \tag{2.8}$$

Adding (2.5) and (2.8), the estimation of sum of left-sided and right-sided fractional integrals formulated in (2.1) can be obtained. □

Some implications of the above result are elaborated in the following.

Corollary 1 *By setting $\mu=\nu$ in* (2.1), *the following fractional integral inequality can be obtained*:

$$\begin{aligned} &{}_g^{\mu}I_{a^+}^k f(x)+{}_g^{\mu}I_{b^-}^k f(x) \\ &\quad\le \frac{(g(x)-g(a))^{\frac{\mu}{k}-1}}{k\Gamma_k(\mu)(x-a)^{\alpha}}\left[(x-a)^{\alpha}\left(mf\left(\frac{x}{m}\right)g(x)-f(a)g(a)\right)\right. \\ &\quad\quad\left.-\Gamma(\alpha+1)\left(mf\left(\frac{x}{m}\right)-f(a)\right)^{\alpha}I_{a^+}g(x)\right] \\ &\quad\quad+\frac{((g(b)-g(x))^{\frac{\mu}{k}-1}}{k\Gamma_k(\mu)(b-x)^{\alpha}}\left[(b-x)^{\alpha}\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)\right)\right. \\ &\quad\quad\left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{x}{m}\right)\right)^{\alpha}I_{b^-}g(x)\right]. \end{aligned} \tag{2.9}$$

Remark 3
(i) If we put $(\alpha, m) = (1, 1)$ and $k = 1$ in (2.1), then [7, Theorem 1] can be obtained.
(ii) If we put $g(x) = x$, $(\alpha, m) = (1, 1)$ and $k = 1$ in (2.1), then [5, Theorem 1] can be obtained.

The next result is the modulus fractional integral inequality which is a generalization of some known results.

Theorem 2 *Let $f, g : [a, b] \subset [0, \infty) \longrightarrow \mathbb{R}$, $a < b$, be differentiable functions. Also let $|f'|$ be (α, m)-convex, and g be strictly increasing on $[a, b]$ with $g' \in L[a, b]$. Then, for $(\alpha, m) \in [0, 1] \times (0, 1]$, the following modulus inequality is valid*:

$$\begin{aligned} &\Big|\Gamma_k(\mu+k)_g^{\mu}I_{a^+}^k f(x) + \Gamma_k(\nu+k)_g^{\nu}I_{b^-}^k f(x) \\ &\quad - \big(\big(g(x)-g(a)\big)^{\frac{\mu}{k}} f(a) + \big(g(b)-g(x)\big)^{\frac{\nu}{k}} f(b)\big)\Big| \\ &\quad \le \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right| + \left|f'(a)\right|\right] \\ &\quad + \frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right| + \left|f'(b)\right|\right] \end{aligned} \tag{2.10}$$

for all $x \in [a, b]$ and $\mu, \nu, k > 0$.

Proof By using (α, m)-convexity of $|f'|$, we have

$$\left|f'(t)\right| \le \left(\frac{x-t}{x-a}\right)^{\alpha}\left|f'(a)\right| + m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.11}$$

From (2.11), we have

$$f'(t) \le \left(\frac{x-t}{x-a}\right)^{\alpha}\left|f'(a)\right| + m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.12}$$

Under given assumptions for the function g, the following inequality holds true:

$$\big(g(x)-g(t)\big)^{\frac{\mu}{k}} \le \big(g(x)-g(a)\big)^{\frac{\mu}{k}} \tag{2.13}$$

for $x \in [a, b]$; $t \in [a, x]$ and $\mu, k > 0$.

From inequalities in (2.12) and (2.13), one can has the following integral inequality:

$$\begin{aligned} &\int_a^x \big(g(x)-g(t)\big)^{\frac{\mu}{k}} f'(t)\,dt \\ &\quad \le \big(g(x)-g(a)\big)^{\frac{\mu}{k}}\left[\left|f'(a)\right|\int_a^x\left(\frac{x-t}{x-a}\right)^{\alpha}dt \right. \\ &\quad \left. + m\left|f'\left(\frac{x}{m}\right)\right|\int_a^x\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)dt\right] \\ &\quad = \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right| + \left|f'(a)\right|\right]. \end{aligned} \tag{2.14}$$

The left hand side of inequality (2.14) is evaluated as follows:

$$\begin{aligned}&\int_a^x \big(g(x)-g(t)\big)^{\frac{\mu}{k}} f'(t)\,dt\\&\quad= f(t)\big(g(x)-g(t)\big)^{\frac{\mu}{k}}\big|_a^x + \frac{\mu}{k}\int_a^x \big(g(x)-g(t)\big)^{\frac{\mu}{k}-1} f(t)g'(t)\,dt\\&\quad= -f(a)\big(g(x)-g(a)\big)^{\frac{\mu}{k}} + \Gamma_k(\mu+k)\,{}_g^{\mu}I_{a^+}^{k} f(x).\end{aligned}$$

Therefore inequality (2.14) takes the following form:

$$\begin{aligned}&\Gamma_k(\mu+k)\,{}_g^{\mu}I_{a^+}^{k} f(x) - f(a)\big(g(x)-g(a)\big)^{\frac{\mu}{k}}\\&\quad\le \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right| + \big|f'(a)\big|\right].\end{aligned} \tag{2.15}$$

Also from (2.11), one can has

$$f'(t) \ge -\left(\left(\frac{x-t}{x-a}\right)^{\alpha}\big|f'(a)\big| + m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|\right). \tag{2.16}$$

Adopting the same procedure as we did for (2.12), the following inequality holds:

$$\begin{aligned}&f(a)\big(g(x)-g(a)\big)^{\frac{\mu}{k}} - \Gamma_k(\mu+k)\,{}_g^{\mu}I_{a^+}^{k} f(x)\\&\quad\le \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right| + \big|f'(a)\big|\right].\end{aligned} \tag{2.17}$$

From (2.15) and (2.17), the following modulus inequality can be obtained:

$$\begin{aligned}&\left|\Gamma_k(\mu+k)\,{}_g^{\mu}I_{a^+}^{k} f(x) - f(a)\big(g(x)-g(a)\big)^{\frac{\mu}{k}}\right|\\&\quad\le \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right| + \big|f'(a)\big|\right].\end{aligned} \tag{2.18}$$

By using (α, m)-convexity of $|f'|$, we also have

$$\big|f'(t)\big| \le \left(\frac{t-x}{b-x}\right)^{\alpha}\big|f'(b)\big| + m\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.19}$$

Now for $x \in [a,b]$; $t \in [x,b]$ and $\nu, k > 0$, the following inequality holds true:

$$\big(g(t)-g(x)\big)^{\frac{\nu}{k}} \le \big(g(b)-g(x)\big)^{\frac{\nu}{k}}. \tag{2.20}$$

By adopting the same way as we have done for (2.12), (2.13) and (2.16) one can get from (2.19) and (2.20) the following inequality:

$$\begin{aligned}&\left|\Gamma_k(\nu+k)\,{}_g^{\nu}I_{b^-}^{k} f(x) - f(b)\big(g(b)-g(x)\big)^{\frac{\nu}{k}}\right|\\&\quad\le \frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right| + \big|f'(b)\big|\right].\end{aligned} \tag{2.21}$$

From inequalities (2.18) and (2.21) via triangular inequality, the inequality in (2.10) can be obtained. □

Some implications of the above theorem are discussed in the following.

Corollary 2 *By setting $\mu = \nu$ in (2.10), the following fractional integral inequality can be obtained*:

$$\begin{aligned}
&\Big|\Gamma_k(\mu+k)\big({}^{\mu}_{g}I^{k}_{a^+}f(x)+{}^{\mu}_{g}I^{k}_{b^-}f(x)\big)\\
&\quad-\big(\big(g(x)-g(a)\big)^{\frac{\mu}{k}}f(a)+\big(g(b)-g(x)\big)^{\frac{\mu}{k}}f(b)\big)\Big|\\
&\quad\le\frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right|+\left|f'(a)\right|\right]\\
&\quad\quad+\frac{(g(b)-g(x))^{\frac{\mu}{k}}(b-x)}{\alpha+1}\left[\alpha m\left|f'\left(\frac{x}{m}\right)\right|+\left|f'(b)\right|\right].
\end{aligned}$$

Remark 4

(i) If we put $(\alpha,m)=(1,1)$ and $k=1$ in (2.10), then [7, Theorem 2] can be obtained.

(ii) If we put $g(x)=x$, $(\alpha,m)=(1,1)$ and $k=1$ in (2.10), then [5, Theorem 2] can be obtained.

We use the following lemma in the proof of the next result.

Lemma 1 *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be an (α,m)-convex function. If $f(x)=f(\frac{a+b-x}{m})$, then the following inequality is valid*:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2^\alpha}\big(1+m\big(2^\alpha-1\big)\big)f(x) \tag{2.22}$$

for all $x\in[a,b]$ and $m\in(0,1]$.

Proof We have

$$\frac{a+b}{2}=\frac{1}{2}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)+m\frac{1}{2}\left(\frac{\frac{x-a}{b-a}a+\frac{b-x}{b-a}b}{m}\right). \tag{2.23}$$

It is given that f is (α,m)-convex, therefore we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)&\le\frac{1}{2^\alpha}f\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)+m\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{\frac{x-a}{b-a}a+\frac{b-x}{b-a}b}{m}\right)\\
&=\frac{1}{2^\alpha}f(x)+m\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{a+b-x}{m}\right).
\end{aligned} \tag{2.24}$$

Using the given condition $f(x)=f(\frac{a+b-x}{m})$ in the above inequality, the inequality in (2.22) can be obtained. □

Theorem 3 *Let $f,g:[a,b]\subset[0,\infty)\longrightarrow\mathbb{R}$, $a<b$, be two functions such that g be differentiable and $f\in L[a,b]$. Also let f be (α,m)-convex, $f(x)=f(\frac{a+b-x}{m})$ and g be strictly increasing on $[a,b]$ with $g'\in L[a,b]$. Then, for $(\alpha,m)\in[0,1]\times(0,1]$, the following estimation of*

Hadamard type is valid:

$$\begin{aligned}
&\frac{2^\alpha k}{(1+m(2^\alpha-1))} f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\mu+k}+\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\nu+k}\right] \\
&\quad \leq k\left(\Gamma_k(\mu+k){}_g^{\mu+k}I_{a^+}^k f(b)+\Gamma_k(\nu+k){}_g^{\nu+k}I_{b^-}^k f(a)\right) \\
&\quad \leq \frac{(g(b)-g(a))^{\frac{\mu}{k}}+(g(b)-g(a))^{\frac{\nu}{k}}}{(b-a)^\alpha}\left[(b-a)^\alpha\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right. \\
&\qquad \left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{a}{m}\right)\right)^\alpha I_{b^-}g(a)\right] \qquad (2.25)
\end{aligned}$$

for all $x \in [a,b]$ *and* $\mu,\nu,k>0$.

Proof Under given assumptions for the function g, the following inequality holds true:

$$g'(x)\left(g(x)-g(a)\right)^{\frac{\nu}{k}} \leq g'(x)\left(g(b)-g(a)\right)^{\frac{\nu}{k}}, \qquad (2.26)$$

for $x \in [a,b]$ and $\nu,k>0$. Since the identity

$$x=\frac{x-a}{b-a}b+m\frac{b-x}{b-a}\frac{a}{m}$$

exists therefore, by using (α,m)-convexity of f, we have

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^\alpha f(b)+m\left(1-\left(\frac{x-a}{b-a}\right)^\alpha\right)f\left(\frac{a}{m}\right). \qquad (2.27)$$

From inequalities in (2.26) and (2.27), one has the following inequality:

$$\begin{aligned}
&\int_a^b \left(g(x)-g(a)\right)^{\frac{\nu}{k}} f(x)g'(x)\,dx \\
&\quad \leq \left(g(b)-g(a)\right)^{\frac{\nu}{k}}\left[f(b)\int_a^b\left(\frac{x-a}{b-a}\right)^\alpha g'(x)\,dx\right. \\
&\qquad \left.+mf\left(\frac{a}{m}\right)\int_a^b\left(1-\left(\frac{x-a}{b-a}\right)^\alpha\right)g'(x)\,dx\right].
\end{aligned}$$

By using (1.10) of Definition 5, and (1.4) of Definition 2, we get

$$\begin{aligned}
&k\Gamma_k(\nu+k){}_g^{\nu+k}I_{b^-}^k f(a) \\
&\quad \leq \frac{(g(b)-g(a))^{\frac{\nu}{k}}}{(b-a)^\alpha}\left[(b-a)^\alpha\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right. \\
&\qquad \left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{a}{m}\right)\right)^\alpha I_{b^-}g(a)\right]. \qquad (2.28)
\end{aligned}$$

Now for $x \in [a,b]$; $t \in [x,b]$ and $\mu,k>0$, the following inequality holds true:

$$g'(x)\left(g(b)-g(x)\right)^{\frac{\mu}{k}} \leq g'(x)\left(g(b)-g(a)\right)^{\frac{\mu}{k}}. \qquad (2.29)$$

By adopting the same way as we have done for (2.26) and (2.27), one can get from (2.27) and (2.29), the following inequality:

$$\begin{aligned} &k\Gamma_k(\mu+k)_g^{\mu+k}I_{a^+}^k f(b) \\ &\quad\leq \frac{(g(b)-g(a))^{\frac{\mu}{k}}}{(b-a)^\alpha}\left[(b-a)^\alpha\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right. \\ &\quad\left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{a}{m}\right)\right)^\alpha I_{b^-}g(a)\right]. \end{aligned} \tag{2.30}$$

Adding inequalities in (2.28) and (2.30), we get the following inequality:

$$\begin{aligned} &k\left(\Gamma_k(\mu+k)_g^{\mu+k}I_{a^+}^k f(b)+\Gamma_k(\nu+k)_g^{\nu+k}I_{b^-}^k f(a)\right) \\ &\quad\leq \frac{(g(b)-g(a))^{\frac{\mu}{k}}+(g(b)-g(a))^{\frac{\nu}{k}}}{(b-a)^\alpha}\left[(b-a)^\alpha\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right. \\ &\quad\left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{a}{m}\right)\right)^\alpha I_{b^-}g(a)\right]. \end{aligned} \tag{2.31}$$

On the other hand multiplying (2.22) with $(g(x)-g(a))^{\frac{\nu}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\int_a^b (g(x)-g(a))^{\frac{\nu}{k}}g'(x)\,dx \\ &\quad\leq \frac{1}{2^\alpha}\left(1+m\left(2^\alpha-1\right)\right)\int_a^b (g(x)-g(a))^{\frac{\nu}{k}}g'(x)f(x)\,dx. \end{aligned} \tag{2.32}$$

By using (1.10) of Definition 5, we get

$$\frac{2^\alpha k(g(b)-g(a))^{\frac{\nu}{k}+1}}{(\nu+k)(1+m(2^\alpha-1))}f\left(\frac{a+b}{2}\right)\leq k\Gamma_k(\nu+k)_g^{\nu+k}I_{b^-}^k f(a). \tag{2.33}$$

Similarly, multiplying (2.22) with $(g(b)-g(x))^{\frac{\mu}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$\frac{2^\alpha k(g(b)-g(a))^{\frac{\mu}{k}+1}}{(\mu+k)(1+m(2^\alpha-1))}f\left(\frac{a+b}{2}\right)\leq k\Gamma_k(\mu+k)_g^{\mu+k}I_{a^+}^k f(b). \tag{2.34}$$

From inequalities in (2.33) and (2.34), we get the following inequality:

$$\begin{aligned} &\frac{2^\alpha k}{(1+m(2^\alpha-1))}f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\mu+k}+\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\nu+k}\right] \\ &\quad\leq k\left(\Gamma_k(\mu+k)_g^{\mu+k}I_{a^+}^k f(b)+\Gamma_k(\nu+k)_g^{\nu+k}I_{b^-}^k f(a)\right) \end{aligned} \tag{2.35}$$

Combining inequalities (2.31) and (2.35), we get inequalities in (2.25). □

Corollary 3 *By setting $\mu = \nu$ in (2.25), the following fractional integral inequality can be obtained*:

$$\frac{2^{\alpha+1}k(g(b)-g(a))^{\frac{\mu}{k}+1}}{(1+m(2^{\alpha}-1))(\mu+k)}f\left(\frac{a+b}{2}\right)$$
$$\leq k\Gamma_k(\mu+k)\left({}^{\mu+k}_{g}I^{k}_{a^+}f(b)+{}^{\mu+k}_{g}I^{k}_{b^-}f(a)\right)$$
$$\leq \frac{2(g(b)-g(a))^{\frac{\mu}{k}}}{(b-a)^{\alpha}}\left[(b-a)^{\alpha}\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right.$$
$$\left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{a}{m}\right)\right)^{\alpha}I_{b^-}g(a)\right].$$

Remark 5

(i) By setting $(\alpha, m) = (1,1)$ and $k = 1$, in (2.25), [7, Theorem 3] can be obtained.

(ii) By setting $g(x) = x$, $(\alpha, m) = (1,1)$ and $k = 1$, in (2.1), [5, Theorem 3] can be obtained.

3 Applications

In this section we give applications of the results proved in the previous section. First we apply Theorem 1 and get the following result.

Theorem 4 *Under the assumptions of Theorem 1, we have*

$$k\left(\Gamma_k(\mu){}^{\mu}_{g}I^{k}_{a^+}f(b)+\Gamma_k(\nu){}^{\nu}_{g}I^{k}_{b^-}f(a)\right)$$
$$\leq \frac{((g(b)-g(a))^{\frac{\mu}{k}-1}}{(b-a)^{\alpha}}\left[(b-a)^{\alpha}\left(mf\left(\frac{b}{m}\right)g(b)-f(a)g(a)\right)\right.$$
$$\left.-\Gamma(\alpha+1)\left(mf\left(\frac{b}{m}\right)-f(a)\right)^{\alpha}I_{a^+}g(b)\right]$$
$$+\frac{((g(b)-g(a))^{\frac{\nu}{k}-1}}{(b-a)^{\alpha}}\left[(b-a)^{\alpha}\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right.$$
$$\left.-\Gamma(\alpha+1)\left(f(b)-mf\left(\frac{a}{m}\right)\right)^{\alpha}I_{b^-}g(a)\right]. \tag{3.1}$$

Proof Setting $x = a$ and $x = b$ in (2.1), respectively, then adding the resulting inequalities, (3.1) can be obtained. □

Corollary 4 *By setting $\mu = \nu$ in (3.1), the following fractional integral inequality can be obtained*:

$$k\left(\Gamma_k(\mu)\left({}^{\mu}_{g}I^{k}_{a^+}f(b)+{}^{\mu}_{g}I^{k}_{b^-}f(a)\right)\right)$$
$$\leq \frac{((g(b)-g(a))^{\frac{\mu}{k}-1}}{(b-a)^{\alpha}}\left[(b-a)^{\alpha}\left(\left(mf\left(\frac{b}{m}\right)-f(b)\right)g(b)\right.\right.$$
$$\left.-\left(f(a)+mf\left(\frac{a}{m}\right)\right)g(a)\right)-\Gamma(\alpha+1)\left(\left(mf\left(\frac{b}{m}\right)-f(a)\right)^{\alpha}I_{a^+}g(b)\right.$$
$$\left.\left.+\left(f(b)-mf\left(\frac{b}{m}\right)\right)^{\alpha}I_{b^-}g(a)\right)\right]. \tag{3.2}$$

Corollary 5 *By setting $\mu = 1$, $(\alpha, m) = (1,1)$ and $g(x) = x$ in* (3.2), *the following integral inequality can be obtained*:

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{3.3}$$

Remark 6 By setting $(\alpha, m) = (1,1)$ and $k = 1$ in (3.1), [7, Theorem 4] can be obtained.

Next we apply Theorem 2 and obtain the following modulus inequalities.

Theorem 5 *Under the assumptions of Theorem* 2, *we have*

$$\begin{aligned}
&\Bigg|\Gamma_k(\mu+k)\,{}_g^{\mu}I_{a^+}^{k}f\left(\frac{a+b}{2}\right) + \Gamma_k(\nu+k)\,{}_g^{\nu}I_{b^-}^{k}f\left(\frac{a+b}{2}\right) \\
&\quad - \left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\mu}{k}} f(a) + \left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\nu}{k}} f(b)\right)\Bigg| \\
&\quad \le \frac{b-a}{2(\alpha+1)}\left[\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\mu}{k}}\left(\alpha m\left|f'\left(\frac{a+b}{2m}\right)\right| + \left|f'(a)\right|\right)\right. \\
&\quad \left. + \left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\nu}{k}}\left(\alpha m\left|f'\left(\frac{a+b}{2m}\right)\right| + \left|f'(b)\right|\right)\right].
\end{aligned} \tag{3.4}$$

Proof If we put $x = \frac{a+b}{2}$ in (2.10), then the resulting inequality (3.4) can be obtained. □

Corollary 6 *By setting $\mu = \nu$ in* (3.4), *the following fractional integral inequality can be obtained*:

$$\begin{aligned}
&\Bigg|\Gamma_k(\mu+k)\left({}_g^{\mu}I_{a^+}^{k}f\left(\frac{a+b}{2}\right) + {}_g^{\mu}I_{b^-}^{k}f\left(\frac{a+b}{2}\right)\right) \\
&\quad - \left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\mu}{k}} f(a) + \left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\mu}{k}} f(b)\right)\Bigg| \\
&\quad \le \frac{b-a}{2(\alpha+1)}\left[\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\mu}{k}}\left(\alpha m\left|f'\left(\frac{a+b}{2m}\right)\right| + \left|f'(a)\right|\right)\right. \\
&\quad \left. + \left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\mu}{k}}\left(\alpha m\left|f'\left(\frac{a+b}{2m}\right)\right| + \left|f'(b)\right|\right)\right].
\end{aligned} \tag{3.5}$$

Corollary 7 *By setting $\mu = 1$, $(\alpha, m) = (1,1)$ and $g(x) = x$ in* (3.5), *the following integral inequality can be obtained*:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right| + 2\left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{3.6}$$

Remark 7

(i) By setting $(\alpha, m) = (1,1)$ and $k = 1$ in (3.4), [7, Theorem 5] can be obtained.

(ii) If f' passes through $x = \frac{a+b}{2}$, then from (3.6) we get [4, Theorem 2.2]. If $f'(x) \le 0$, then (3.6) gives the refinement of [4, Theorem 2.2].

Remark 8 By applying Theorem 3 similar relations can be established; we leave these for the reader.

3.1 Concluding remarks

This research investigates fractional integral inequalities, which provide the bounds of a compact form of classical fractional integral. A modulus fractional integral inequality for this compact fractional integral operator is studied. Moreover, bounds of Hadamard type of this classical fractional integral operator are obtained. It is interesting that all results of this paper are obtainable at once in particular for present-day fractional integral operators which are comprised in Remark 2. In the future the authors are interested in further estimations of generalized integral operators for other related classes of functions.

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