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Approximate coincidence point and common fixed point results for a hybrid pair of mappings with constraints in partially ordered Menger PM-spaces

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Abstract

We study an approximate coincidence point and a common fixed point problem for a hybrid pair of mappings with constraints in Menger PM-spaces, and obtain some new results. We derive interesting consequences of the main results by using the properties of a Menger–Hausdorff metric, and analogous results based on graphs instead of partial orders can be similarly formulated. Moreover, we construct two examples to reveal that the main results are valid, and show that the main results can be used to explore the existence of solutions to a system of nonlinear integral equations.

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1 Introduction

A statistical metric was defined by Menger and later revisited by other authors, which led further to the emergence of the definition of a probabilistic metric space [1]. Many efforts have been devoted to the study of fixed point and optimization problems in PM-spaces since the formulation of PM-space theory [2–8]. Fixed point and related problems in the framework of different types of spaces equipped with a partial order have also been explored [9–17].

Let $(X, d, \preceq_1, \preceq_2)$ be a partially ordered metric space, where $\preceq_1$ and $\preceq_2$ are two partial orders, and $S, O, P, Q, R : X \to X$ be self-mappings. In [18], the authors raised the problem of seeking $x \in X$ satisfying

$$\begin{cases} x = Sx, \\ Ox \preceq_1 Px, \\ Qx \preceq_2 Rx. \end{cases} \tag{1.1}$$

In [18], the authors gave sufficient conditions for the existence of solutions to problem (1.1), in which the continuity of O, P, Q, and R is required. This requirement is weakened to the continuity of O and P (or Q and R) in [19], and the result in [18] is also extended under general contractive conditions by making use of a general class of functions. Recently, the main results of [18] and [19] were generalized to the framework of a Menger PM-space [20].

Let $(X, d, \preceq_1, \preceq_2)$ be a partially ordered metric space, where $\preceq_1$ and $\preceq_2$ are two partial orders, $S : X \to \mathcal{N}(X)$ be a set-valued mapping, where $\mathcal{N}(X)$ denotes the collection of all nonempty subsets of X, and $O, P, Q, R : X \to X$ be self-mappings. The authors in [21] investigated the following approximate fixed point problem for a set-valued mapping with constraints: seeking $x \in X$ satisfying

$$\begin{cases} x \in \overline{Sx}, \\ Ox \preceq_1 Px, \\ Qx \preceq_2 Rx. \end{cases} \tag{1.2}$$

The authors in [21] studied problem (1.2) and obtained an interesting result.

Let $(X, \mathfrak{F}, \Delta, \preceq_1, \preceq_2)$ be a partially ordered Menger PM space, where $\preceq_1$ and $\preceq_2$ are partial orders, $S : X \to \mathcal{N}(X)$ be a set-valued mapping, and $f, O, P, Q, R : X \to X$ be self-mappings. Now, consider the following two problems. The first one is to seek $x \in X$ satisfying

$$\begin{cases} fx \in \overline{Sx}, \\ Ofx \preceq_1 Pfx, \\ Qfx \preceq_2 Rfx. \end{cases} \tag{1.3}$$

The second one is to seek $x \in X$ satisfying

$$\begin{cases} x = fx \in Sx, \\ Ox \preceq_1 Px, \\ Qx \preceq_2 Rx. \end{cases} \tag{1.4}$$

In this paper, we investigate the above-mentioned two problems, which are in fact an approximate coincidence point problem and a common fixed point problem for a hybrid pair of mappings (i.e., a single-valued one and a set-valued one) with constraints in the framework of a partially ordered Menger PM-space, respectively. The rest of the paper is arranged as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we provide an approximate coincidence point theorem and a common fixed point theorem for a hybrid pair of mappings in partially ordered Menger PM-spaces and give two examples. In Sect. 4, we derive some consequent results of the theorems proved in Sect. 3. An application of the main results in discussing the solutions to a system of Volterra integral equations is presented in Sect. 5. Finally, we summarize the paper with some concluding remarks.

2 Preliminaries

A *distribution function* is a mapping $F : \mathbb{R} \to \mathbb{R}^+$ satisfying nondecreasingness, left-continuity, $\sup_{u\in\mathbb{R}} F(u) = 1$, and $\inf_{u\in\mathbb{R}} F(u) = 0$. We denote by $\mathfrak{D}$ the collection of all distribution functions and by γ the following special distribution function:

$$\gamma(u) = \begin{cases} 0, & u \leq 0, \\ 1, & u > 0. \end{cases}$$

Definition 2.1 ([3]) A *triangular norm* (a *t-norm*) is a mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ satisfying $\Delta(x,1) = x$, $\Delta(x,y) = \Delta(y,x)$, $\Delta(x_1,x_2) \geq \Delta(y_2,y_2)$ for $x_1 \geq y_1$, $x_2 \geq y_2$ and $\Delta(x, \Delta(y,z)) = \Delta(\Delta(x,y),z)$.

Definition 2.2 ([3]) A *Menger probabilistic metric space* (a *Menger PM-space*) is a triplet $(X, \mathfrak{F}, \Delta)$, where X is a nonempty set, Δ is a t-norm, and $\mathfrak{F} : X \times X \to \mathfrak{D}$ is a mapping satisfying (we rewrite $\mathfrak{F}(a,b)$ as $F_{a,b}$)

(MPM-1) $F_{a,b}(w) = \gamma(w)$ for all $w \in \mathbb{R}$ if and only if $a = b$;
(MPM-2) $F_{a,b}(w) = F_{b,a}(w)$ for all $w \in \mathbb{R}$;
(MPM-3) $F_{a,b}(w + v) \geq \Delta(F_{a,c}(w), F_{c,b}(v))$ for all $a, b, c \in X$ and $w, v \geq 0$.

Note that if $\sup_{0<u<1} \Delta(u,u) = 1$, then $(X, \mathfrak{F}, \Delta)$ is a Hausdorff topological space in the (ε, λ)-topology τ, i.e., the family of sets $\{U_a(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0,1]\}$ $(a \in X)$ is a basis of neighborhoods of a point a in X for τ, where $U_a(\varepsilon, \lambda) = \{b \in X : F_{a,b}(\varepsilon) > 1 - \lambda\}$. The concepts of τ-convergence of a sequence, τ-Cauchy sequence in $(X, \mathfrak{F}, \Delta)$, and the τ-completeness of $(X, \mathfrak{F}, \Delta)$ can thus be introduced with respect to this topology. For more details, please refer to [3].

Let (X, d) be a metric space, $CL(X)$ be the collection of all nonempty closed subsets of X, and H be the *Hausdorff metric* which is defined by

$$H(\Theta, \Xi) = \max\Big\{\sup_{a\in\Theta} d(a, \Xi), \sup_{b\in\Xi} d(b, \Theta)\Big\},$$

for any $\Theta, \Xi \in CL(X)$, where $d(a, \Theta) = \inf_{b\in\Theta} d(a,b)$.

Let $(X, \mathfrak{F}, \Delta)$ be a Menger PM-space, $\mathcal{N}(X)$ be the collection of all nonempty subsets of X, and $\mathcal{CL}(X)$ be the collection of all nonempty τ-closed subsets of X. For any $\Theta, \Xi \in \mathcal{N}(X)$, define

$$\tilde{\mathfrak{F}}(\Theta, \Xi)(u) = \tilde{F}_{\Theta,\Xi}(u) = \sup_{v<u} \Delta\Big(\inf_{a\in\Theta} \sup_{b\in\Xi} F_{a,b}(v), \inf_{b\in\Xi} \sup_{a\in\Theta} F_{a,b}(v)\Big), \quad v, u \in \mathbb{R},$$

$$\mathfrak{F}(a, \Theta)(u) = F_{a,\Theta}(u) = \sup_{v<u} \sup_{b\in\Theta} F_{a,b}(v), \quad v, u \in \mathbb{R},$$

where $\tilde{\mathfrak{F}}$ is called the *Menger–Hausdorff metric*.

Remark 2.1 ([3])

(1) Let (X, d) be a metric space. Define

$$\mathfrak{F}(a,b)(u) = F_{a,b}(u) = \gamma\big(u - d(a,b)\big) \quad \text{for all } a, b \in X \text{ and } u \in \mathbb{R}. \tag{2.1}$$

Then $(X, \mathfrak{F}, \Delta_{\min})$ is a Menger PM-space, where $\Delta_{\min}$ is the t-norm defined by $\Delta_{\min}(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$. Furthermore, the completeness of (X, d) implies the τ-completeness of $(X, \mathfrak{F}, \Delta_{\min})$;

(2) Let $H(\cdot,\cdot)$ be the Hausdorff metric, and define

$$\begin{aligned}\tilde{\mathfrak{F}}(\Theta,\Xi)(u) &= \tilde{F}_{\Theta,\Xi}(u) \\ &= \gamma\bigl(u - H(\Theta,\Xi)\bigr) \quad \text{for all } \Theta,\Xi \in \mathcal{CL}(X) \text{ and } u \in \mathbb{R}. \end{aligned} \tag{2.2}$$

Then $\tilde{\mathfrak{F}}$ is the Menger–Hausdorff metric induced by $\mathfrak{F}$. Moreover, the τ-completeness of $(\mathcal{CL}(X),\tilde{\mathfrak{F}},\Delta)$ follows from the τ-completeness of $(X,\mathfrak{F},\Delta)$ provided that $\Delta \geq \Delta_m$, where $\Delta_m(x,y) = \max\{x+y-1,0\}$ for all $x,y \in [0,1]$.

Lemma 2.1 ([3]) *Let $(X,\mathfrak{F},\Delta)$ be a Menger PM-space. For any $\Theta,\Xi \in \mathcal{CL}(X)$ and any $a,b \in X$, the following statements hold*:

(i) *For any $a \in \Theta$, $F_{a,\Xi}(u) \geq \tilde{F}_{\Theta,\Xi}(u)$ for all $u \geq 0$;*

(ii) *$F_{a,\Theta}(w+v) \geq \Delta(F_{a,b}(w),F_{y,\Theta}(v))$ for all $w,v \geq 0$;*

(iii) *$F_{a,\Theta}(w+v) \geq \Delta(F_{a,\Xi}(w),\tilde{F}_{\Theta,\Xi}(v))$ for all $w,v \geq 0$.*

The concept of F-regularity was given in [20] as follows.

Definition 2.3 ([20]) Let $(X,\mathfrak{F},\Delta,\preceq)$ be a partially ordered Menger PM-space. We say that $\preceq$ is *F-regular*, if for sequences $\{c_n\},\{d_n\} \subset X$ and $(c,d) \in X \times X$, we have

$$\lim_{n\to\infty} F_{c_n,c}(u) = \lim_{n\to\infty} F_{d_n,d}(u) = 1 \quad \text{for all } u > 0 \text{ and } c_n \preceq d_n \text{ for all } n \in \mathbb{N} \Longrightarrow c \preceq d.$$

For a hybrid pair of mappings (f,S), where $f: X \to X$ is a self-mapping and $S: X \to \mathcal{N}(X)$ is a set-valued one, and $\xi \in (0,1)$, we introduce the following quantity:

$$J_\xi^x(f,S) := \left\{ fy \in Sx \,\middle|\, F_{fx,Sx}(u) \leq F_{fx,fy}\left(\frac{u}{\xi}\right) \text{ for all } u > 0 \right\}, \quad x \in X.$$

Note that $J_\xi^x(f,S) \neq \emptyset$ for arbitrary $x \in X$. Moreover, for self-mappings M,N and a partial order $\preceq$ on X, if $Mfy \preceq Nfy$ for some $fy \in J_\xi^x(f,S)$, then we write $MSx \preceq NSx$.

The following definition generalizes the corresponding one in [21] in two aspects. On the one hand, the interpretation of $QSx \preceq_2 RSx$ relies on the set $J_\xi^x(f,S)$ which involves the probabilistic metric. On the other hand, the quantity $J_\xi^x(f,S)$ is defined with respect to two mappings instead of one.

Definition 2.4 Let X be a nonempty set equipped with partial orders $\preceq_1$ and $\preceq_2$. Let $f,O,P,Q,R: X \to X$ be self-mappings, $S: X \to \mathcal{N}(X)$ be a set-valued one, and $\xi \in (0,1)$. We say that the hybrid pair of mappings (f,S) is ξ-$(O,P,Q,R,\preceq_1,\preceq_2)$-*stable* if

$$x \in X, \quad Ofx \preceq_1 Pfx \quad \Rightarrow \quad QSx \preceq_2 RSx.$$

For a self-mapping $f: X \to X$ and a set-valued mapping $S: X \to \mathcal{N}(X)$ defined on X, if $fw \in Sw$ for $w \in X$, then we say that w is a *coincidence point* of f and S, and if $fw \in \overline{Sw}$, then we say that w is an *approximate coincidence point* of f and S. Furthermore, if $w \in X$ satisfies $w = fw \in Sw$, then we say that w is a *common fixed point* of f and S. We denote by $C(f,S)$ the set of coincidence points of f and S and by $F(f,S)$ the set of common fixed points of the two mappings.

Definition 2.5 Let $(X, \mathfrak{F}, \Delta)$ be a Menger PM-space, $f : X \to X$ be a self-mapping, and $S : X \to \mathcal{N}(X)$ be a set-valued one. S is called *τ-closed* with respect to f if $G(S)_f$ is a τ-closed subset of $(X \times X, F^*)$, where

$$F^*_{(fa,fb),(\mu,\nu)}(u) = \Delta\left(F_{fa,u}\left(\frac{u}{2}\right), F_{fb,\nu}\left(\frac{u}{2}\right)\right),$$

for all $(a,b),(\mu,\nu) \in X \times X$ and $u > 0$, and $G(S)_f = \{(a,fb) : a \in X, fb \in Sa\}$.

For the sake of brevity, $\overline{S}$ shall denote the set-valued mapping satisfying $\overline{S}a = \overline{Sa}$ for all $a \in X$.

3 Main results

In this section, we shall present and prove the main results of this paper. We first list some assumptions.

(A_1) If there exists a sequence $\{a_n\} \subset X$, such that $\{fa_n\}$ is τ-convergent to fa for $a \in X$, $fa_n \in Sa_{n-1}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} F_{fa_n,Sa_n}(u) = 1$ for all $u > 0$, then $F_{fa,Sa}(u) = 1$ for all $u > 0$.

(A_2) The following implication holds for any $b \in X$:

$$fb \notin \overline{Sb} \quad \Rightarrow \quad \sup_{a\in X} \Delta\big(F_{fa,fb}(u), F_{fa,Sa}(u)\big) < 1 \quad \text{for all } u > 0.$$

(A_3) $\overline{S}$ is a τ-closed set-valued mapping with respect to f.

The following two theorems are the main results of this paper.

Theorem 3.1 *Let $(X, \mathfrak{F}, \Delta, \preceq_1, \preceq_2)$ be a τ-complete partially ordered Menger PM-space, where $\preceq_1$ and $\preceq_2$ are partial orders on X and Δ is a continuous t-norm, $f, O, P, Q, R : X \to X$ be self-mappings satisfying that $f(X)$ is τ-closed and O, P, Q, R are τ-continuous, and $S : X \to \mathcal{N}(X)$ be a set-valued mapping. Furthermore, suppose that one of assumptions (A_1)–(A_3) holds, and the following hypotheses hold for some $\xi \in (0,1)$:*

(i) *$\preceq_1$ and $\preceq_2$ are F-regular;*
(ii) *$Ofx_0 \preceq_1 Pfx_0$ for some $x_0 \in X$;*
(iii) *(f,S) is ξ-$(O,P,Q,R,\preceq_1,\preceq_2)$-stable;*
(iv) *(f,S) is ξ-$(Q,R,O,P,\preceq_2,\preceq_1)$-stable;*
(v) *$F_{fy,Sy}(u) \geq F_{fx,fy}(\frac{u}{\eta})$ for all $x \in X$ and $fy \in J^x_\xi(f,S)$ with ($Ofx \preceq_1 Pfx$ and $Qfy \preceq_2 Rfy$) or ($Ofy \preceq_1 Pfy$ and $Qfx \preceq_2 Rfx$) and $u > 0$, where $\eta \in (0,\xi)$.*

Then there exists at least one solution to problem (1.3).

Proof By condition (ii), $Ofx_0 \preceq_1 Pfx_0$ for some $x_0 \in X$. Using (iii) and Definition 2.4, we get $QTx_0 \preceq_2 RTx_0$, which implies that there exists $fx_1 \in J^{x_0}_\xi(f,S)$ such that $Qfx_1 \preceq_2 Rfx_1$. Utilizing (iv) and Definition 2.4, we get $OTx_1 \preceq_1 PTx_1$, which implies that there exists $fx_2 \in J^{x_1}_\xi(f,S)$ such that $Ofx_2 \preceq_2 Pfx_2$. Hence, we can inductively construct a sequence $\{x_n\} \subset X$ satisfying

$$Ofx_{2n} \preceq_1 Pfx_{2n}, \qquad Qfx_{2n+1} \preceq_2 Rfx_{2n+1}, \qquad fx_{n+1} \in J^{x_n}_\xi(f,S), n \in \mathbb{N} \cup \{0\}, \tag{3.1}$$

and

$$F_{fx_{n-1},fx_n}\left(\frac{u}{\eta}\right) \le F_{fx_n,Sx_n}(u) \quad \text{for all } u > 0 \text{ and } n \in \mathbb{N}. \tag{3.2}$$

Noting that $fx_{n+1} \in J_\xi^{x_n}(f,S)$, we get

$$F_{fx_n,Sx_n}(u) \le F_{fx_n,fx_{n+1}}\left(\frac{u}{\xi}\right) \quad \text{for all } u > 0 \text{ and } n \in \mathbb{N}. \tag{3.3}$$

It immediately follows from (3.2) and (3.3) that

$$F_{fx_n,fx_{n+1}}(u) \ge F_{fx_{n-1},fx_n}\left(\frac{\xi u}{\eta}\right) \quad \text{for all } u > 0 \text{ and } n \in \mathbb{N}. \tag{3.4}$$

It is easy to prove from (3.4) that $\{fx_n\}$ is a τ-Cauchy sequence. Therefore, by the τ-completeness of $(X,\mathfrak{F},\Delta)$, there exists $u \in X$ such that $fx_n \xrightarrow{\tau} a(n \to \infty)$. By the τ-closedness of $f(X)$, we have $a = fb$ for some $b \in X$, i.e., $fx_n \xrightarrow{\tau} fb(n \to \infty)$.

Since O, P, Q, R are τ-continuous, by (3.1) and (i), we obtain

$$Ofb \preceq_1 Pfb \tag{3.5}$$

and

$$Qfb \preceq_1 Rfb. \tag{3.6}$$

Now, if there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ satisfying that $Sx_{n_l} = Sb$ for all $l \in \mathbb{N}$. Note that

$$F_{fb,Sb}(u) \ge \Delta\left(F_{fb,fx_{n_l+1}}\left(\frac{u}{2}\right), F_{fx_{n_l+1},Sv}\left(\frac{u}{2}\right)\right) \quad \text{for all } u > 0.$$

Thus, it follows from $fx_{n_l} \in Sb$ and $fx_n \xrightarrow{\tau} fb(n \to \infty)$ that $fb \in \overline{Sb}$. Suppose that this is not true, then $Sx_n \neq Sb$ for all $n \in \mathbb{N}$. Using (3.2) and the fact that $fx_{n+1} \in J_\xi^{x_n}(f,S)$, we get

$$F_{fx_n,Sx_n}(u) \ge F_{fx_{n-1},Sx_{n-1}}\left(\frac{\xi u}{\eta}\right) \quad \text{for all } u > 0 \text{ and } n \in \mathbb{N}.$$

Hence, we get $\lim_{n\to\infty} F_{fx_n,Sx_n}(u) = 1$ for all $u > 0$ by letting $n \to \infty$. Now, we proceed the proof by considering the following three cases.

Case 1. Suppose that assumption (A_1) holds. By the inductive construction of $\{x_n\}$ and (A_1), we have $F_{fb,Sb}(u) = 1$ for all $u > 0$. So $fb \in \overline{Sb}$.

Case 2. Suppose that assumption (A_2) holds. Now, suppose $fb \notin \overline{Sb}$. Then we have

$$1 > \sup_{a\in X} \Delta\bigl(F_{fa,fb}(u), F_{fa,Sa}(u)\bigr) \ge \sup_{n\in\mathbb{N}} \Delta\bigl(F_{fx_n,fb}(u), F_{fx_n,Sx_n}(u)\bigr) = 1 \quad \text{for all } u > 0,$$

which cannot be true. Therefore, $fb \in \overline{Sb}$.

Case 3. Suppose that assumption (A_3) holds. Noting that Δ is continuous, by Definition 2.5, we obtain

$$\lim_{n\to\infty} F^*_{(fx_n,fx_{n+1}),(b,b)}(u) = \lim_{n\to\infty} \Delta\left(F_{fx_n,b}\left(\frac{u}{2}\right), F_{fx_{n+1},b}\left(\frac{u}{2}\right)\right) = 1 \quad \text{for all } u > 0.$$

It follows from the above assertion that $(b,b) \in G(\overline{S})_f$, that is,

$$fb \in \overline{Sb}. \tag{3.7}$$

Combining (3.5), (3.6), and (3.7), we conclude that b is a solution to problem (1.3). □

We shall further study the existence of solutions to problem (1.4).

Theorem 3.2 *If the range of the mapping S is assumed to be $\mathcal{CL}(X)$ rather than $\mathcal{N}(X)$, and O, P (or Q, R) are τ-continuous, while the other conditions are the same as the ones in Theorem 3.1, then there exists $b \in X$ such that $fb \in Sb$ with constraints $Ofb \preceq_1 Pfb$ and $Qfb \preceq_2 Rfb$. Furthermore, if we assume that $ffx = fx$ for $x \in C(f,S)$, then there exists at least one solution to problem* (1.4).

Proof We first assume that O, P are τ-continuous. Then, following the proof in Theorem 3.1, we can get Eqs. (3.5) and (3.7). Since Sb is τ-closed now, we get

$$fb \in Sb. \tag{3.8}$$

Now, we prove (3.6) holds. In fact, by (iv) and (3.5), we obtain

$$QSb \preceq_2 RSb,$$

which implies that there exists $fy \in J^b_\xi(f,S)$ such that $Qy \preceq_2 Ry$. Noting that $fb \in Sb$, we can easily verify that

$$J^b_\xi(f,S) = \left\{ fy \in Sb \,\middle|\, F_{fb,Sb}(u) \le F_{fb,fy}\left(\frac{u}{\xi}\right) \text{ for all } u > 0 \right\} = \{fb\}.$$

Therefore, we deduce that (3.6) holds. If Q and R are τ-continuous, the conclusion can be proved by using the same method. □

We next prove the second part of the theorem. From (3.8), we get $b \in C(f,S)$ and so $a = fb = ffb = fa \in Sb$. Since $J^b_\xi(f,S) = \{fb\}$, we have $fa \in J^b_\xi(f,S)$. Noting that (3.5) and (3.6) hold, it follows from condition (v) of Theorem 3.1 that

$$F_{fa,Sa}(u) \ge F_{fb,fa}\left(\frac{u}{\eta}\right) = 1 \quad \text{for all } u > 0.$$

Therefore, we have $fa \in Sa$, and thus $a = fa \in Sa$. Also, by (3.5) and (3.6), we have $Oa \preceq_1 Pa$ and $Qa \preceq_1 Ra$. Hence, a is a solution to problem (1.4).

Now, we give two examples to show that the above two theorems are valid.

Example 3.1 Let $X = \{2 + \frac{1}{4^n} : n \in \mathbb{N}\} \cup \{2, 3\}$, and the following partial order $\preceq$ is imposed on X: $x \preceq y$ iff $x \leq y$ for $x, y \in X$. Define $f, O, P, Q, R : X \to X$ and $S : X \to \mathcal{CL}(X)$ as follows:

$$Ox = \begin{cases} \frac{1}{16}x + \frac{15}{8}, & x = 2 + \frac{1}{4^{n+1}}, n \in \mathbb{N}, \\ 3, & x = 2, 3, \frac{9}{4}, \end{cases} \qquad Px = \begin{cases} \frac{1}{4}x + \frac{3}{2}, & x = 2 + \frac{1}{4^{n+1}}, n \in \mathbb{N}, \\ 2, & x = 2, 3, \frac{9}{4}, \end{cases}$$

$$Qx = \begin{cases} (x-2)^4 + 2, & x = 2 + \frac{1}{4^{n+1}}, n \in \mathbb{N}, \\ 3, & x = 2, 3, \frac{9}{4}, \end{cases} \qquad Rx = \begin{cases} (x-2)^2 + 2, & x = 2 + \frac{1}{4^{n+1}}, n \in \mathbb{N}, \\ 2, & x = 2, 3, \frac{9}{4}, \end{cases}$$

$$fx = \begin{cases} 2 + \frac{1}{4^{n+1}}, & x = 2 + \frac{1}{4^n}, n \in \mathbb{N}, \\ 2, & x = 2, \\ \frac{33}{16}, & x = 3, \end{cases}$$

and

$$Sx = \begin{cases} \{2 + \frac{1}{4^{n+2}}, 3\}, & x = 2 + \frac{1}{4^n}, n \in \mathbb{N}, \\ \{2, \frac{33}{16}\}, & x = 2, \\ \{\frac{33}{16}, 3\}, & x = 3. \end{cases}$$

Let $\mathfrak{F}$ be defined by (2.1), where $d(x, y) = |x - y|$. Then $(X, \mathfrak{F}, \Delta_{\min})$ is τ-complete. Direct calculation shows that

$$F_{fx,Sx}(u) = \begin{cases} \gamma(u - \frac{3}{4^{n+2}}), & x = 2 + \frac{1}{4^n}, n \in \mathbb{N}, \\ \gamma(u), & x = 2, 3, \end{cases}$$

for all $u > 0$, and so for any $\xi \in (\frac{1}{4}, 1)$, we have $J_\xi^{2+\frac{1}{4^n}}(f, S) = \{2 + \frac{1}{4^{n+1}}\}$ for all $n \in \mathbb{N}$, $J_\xi^2(f, S) = \{2\}$ and $J_\xi^3(f, S) = \{\frac{33}{16}\}$. Moreover, for each $x \in X$ and $fy \in J_\xi^x(f, S)$, the inequality $F_{fy,Sy}(u) \geq F_{fx,fy}(\frac{u}{\eta})$ holds for $\eta = \frac{1}{4} \in (0, \xi)$.

Noting that when $Ofx \leq Pfx$, we have $x \in \{2 + \frac{1}{4^n}, n \in \mathbb{N}\}$, and there exists $fy \in Sx$ such that $F_{fx,Sx}(u) \leq F_{fx,fy}(\frac{u}{\xi})$ for all $\xi \in (\frac{1}{4}, 1)$ and $u > 0$, and $Qfy \leq Rfy$, we observe that (f, S) is ξ-$(O, P, Q, R, \preceq, \preceq)$-stable, where $\xi \in (\frac{1}{4}, 1)$. Similarly, (f, S) is ξ-$(Q, R, O, P, \preceq, \preceq)$-stable with $\xi \in (\frac{1}{4}, 1)$. Besides, it is easy to check that assumption (A_1) holds. By Theorem 3.1, it is claimed that $C(f, S) \neq \emptyset$. In this example, $C(f, S) = \{2, 3\}$. Furthermore, it is obvious that $ff2 = f2$ and $ff3 \neq f3$, so by Theorem 3.2 $F(f, S) \neq \emptyset$. Here, $F(f, S) = \{2\}$.

Example 3.2 Let $X = [0, \frac{\pi}{2}]$, and the partial order $\preceq$ is defined in the same way as in Example 3.1. Define $O, P, Q, R : X \to X$ and $S : X \to \mathcal{CL}(X)$ as follows:

$$Ox = \begin{cases} \frac{1}{4}e^{\frac{1}{2}x}, & x \in [0, \frac{\pi}{2}), \\ \frac{3}{2}, & x = \frac{\pi}{2}, \end{cases} \qquad Px = \begin{cases} \frac{1}{3}x + \frac{1}{2}, & x \in [0, \frac{\pi}{2}), \\ 1, & x = \frac{\pi}{2}, \end{cases}$$

$$Qx = \begin{cases} \sin\frac{x}{2}, & x \in [0, \frac{\pi}{2}), \\ \frac{5}{4}, & x = \frac{\pi}{2}, \end{cases} \qquad Rx = \begin{cases} \cos\frac{x}{2}, & x \in [0, \frac{\pi}{2}), \\ \frac{3}{4}, & x = \frac{\pi}{2}, \end{cases}$$

$$fx = \begin{cases} \frac{x}{2}, & x \in [0, \frac{\pi}{2}), \\ \frac{\pi}{2}, & x = \frac{\pi}{2}, \end{cases}$$

and

$$Sx=\begin{cases}[0,\frac{x}{4}], & x\in[0,\frac{\pi}{2}),\\ [\frac{\pi}{8},\frac{\pi}{2}], & x=\frac{\pi}{2}.\end{cases}$$

Let $\mathfrak{F}$ be defined as the same one in Example 3.1. Then $(X,\mathfrak{F},\Delta_{\min})$ is τ-complete. By direct calculation, we get

$$F_{fx,Sx}(u)=\begin{cases}\gamma(u-\frac{x}{4}), & x\in[0,\frac{\pi}{2}),\\ \gamma(u), & x=\frac{\pi}{2},\end{cases}$$

for all $u>0$, and so for all $\xi\in(\frac{1}{2},1)$, we have $J_\xi^x(f,S)=[(\frac{1}{2}-\frac{1}{4\xi})x,\frac{x}{4}]$ for $x\in[0,\frac{\pi}{2})$ and $J_\xi^{\frac{\pi}{2}}(f,S)=\{\frac{\pi}{2}\}$. Moreover, for $x\in X$ and $fy\in J_\xi^x(f,S)$, the inequality $F_{fy,Sy}(u)\geq F_{fx,fy}(\frac{u}{\eta})$ holds for $\eta=\frac{1}{2}\in(0,\xi)$ and $u>0$.

We can easily check that (f,S) are ξ-$(O,P,Q,R,\preceq,\preceq)$-stable and ξ-$(Q,R,O,P,\preceq,\preceq)$-stable for $\xi\in(\frac{1}{2},1)$. Besides, it is obvious that assumption (A_1) holds. It can be seen that all the hypotheses of Theorem 3.1 and Theorem 3.2 hold. So we claim that $F(f,S)\neq\emptyset$. In this example, $F(f,S)=\{0,\frac{\pi}{2}\}$.

4 Consequent results

In this section, we give some results as consequences of Theorem 3.1 and Theorem 3.2. First, if we use another contraction condition using Menger–Hausdorff metric, we can get the following result which can be viewed as a corollary of Theorem 3.1.

Corollary 4.1 *Let $(X,\mathfrak{F},\Delta,\preceq_1,\preceq_2)$ be a τ-complete partially ordered Menger PM-space, where $\preceq_1$ and $\preceq_2$ are partial orders on X and Δ is a continuous t-norm, $f,O,P,Q,R:X\to X$ be self-mappings satisfying $f(X)$ is τ-closed and O,P,Q,R are τ-continuous, and $S:X\to\mathcal{N}(X)$ be a set-valued mapping. Furthermore, suppose that the following hypotheses hold for some $\xi\in(0,1)$:*

(i) *$\preceq_1$ and $\preceq_2$ are F-regular;*
(ii) *$Ofx_0\preceq_1 Pfx_0$ for some $x_0\in X$;*
(iii) *(f,S) is ξ-$(O,P,Q,R,\preceq_1,\preceq_2)$-stable;*
(iv) *(f,S) is ξ-$(Q,R,O,P,\preceq_2,\preceq_1)$-stable;*
(v) *$\tilde{F}_{Sx,Sy}(u)\geq F_{fx,fy}(\frac{u}{\eta})$ for all $x,y\in X$ and $u>0$, where $\eta\in(0,\xi)$.*

Then there exists at least one solution to problem (1.3).

Proof First, by (v) and Lemma 2.1 (i), for all $x\in X$ and $fy\in J_\xi^x(f,S)$ and $u>0$, we have

$$F_{fy,Sy}(u)\geq\tilde{F}_{Sx,Sy}(u)\geq F_{fx,fy}\left(\frac{u}{\eta}\right).$$

Thus, conditions (i)–(v) of Theorem 3.1 are satisfied. We next show that assumption (A_1) holds. In fact, combining the definition of Menger–Hausdorff metric and Lemma 2.1 (ii) and (iii), for all $x,y\in X$ and $u>0$, we get

$$F_{fx,Sx}(u)\geq\Delta\left(F_{fx,fy}\left(\frac{u}{2}\right),\Delta\left(F_{fy,Sx}\left(\frac{u}{2}\right),\right)\right)$$

$$\geq \Delta\left(F_{fx,fy}\left(\frac{u}{2}\right), \Delta\left(F_{fy,Sy}\left(\frac{u}{4}\right), \tilde{F}_{Sy,Sx}\left(\frac{u}{4}\right)\right)\right)$$
$$\geq \Delta\left(F_{fx,fy}\left(\frac{u}{2}\right), \Delta\left(F_{fy,Sy}\left(\frac{u}{4}\right), F_{fx,fy}\left(\frac{u}{4\eta}\right)\right)\right).$$

Taking $y = x_n$ yields that

$$F_{fx,Tx}(u) \geq \Delta\left(F_{fx,fx_n}\left(\frac{u}{2}\right), \Delta\left(F_{fx_n,Tx_n}\left(\frac{u}{4}\right), F_{fx,fx_n}\left(\frac{u}{4\eta}\right)\right)\right).$$

We deduce that assumption (A_1) holds by letting $n \to \infty$. Thus, it follows from Theorem 3.1 that the conclusion holds. □

Corollary 4.2 *If the range of the mapping S is assumed to be $\mathcal{CL}(X)$ rather than $\mathcal{N}(X)$, and O, P (or Q, R) are τ-continuous, while the other conditions are the same as the ones in Corollary 4.1, then there exist $b \in X$ such that $fb \in Sb$ with constraints $Ofb \preceq_1 Pfb$ and $Qfb \preceq_2 Rfb$. Furthermore, if we assume that $ffx = fx$ for $x \in C(f, S)$, then there exists at least one solution to problem (1.4).*

We can also derive some other consequent results by posing restrictions on the mappings or the partial orders from Corollary 4.1 and Corollary 4.2. For the sake of brevity, we omit them here.

Remark 4.1 Example 3.1 and Example 3.2 cannot be used to illustrate Corollary 4.1, since in Example 3.1,

$$\tilde{F}_{S(2),S(2+\frac{1}{4^n})}(u) = \gamma\left(u - \frac{15}{16}\right) \leq \gamma\left(t - \frac{1}{4^{n+1}}\right) = F_{f(2),f(2+\frac{1}{4^n})}(u) \quad \text{for all } n \in \mathbb{N} \text{ and } u > 0,$$

and in Example 3.2, it holds that $\tilde{F}_{S(0),S(\frac{\pi}{2})}(u) = \gamma(u - \frac{\pi}{2}) = F_{f(0),f(\frac{\pi}{2})}(u)$ for all $u > 0$.

By considering a graph instead of a partial order, one can establish analogous results as above. We first recall the concept of a directed graph.

Definition 4.1 ([22]) An ordered pair (X, E), where X is a nonempty set and $E \subset X \times X$ is a binary relation, is called a directed graph G.

We next introduce the following definitions based on graphs.

Definition 4.2 Let $(X, \mathfrak{F}, \Delta, G)$ be a graph-based Menger PM-space, where $G = (X, E)$ is a graph. The graph G is called *F_G-regular*, if for sequences $\{c_n\}, \{d_n\} \subset X$ and $(c, d) \in X \times X$, we have

$$\lim_{n\to\infty} F_{c_n,c}(u) = \lim_{n\to\infty} F_{d_n,d}(u)$$
$$= 1 \quad \text{for all } u > 0, \text{ and } (c_n, d_n) \in E \text{ for all } n \in \mathbb{N} \Longrightarrow (c, d) \in E.$$

Similarly, if there exists $fy \in J^x_\xi(f, S)$ such that $(My, Ny) \in E$, where $\xi \in (0, 1)$, then we write $(MSx, NSx) \in E$.

Definition 4.3 Let X be a nonempty set with graphs $G_1 = (X, E_1)$ and $G_2 = (X, E_2)$ imposed on it, $f, O, P, Q, R : X \to X$ be self-mappings, and $S : X \to \mathcal{N}(X)$ be a set-valued mapping. The hybrid pair of mappings (f, S) is called ξ-(O, P, Q, R, G_1, G_2)-*graph-stable* if

$$x \in X, \quad (Ofx, Pfx) \in E_1 \quad \Longrightarrow \quad (QSx, RSx) \in E_2.$$

Based on the above definitions, we can prove the following two theorems using the same method in Theorem 3.1 and Theorem 3.2, and also derive corresponding corollaries.

Theorem 4.1 *Let $(X, \mathfrak{F}, \Delta, G_1, G_2)$ be a τ-complete graph-based Menger PM-space, where G_1 and G_2 are two graphs and Δ is a continuous t-norm, $f, O, P, Q, R : X \to X$ be self-mappings satisfying $f(X)$ is τ-closed and O, P, Q, R are τ-continuous, and $S : X \to \mathcal{N}(X)$ be a set-valued mapping. Suppose that one of assumptions (A_1)–(A_3) holds, and the following hypotheses hold for some $\xi \in (0, 1)$:*

(i) *G_1 and G_2 are F_G-regular;*
(ii) *$(Ofx_0, Pfx_0) \in G_1$ for some $x_0 \in X$;*
(iii) *(f, S) is ξ-(O, P, Q, R, G_1, G_2)-graph-stable;*
(iv) *(f, S) is ξ-(Q, R, O, P, G_2, G_1)-graph-stable;*
(v) *$F_{fy,Sy}(u) \geq F_{fx,fy}(\frac{u}{\eta})$ for all $x \in X$ and $fy \in J_\xi^x(f, S)$ with $((Ofx, Pfx) \in E_1$ and $(Qfy, Rfy) \in E_2)$ or $((Ofy, Pfy) \in E_1$ and $(Qfx, Rfx) \in E_2)$ and $u > 0$, where $\eta \in (0, \xi)$.*

Then there exists $a \in X$ such that $fa \in \overline{Sa}$, $(Ofa, Pfa) \in E_1$ and $(Qfa, Rfa) \in E_2$.

Theorem 4.2 *If the range of the mapping S is assumed to be $\mathcal{CL}(X)$ rather than $\mathcal{N}(X)$, and O, P (or Q, R) are τ-continuous, while the other conditions are the same as the ones in Corollary 4.1, then there exists $v \in X$ such that $fv \in Sv$ with constraints $Ofv \preceq_1 Pfv$ and $Qfv \preceq_2 Rfv$. Furthermore, if we assume that $ffx = fx$ for $x \in C(f, S)$, then there exists at least one solution to problem (1.4).*

5 An application

In this section, we utilize the main results in Sect. 3 to investigate the existence of solutions for a system of nonlinear integral equations.

Let $X = C([p, q], \mathbb{R})$, where $C([p, q], \mathbb{R})$ denotes the space of all continuous functions on $[p, q]$ with $q > p > 0$, and impose the following norm on X:

$$\|x\|_1 = \max_{u \in [p,q]} \bigl|x(u)\bigr|, \quad x \in X.$$

Then $(X, \|\cdot\|_1)$ is a Banach space.

Consider another norm

$$\|x\|_2 = \max_{u \in [p,q]} e^{-ku}\bigl|x(u)\bigr|, \quad x \in X, k > 0. \tag{5.1}$$

Note that the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent (see [23]), which implies that $(X, \|\cdot\|_2)$ is also a Banach space.

Define $\mathfrak{F} : X \times X \to \mathfrak{D}$ by

$$\mathfrak{F}(x, y)(w) = F_{x,y}(w) = \gamma\bigl(w - \|x - y\|_2\bigr), \quad x, y \in X, w > 0, \tag{5.2}$$

and impose on X the following partial order:

$$x \preceq y \iff x(u) \le y(u) \quad \text{for all } u \in [p,q]. \tag{5.3}$$

Now, we consider the problem of the existence of solutions for the following system of Volterra integral equations:

$$x_i(u) = h(u) + \int_0^u \Phi_i\bigl(u,v,x(v)\bigr)\,dv, \quad i = 1,2, \tag{5.4}$$

for all $u \in [p,q]$, where $q > p > 0$, $h, x_i \in X$ and $\Phi_i \in C([p,q] \times [p,q] \times X, \mathbb{R})$.

Theorem 5.1 *Let $(X, \mathfrak{F}, \Delta_{\min})$ be the Menger PM-space induced by the Banach space X, where $\mathfrak{F}$ is defined by (5.2), and X is equipped with the partial order $\preceq$ defined by (5.3), $\Phi_i \in C([p,q] \times [p,q] \times X, \mathbb{R})$, $i = 1,2$, φ_1 and φ_2 are two functionals on X, and the following hypotheses hold*:

(i) $\|\Phi_j\|_\infty = \sup_{u,v\in[p,q],x\in C([p,q],\mathbb{R})} |\Phi_i(u,v,x(v))| < \infty$ *for* $j \in \{1,2\}$;

(ii) $\varphi_1(fx_0) > 0$ *for some* $x_0 \in X$;

(iii) *If* $\varphi_1(fx) > 0$, *then there exists* $y \in X$ *such that* $\int_0^u \Phi_1(u,v,y(v))\,dv = \int_0^u \Phi_2(u,v,x(v))\,dv$ *and* $\varphi_2(fy) > 0$; *if* $\varphi_2(fx) > 0$, *then there exists* $y \in X$ *such that* $\int_0^u \Phi_1(u,v,y(v))\,dv = \int_0^u \Phi_2(u,v,x(v))\,dv$ *and* $\varphi_1(fy) > 0$;

(iv) *There exist* $k > 0$ *and* $0 < \eta < \xi$ *with* $\xi \in (0,1)$ *such that, for all* $x, y \in X$ *and all* $u, v \in [p,q]$, $1 - e^{-kq} \le \eta$, *and whenever* $x \in X$, $\int_0^u \Phi_1(u,v,y(v))\,dv = \int_0^u \Phi_2(u,v,x(v))\,dv$ *with* $\varphi_1(fx) > 0$ *and* $\varphi_2(fy) > 0$, *we have*

$$\bigl|\Phi_1\bigl(u,v,y(v)\bigr) - \Phi_2\bigl(u,v,y(v)\bigr)\bigr| \le k\bigl|fx(v) - fy(v)\bigr|,$$

where $f, S : X \to X$ *are defined by* $fx(u) = h(u) + \int_0^u \Phi_1(u,v,x(v))\,dv$ *and* $Sx(u) = h(u) + \int_0^u \Phi_2(u,v,x(v))\,dv$ *for all* $u \in [p,q]$, *respectively*;

(v) *There exists a sequence* $\{x_n\}$ *in* X *such that* $\int_0^u \Phi_2(u,v,x_n(v))\,dv \to \int_0^u \Phi_2(u,v,x(v))\,dv$ $(n \to \infty)$;

(vi) *If* $\int_0^u \Phi_1(u,v,z(v))\,dv = \int_0^u \Phi_2(u,v,z(v))\,dv$, *then* $\int_0^u \Phi_1(u,v,fz(v))\,dv = \int_0^u \Phi_1(u,v,z(v))\,dv$.

Then (5.4) *has a solution* z' *in* X *with constraints* $Az' \preceq Bz'$ *and* $Cz' \preceq Dz'$.

Proof It is obvious that $(X, \mathfrak{F}, \Delta_{\min})$ is τ-complete and the partial order $\preceq$ is F-regular. Define $O, P, Q, R : X \to X$ as follows:

$$Ox = \begin{cases} |x|, & \varphi_1(x) > 0, \\ 2, & \text{otherwise}, \end{cases} \qquad Px = \begin{cases} 2|x|, & \varphi_1(x) > 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$Qx = \begin{cases} 3|x|, & \varphi_2(x) > 0, \\ 4, & \text{otherwise}, \end{cases} \qquad Rx = \begin{cases} 4|x|, & \varphi_2(x) > 0, \\ 3, & \text{otherwise}, \end{cases}$$

where $|x| \in X$ is the function defined by $|x|(u) = |x(u)|$ for all $u \in [p,q]$.

By condition (ii), we know that $Afx_0 \preceq Bfx_0$ for some $x_0 \in X$, and by (iii), (f, S) are both ξ-$(A,B,C,D,\preceq,\preceq)$-stable and ξ-$(C,D,A,B,\preceq,\preceq)$-stable for $\xi \in (0,1)$.

Consider the norm defined by (5.1), where k satisfies condition (iv). It follows from (iv) that for $x \in X$, $fy = Sx$ with $Ofx \preceq Pfx$ and $Qfy \preceq Rfy$, we obtain

$$\begin{aligned}\|fy - Sy\|_2 &\le \max_{u\in[p,q]} \int_0^u e^{-kv} \big|\Phi_1\big(u,v,y(v)\big) - \Phi_2\big(u,v,y(v)\big)\big| e^{k(v-u)}\,dv \\ &\le k\|fx - fy\|_2 \max_{u\in[p,q]} \int_0^u e^{k(v-u)}\,dv \\ &\le \big(1 - e^{-kq}\big)\|fx - fy\|_2 \\ &\le \eta\|fx - fy\|_2,\end{aligned}$$

which implies that

$$F_{fy,Sy}(w) \ge F_{fx,fy}\left(\frac{w}{\eta}\right)$$

for some $\eta \in (0,\xi)$ and any $w > 0$ by (5.2).

Moreover, it follows from condition (v) that assumption (A_1) holds. And by (vi), it holds that $ffz = fz$ for $z \in C(f,S)$. Therefore, all the hypotheses of Theorem 3.1 and Theorem 3.2 hold, and so there exists a point $z' \in F(f,S)$ in X with constraints $Az' \preceq Bz'$ and $Cz' \preceq Dz'$, which means that the system of integral equations (5.4) has a solution z' with constraints $Az' \preceq Bz'$ and $Cz' \preceq Dz'$. □

6 Conclusions

In this paper, we have studied a common fixed point problem for a hybrid pair of mappings (a self-mapping and a set-valued mapping) under constraints in the framework of a Menger PM-space, by first investigating a related approximate coincidence point problem under certain constraints, and have derived some new results. It is worth noting that for the existence result of a solution to the common fixed point problem (1.4), the τ-continuity of two mappings O and P (or Q and S) is required rather than posing the τ-continuity on the four self-mappings. We have also constructed some examples and explored an application of the main results. The obtained results in this paper may shed some new light on the study of approximate coincidence point problems and common fixed point problems for a hybrid pair of mappings in the framework of Menger PM-spaces.

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The authors declare that they have no competing interests.

Authors' contributions

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References

1. Schweizer, B., Sklar, A.: Probabilistic Metric Spaces. North-Holland, Amsterdam (1983)
2. Hadžić, O., Pap, E.: Fixed Point Theory in Probabilistic Metric Spaces. Kluwer Academic, Dordrecht (2001)
3. Chang, S., Cho, Y.J., Kang, S.M.: Nonlinear Operator Theory in Probabilistic Metric Spaces. Nova Science Publishers, New York (2001)
4. Guillen, B.L., Harikrishnan, P.: Probabilistic Normed Spaces. Imperial College Press, London (2014)
5. Wu, Z., Zhu, C., Li, J.: Common fixed point theorems for two hybrid pairs of mappings satisfying the common property (E.A) in Menger PM-spaces. Fixed Point Theory Appl. **2013** 25 (2013)
6. Wu, Z., Zhu, C., Zhang, X.: Some new fixed point theorems for single and set-valued admissible mappings in Menger PM-spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **110**, 755–769 (2016)
7. Wu, Z., Zhu, C., Yuan, C.: Fixed point results for (α,η,ψ,ξ)-contractive multi-valued mappings in Menger PM-spaces and their applications. Filomat **31**(16), 5357–5368 (2017)
8. Wu, Z., Zhu, C., Yuan, C.: Fixed point results for cyclic contractions in Menger PM-spaces and generalized Menger PM-spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**, 449–462 (2018)
9. Lakshmikantham, V., Ćirić, L.: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341–4349 (2009)
10. Samet, B., Vetro, C.: Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces. Nonlinear Anal. **74**, 4260–4268 (2011)
11. Arutyunova, A.V., Zhukovskiyb, E.S., Zhukovskiya, S.E.: Coincidence points principle for mappings in partially ordered spaces. Topol. Appl. **179**, 13–33 (2015)
12. Juan, J.N., Abdelghani, O., Rosana, R.-L.: Random fixed point theorems in partially ordered metric spaces. Fixed Point Theory Appl. **2016**, 98 (2016)
13. Fomenko, T.N., Podoprikhin, D.A.: Fixed points and coincidences of mappings of partially ordered sets. J. Fixed Point Theory Appl. **18**(4), 823–842 (2016)
14. Durmaz, G., Minaky, G., Altunz, I.: Fixed points of ordered F-contractions. Hacet. J. Math. Stat. **45**(1), 15–21 (2016)
15. Alsamira, H., Noorania, M.S.M., Shatanawib, W., Abodyahc, K.: Common fixed point results for generalized (ψ,β)-Geraghty contraction type mapping in partially ordered metric-like spaces with application. Filomat **31**(17), 5497–5509 (2017)
16. Arslan, H.A., Diana, D.-D., Tatjana, D., Stojan, R.: Coupled coincidence point theorems for $(\alpha$-μ-ψ-H-$\mathcal{F})$-two sided-contractive type mappings in partially ordered metric spaces using compatible mappings. Filomat **31**(9), 2657–2673 (2017)
17. Zhang, J., Agarwal, R.P., Jiang, N.: *N* fixed point theorems and *N* best proximity point theorems for generalized contraction in partially ordered metric spaces. J. Fixed Point Theory Appl. **20**, 18 (2018)
18. Jleli, M., Samet, B.: A fixed point problem under two constraint inequalities. Fixed Point Theory Appl. **2016**, 18 (2016)
19. Ansari, A.H., Kumam, P., Samet, B.: A fixed point problem with constraint inequalities via an implicit contraction. J. Fixed Point Theory Appl. **19**, 1145–1163 (2017)
20. Wu, Z., Zhu, C., Yuan, C.: Fixed point results under constraint inequalities in Menger PM-spaces. J. Comput. Anal. Appl. **25**(7), 1324–1336 (2018)
21. Samet, B., Vetro, C., Vetro, F.: An approximate fixed point result for multivalued mappings under two constraint inequalities. J. Fixed Point Theory Appl. **19**, 2095–2107 (2017)
22. Jachymski, J.: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **136**, 1359–1373 (2008)
23. Chang, S.S.: On the theory of probabilistic metric spaces with applications. Acta Math. Sin. Engl. Ser. **1**, 366–377 (1985)