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Applications of differential subordinations involving a generalized fractional differintegral operator

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Abstract

Using the third-order differential subordination basic results, we introduce certain classes of admissible functions and investigate some applications of third-order differential subordination for p-valent functions associated with generalized fractional differintegral operator.

MSC: 30C45; 30C50

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1 Introduction and preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote the space of analytic functions in the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{H}[a,n]$ denote the subclass of functions $f \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad z \in \mathbb{U}, \quad \bigl(a \in \mathbb{C}, n \in \mathbb{N} := \{1,2,\ldots\}\bigr),$$

and $\mathcal{H}_p := \mathcal{H}[0,p]$. Also, let $\mathcal{A}(p)$ be the subclass of functions $f \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad z \in \mathbb{U}, \quad (p \in \mathbb{N}) \tag{1.1}$$

and set $\mathcal{A} := \mathcal{A}(1)$.

For two functions $f, g \in \mathcal{H}(\mathbb{U})$, we say that the function f is *subordinate* to g, written $f \prec g$, if there exists a *Schwarz function* w, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [9] and [12]):

$$f(z) \prec g(z) \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi(r,s,t;z):\mathbb{C}^4\times\mathbb{U}\to\mathbb{C}$ and h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the *third-order differential subordination*

$$\phi\bigl(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\bigr)\prec h(z), \tag{1.2}$$

then p is called a solution of the differential subordination (1.2). The univalent function q is said to be a *dominant* of (1.2) if $p(z)\prec q(z)$ for all p that satisfy (1.2). A dominant $\widetilde{q}$ is called the *best dominant* if $\widetilde{q}(z)\prec q(z)$ for all dominants q of (1.2).

We recall here the following *generalized fractional integral and generalized fractional derivative operators* due to Srivastava et al. [22] (see also [16, 17]).

Definition 1.1 ([22, Definition 3]) For $\lambda>0$ and μ,η real numbers, the *Srivastava–Saigo–Owa hypergeometric fractional integral operator* $I_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$I_{0,z}^{\lambda,\mu,\eta}f(z)=\frac{z^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^z(z-\zeta)^{\lambda-1}F\biggl(\mu+\lambda,-\eta;\lambda;1-\frac{\zeta}{z}\biggr)f(\zeta)\,d\zeta,$$

where f is an analytic function in a simply-connected region of the complex z-plane containing the origin with the order $f(z)=O(|z|^{\varepsilon})$, $z\to0$, where $\varepsilon>\max\{0,\mu-\eta\}-1$ and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$. Also, Γ is the well-known *gamma function*, while the function F denotes the *Gauss hypergeometric function*, that is,

$${}_2F_1(a,b;c:z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_nn!}z^n,\qquad|z|<1,$$

and its analytic continuation into $|\arg(1-z)|<\pi$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$.

Definition 1.2 Under the hypotheses of Definition 1.1, the *Srivastava–Saigo–Owa hypergeometric fractional derivative operator* $J_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$J_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{d}{dz}\Bigl[\frac{z^{\lambda-\mu}\int_0^z(z-\zeta)^{-\lambda}f(\zeta){}_2F_1(\mu-\lambda,1-\eta;1-\lambda;1-\frac{\zeta}{z})\,d\zeta}{\Gamma(1-\lambda)}\Bigr],\\ \qquad\qquad\text{if }0\le\lambda<1,\\ \frac{d^n}{dz^n}J_{0,z}^{\lambda-n,\mu,\eta}f(z),\quad\text{if }n\le\lambda<n+1,n\in\mathbb{N},\end{cases}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.1.

We note that

$$I_{0,z}^{\lambda,-\lambda,\eta}f(z)=D_z^{-\lambda}f(z),\quad\text{if }\lambda>0,\quad\text{and}\quad J_{0,z}^{\lambda,\lambda,\eta}f(z)=D_z^{\lambda}f(z),\quad\text{if }0\le\lambda<1,$$

where $D_z^{-\lambda}$ denotes the *fractional integral operator*, and D_z^{λ} denotes the *fractional derivative operator* studied by Owa [13].

In relation to the Srivastava–Saigo–Owa hypergeometric fractional derivative operator, Goyal and Prajapat [10] (see also [15]) defined the operator

$$S_{0,z}^{\lambda,\mu,\eta,p}f(z)=\begin{cases}\frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)}z^{\mu}J_{0,z}^{\lambda,\mu,\eta}f(z),\\ \quad \text{if } 0\le\lambda<\eta+p+1, z\in\mathbb{U},\\ \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)}z^{\mu}I_{0,z}^{-\lambda,\mu,\eta}f(z),\\ \quad \text{if } -\infty<\lambda<0, z\in\mathbb{U}.\end{cases}$$

Thus, for a function $f\in\mathcal{A}(p)$ of the form (1.1), we have

$$\begin{aligned}S_{0,z}^{\lambda,\mu,\eta,p}f(z)&=z^p\,{}_3F_2(1,1+p,1+p+\eta-\mu;1+p-\mu,1+p+\eta-\lambda;z)*f(z)\\&=z^p+\sum_{n=1}^{\infty}\frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}a_{p+n}z^{p+n},\quad z\in\mathbb{U},\\&(p\in\mathbb{N},\mu,\eta\in\mathbb{R},\mu<p+1,\infty<\lambda<\eta+p+1),\end{aligned}$$

where ${}_qF_s$, with $q\le s+1$ and $q,s\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, is the well-known *generalized hypergeometric function* (for more details, see [14] and [20]) and $(\nu)_n$ is the *Pochhammer symbol* defined by

$$(\nu)_n=\begin{cases}1, & \text{if } n=0,\\ \nu(\nu+1)(\nu+2)\cdots(\nu+n-1), & \text{if } n\in\mathbb{N}.\end{cases}$$

Let

$$G_{p,\eta,\mu}^{\lambda}(z)=z^p+\sum_{n=1}^{\infty}\frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}z^{p+n},\quad z\in\mathbb{U},$$
$$(p\in\mathbb{N},\mu,\eta\in\mathbb{R},\mu<p+1,-\infty<\lambda<\eta+p+1),$$

and define the new function $[G_{p,\eta,\mu}^{\lambda}]^{-1}$ by means of the *Hadamard (or convolution) product*

$$G_{p,\eta,\mu}^{\lambda}(z)*\left[G_{p,\eta,\mu}^{\lambda}(z)\right]^{-1}=\frac{z^p}{(1-z)^{\delta+p}},\quad z\in\mathbb{U},\quad(\delta>-p).$$

Using the above defined function, Tang et al. [24] (see also Aouf et al. [4, 7] and [8]) defined the operator $H_{p,\eta,\mu}^{\lambda,\delta}:\mathcal{A}(p)\to\mathcal{A}(p)$ by

$$H_{p,\eta,\mu}^{\lambda,\delta}f(z)=\left[G_{p,\eta,\mu}^{\lambda}(z)\right]^{-1}*f(z).$$

It is easy to check that, for a function $f\in\mathcal{A}(p)$ of the form (1.1), we have

$$H_{p,\eta,\mu}^{\lambda,\delta}f(z)=z^p+\sum_{n=1}^{\infty}\frac{(\delta+p)_n(p+1-\mu)_n(p+1-\lambda+\eta)_n}{(1)_n(p+1)_n(p+1-\mu+\eta)_n}a_{p+n}z^{p+n},\quad z\in\mathbb{U}.$$

For $k \in \mathbb{N}_0$ and $\zeta > 0$, Aouf et al. [6] defined the operator $\mathcal{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta} : \mathcal{A}(p) \to \mathcal{A}(p)$ as follows:

$$\begin{aligned}
\mathcal{N}^{0,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) &= H^{\lambda,\delta}_{p,\eta,\mu} f(z),\\
\mathcal{N}^{1,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) &=: \mathcal{N}^{\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) = (1-\zeta) H^{\lambda,\delta}_{p,\eta,\mu} f(z) + \zeta \frac{z}{p}\big[H^{\lambda,\delta}_{p,\eta,\mu} f(z)\big]'\\
&= z^p + \sum_{n=1}^{\infty} \frac{p+\zeta n}{p} \frac{(\delta+p)_n (p+1-\mu)_n (p+1-\lambda+\eta)_n}{(1)_n (p+1)_n (p+1-\mu+\eta)_n} a_{p+n} z^{p+n},
\end{aligned}$$

and, in general,

$$\begin{aligned}
\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) &=: \mathcal{N}^{\delta,\zeta}_{p,\lambda,\mu,\eta}\big(\mathcal{N}^{k-1,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z)\big)\\
&= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\zeta n}{p}\right)^k \frac{(\delta+p)_n (p+1-\mu)_n (p+1-\lambda+\eta)_n}{(1)_n (p+1)_n (p+1-\mu+\eta)_n} a_{p+n} z^{p+n}.
\end{aligned} \tag{1.3}$$

Remark 1.1 1. We note that the operator $\mathcal{N}^{m,\delta,\zeta}_{p,\lambda,\mu,\eta}$ generalizes many other remarkable previously studied operators, like:

(i) $\mathcal{N}^{0,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) =: H^{\lambda,\delta}_{p,\eta,\mu} f(z)$ (see [24]);

(ii) $\mathcal{N}^{k,1,\zeta}_{p,p,p,0} f(z) =: D^k_{\zeta,p} f(z)$ (see [3]);

(iii) $\mathcal{N}^{k,1,1}_{p,p,p,0} f(z) =: D^k_p f(z)$ (see [5] and [11]);

(iv) $\mathcal{N}^{k,1,\zeta}_{1,1,1,0} f(z) =: D^k_\zeta f(z)$ (see [1]);

(v) $\mathcal{N}^{k,1,1}_{1,1,1,0} f(z) =: D^k f(z)$ (see [18]).

2. Also, we remark the following special cases of this operator:

(i) $$\begin{aligned}\mathcal{N}^{k,1,\zeta}_{p,\lambda,\mu,\eta} f(z) &=: \mathcal{N}^{k,\zeta}_{p,\lambda,\mu,\eta} f(z)\\ &= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\zeta n}{p}\right)^k \frac{(p+1-\mu)_n (p+1-\lambda+\eta)_n}{(1)_n (p+1-\mu+\eta)_n} a_{p+n} z^{p+n};\end{aligned}$$

(ii) $$\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\lambda,\eta} f(z) =: \mathcal{N}^{k,\delta,\zeta}_{p,\lambda} f(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{p+\zeta n}{p}\right)^k \frac{(\delta+p)_n (p+1-\lambda)_n}{(1)_n (p+1)_n} a_{p+n} z^{p+n};$$

(iii) $$\mathcal{N}^{k,1,\zeta}_{p,\mu,\mu,\eta} f(z) =: \mathcal{N}^{k,\delta,\zeta}_{p,\mu} f(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{p+\zeta n}{p}\right)^k \frac{(\delta+p)_n (p+1-\mu)_n}{(1)_n (p+1)_n} a_{p+n} z^{p+n}.$$

Moreover, it is easy to verify from (1.3) that

$$\begin{aligned}
&\zeta z\big(\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z)\big)' = p\mathcal{N}^{k+1,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) - p(1-\zeta)\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) \quad \text{for } \zeta > 0,\\
&z\big(\mathcal{N}^{k,\delta,\zeta}_{p,\lambda+1,\mu,\eta} f(z)\big)' = (p+\eta-\lambda)\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z) - (\eta-\lambda)\mathcal{N}^{k,\delta,\zeta}_{p,\lambda+1,\mu,\eta} f(z),
\end{aligned} \tag{1.4}$$

and

$$z\big(\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z)\big)' = (\delta+p)\mathcal{N}^{k,\delta+1,\zeta}_{p,\lambda,\mu,\eta} f(z) - \delta\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta} f(z).$$

To obtain our results, we need to use the following definitions and theorems.

Definition 1.3 ([2, p. 441]) Let $\mathcal{Q}$ be the set of all functions q that are analytic and univalent on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{\xi \in \partial\mathbb{U} : \lim_{z\to\xi} q(z) = \infty\right\},$$

and are such that

$$\min\left|q'(\xi)\right| = \rho > 0 \quad \text{for } \xi \in \partial\mathbb{U} \setminus E(q).$$

Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ and $\mathcal{Q}(0) =: \mathcal{Q}_0$.

Definition 1.4 ([2, p. 449]) Let Ω be a subset of $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \geq 2$. The class of admissible operators $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition

$$\psi(r, s, t, u; z) \notin \Omega,$$

whenever

$$r = q(w), \qquad s = nwq'(w), \qquad \operatorname{Re}\frac{t}{s} + 1 \geq n \operatorname{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right),$$

and

$$\operatorname{Re}\frac{u}{s} \geq n^2 \operatorname{Re}\frac{w^2 q'''(w)}{q'(w)}, \quad z \in \mathbb{U}, w \in \partial\mathbb{U} \setminus E(q).$$

Lemma 1.1 ([2, p. 449]) *Let Ω be a subset of $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and $p \in \mathcal{H}[a, n]$ with $n \geq 2$. If $q \in \mathcal{Q}(a)$ and satisfies the following conditions*

$$\operatorname{Re}\frac{wq''(w)}{q'(w)} \geq 0 \quad \text{and} \quad \left|\frac{zp'(z)}{q'(w)}\right| \leq n, \quad z \in \mathbb{U}, w \in \partial\mathbb{U} \setminus E(q),$$

then

$$\psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) \in \Omega, \quad z \in \mathbb{U},$$

implies $p(z) \prec q(z)$.

The aim of the present article is to use the recent works by Tang et al. (see [25, 26]) to systematically investigate the third-order differential subordination general theory to a suitable classes of admissible functions. We obtained new results for a wide class of operators defined by convolution products with Srivastava–Saigo–Owa generalized fractional integral and generalized fractional derivative operators. Our results give interesting new properties and, together with other papers that appeared in the last years, could emphasize the perspective of the importance of the third-order subordination theory and the Srivastava–Saigo–Owa generalized operators.

We emphasize that in recent years, several authors obtained many interesting results involving different linear and convolution operators associated with second-order differential subordinations (see [19, 23]) and regarding the third-order differential subordinations (see [21]) for the above mentioned operator.

2 Main results

Unless otherwise mentioned, we assume throughout this paper that $f \in \mathcal{A}(p)$, $\zeta > 0$, $p \geq 2$, $w \in \partial\mathbb{U} \setminus E(q)$, $\theta \in [0, 2\pi]$ and $z \in \mathbb{U}$.

Definition 2.1 Let Ω be a subset of $\mathbb{C}$ and $q \in \mathcal{Q}_0$. The class of admissible operators $\Phi_p[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\alpha, \beta, \gamma, \varepsilon; z) \notin \Omega,$$

whenever

$$\alpha = q(w), \qquad \beta = w\zeta q'(w) + (1-\zeta)q(w),$$

$$\operatorname{Re}\left(\frac{p}{\zeta}\frac{\gamma + (1-\zeta)(1-\zeta+\frac{\zeta}{p})\alpha - (2-2\zeta+\frac{\zeta}{p})\beta}{\beta-(1-\zeta)\alpha} + 1\right) \geq p\operatorname{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right),$$

and

$$\frac{p^2}{\zeta^2}\operatorname{Re}\left(\frac{\begin{array}{c}\varepsilon - 3(1-\zeta+\frac{\zeta}{p})\gamma - (1-\zeta)(1-\zeta+\frac{\zeta}{p})(1-\zeta+\frac{2\zeta}{p})\alpha \\ + [2(1-\zeta+\frac{\zeta}{p})(2-2\zeta+\frac{\zeta}{p}) - (1-\zeta)^2]\beta\end{array}}{\beta-(1-\zeta)\alpha}\right)$$
$$\geq p^2\operatorname{Re}\frac{w^2q'''(w)}{q'(w)}.$$

Theorem 2.1 *Let Ω be a subset of $\mathbb{C}$ and $\phi \in \Phi_p[\Omega, q]$. If $q \in \mathcal{Q}_0$ satisfies the following conditions*:

$$\operatorname{Re}\frac{wq''(w)}{q'(w)} \geq 0 \quad \textit{and} \quad \left|\frac{z(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z))'}{q'(w)}\right| \leq p, \tag{2.1}$$

then

$$\left\{\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta}f(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega \tag{2.2}$$

implies

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z) \prec q(z).$$

Proof Defining the function p by

$$p(z) = \mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z), \tag{2.3}$$

then p is analytic in $\mathbb{U}$. Differentiating three times (2.3) with respect to z and using (1.4), we obtain the following relations, respectively:

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta}f(z)=\frac{\zeta}{p}zp'(z)+(1-\zeta)p(z),\tag{2.4}$$

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta}f(z)=\frac{\zeta^2}{p^2}z^2p''(z)+\frac{\zeta}{p}\left(2-2\zeta+\frac{\zeta}{p}\right)zp'(z)+(1-\zeta)^2p(z),\tag{2.5}$$

$$\begin{aligned}\mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta}f(z)=&\frac{\zeta^3}{p^3}z^3p'''(z)+\frac{3\zeta^2}{p^2}\left(1-\zeta+\frac{\zeta}{p}\right)z^2p''(z),\\&+\frac{\zeta}{p}\left[(1-\zeta)^2+\left(1-\zeta+\frac{\zeta}{p}\right)\left(2-2\zeta+\frac{\zeta}{p}\right)\right]zp'(z)+(1-\zeta)^3p(z).\end{aligned}\tag{2.6}$$

Letting

$$\begin{aligned}&\alpha(r,s,t,u)=r,\qquad \beta(r,s,t,u)=\frac{\zeta}{p}s+(1-\zeta)r,\\&\gamma(r,s,t,u)=\frac{\zeta^2}{p^2}t+\frac{\zeta}{p}\left(2-2\zeta+\frac{\zeta}{p}\right)s+(1-\zeta)^2r,\\&\varepsilon(r,s,t,u)=\frac{\zeta^3}{p^3}u+\frac{3\zeta^2}{p^2}\left(1-\zeta+\frac{\zeta}{p}\right)t\\&\qquad+\frac{\zeta}{p}\left[(1-\zeta)^2+\left(1-\zeta+\frac{\zeta}{p}\right)\left(2-2\zeta+\frac{\zeta}{p}\right)\right]s+(1-\zeta)^3r,\end{aligned}$$

we will define the transformation $\psi:\mathbb{C}^4\times\mathbb{U}\to\mathbb{C}$ by

$$\psi(r,s,t,u;z)=\phi(\alpha,\beta,\gamma,\varepsilon;z).\tag{2.7}$$

Then, using relations (2.3), (2.4), (2.5) and (2.6), we have

$$\begin{aligned}&\psi\left(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\right)\\&\quad=\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z),\mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta}f(z),\mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta}f(z),\mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta}f(z);z\right).\end{aligned}\tag{2.8}$$

Since

$$\frac{t}{s}+1=\frac{p}{\zeta}\frac{\gamma+(1-\zeta)(1-\zeta+\frac{\zeta}{p})\alpha-(2-2\zeta+\frac{\zeta}{p})\beta}{\beta-(1-\zeta)\alpha}+1,$$

and

$$\frac{u}{s}=\frac{p^2}{\zeta^2}\left(\frac{\begin{array}{c}\varepsilon-3(1-\zeta+\frac{\zeta}{p})\gamma-(1-\zeta)(1-\zeta+\frac{\zeta}{p})(1-\zeta+\frac{2\zeta}{p})\alpha\\+[2(1-\zeta+\frac{\zeta}{p})(2-2\zeta+\frac{\zeta}{p})-(1-\zeta)^2]\beta\end{array}}{\beta-(1-\zeta)\alpha}\right),$$

the admissibility condition for $\phi\in\Phi_p[\Omega,q]$ of Definition 2.1 is equivalent to the admissibility condition for ψ as given in Definition 1.4. Thus, the proof follows from Lemma 1.1 by setting $a=0$, $n=p$ and $a_p=1$. □

The next result is an extension of Theorem 2.1 for the case when the behavior of q on $\partial\mathbb{U}$ is unknown.

Corollary 2.1 *Let Ω be a subset of $\mathbb{C}$ and q be univalent in $\mathbb{U}$ with $q \in \mathcal{Q}_0$. Let $\phi \in \Phi_p[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If q_ρ satisfies the following conditions*:

$$\operatorname{Re}\frac{wq_\rho''(w)}{q_\rho'(w)} \geq 0 \quad \text{and} \quad \left|\frac{z(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z))'}{q_\rho'(w)}\right| \leq p, \quad w \in \partial\mathbb{U} \setminus E(q_\rho), \tag{2.9}$$

then

$$\left\{\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta}f(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega$$

implies

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z) \prec q(z).$$

Proof From Theorem 2.1 we obtain $\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z) \prec q_\rho(z)$ and since $q_\rho(z) \prec q(z)$, we conclude that $\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z) \prec q(z)$. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_p[h(\mathbb{U}), q]$ will be written as $\Phi_p[h, q]$. The following two results are direct consequences of Theorem 2.1 and Corollary 2.1.

Corollary 2.2 *Let $\phi \in \Phi_p[h, q]$, where h is univalent in $\mathbb{U}$ and suppose that $q \in \mathcal{Q}_0$ satisfies conditions* (2.1). *Then*

$$\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta}f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta}f(z); z\right) \prec h(z) \tag{2.10}$$

implies

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z) \prec q(z).$$

Corollary 2.3 *Let q be univalent in $\mathbb{U}$ with $q \in \mathcal{Q}_0$ and $\phi \in \Phi_p[h, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If q_ρ satisfies conditions* (2.9), *then the subordination* (2.10) *implies that*

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta}f(z) \prec q(z).$$

We next show the relation between the best dominant of a differential subordination and the solution of a corresponding differential equation.

Corollary 2.4 *Let h be univalent in $\mathbb{U}$ and ψ be given by* (2.8) *where $\phi \in \Phi_p[h, q]$. Suppose that the differential equation*

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z)$$

has a solution q with $q \in \mathcal{Q}_0$ that satisfies conditions (2.1). Then subordination (2.10) implies that

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z) \prec q(z),$$

and q is the best dominant of (2.10).

Proof Since

$$\begin{aligned}
&\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta} f(z); z\right) \\
&\quad = \psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) \prec h(z),
\end{aligned} \tag{2.11}$$

then p is a solution of (2.11), and from Corollary 2.2 we obtain that $p(z) \prec q(z)$, that is, q is a dominant of (2.11). Also,

$$\begin{aligned}
&\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta} f(z); z\right) \\
&\quad = \psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) \prec h(z) = \psi\left(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z\right),
\end{aligned}$$

which means that q is the best dominant of (2.11). □

3 Special Cases

We specialize the class of admissible functions and corresponding theorems for the case when $q(\mathbb{U})$ is the disk $\mathbb{U}_M := \{w \in \mathbb{C} : |w| < M\}$. First, we remark that the function

$$q(z) = Mz, \quad M > 0, z \in \mathbb{U}, \tag{3.1}$$

is univalent in $\overline{\mathbb{U}}$ and satisfies $q(\mathbb{U}) = \mathbb{U}_M$, $q \in \mathcal{Q}_0$ and $E(q) = \emptyset$.

Definition 3.1 Let Ω be a subset of $\mathbb{C}$ and q be given by (3.1). The class of admissible operators $\Phi_p[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\alpha_1, \beta_1, L, N; z) \notin \Omega, \tag{3.2}$$

whenever

$$\begin{aligned}
&\alpha_1 = \beta_1 = Me^{i\theta}, \quad M > 0, \\
&\operatorname{Re}\left(\frac{p}{\zeta^2} \frac{[L + (1-\zeta)(1-\zeta+\frac{\zeta}{p})\alpha_2 - (2-2\zeta+\frac{\zeta}{p})\beta_2]}{\alpha_1} + 1\right) \geq p,
\end{aligned}$$

and

$$\frac{p^2}{\zeta^2} \operatorname{Re}\begin{pmatrix} N - 3(1-\zeta+\frac{\zeta}{p})L - (1-\zeta)(1-\zeta+\frac{\zeta}{p})(1-\zeta+\frac{2\zeta}{p})\alpha_2 \\ + [2(1-\zeta+\frac{\zeta}{p})(2-2\zeta+\frac{\zeta}{p}) - (1-\zeta)^2]\beta_2 \end{pmatrix} \geq 0,$$

where $\operatorname{Re}(Le^{-i\theta}) \geq p(p-1)M$ and $\operatorname{Re}(Ne^{-i\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$ and $p \geq 2$.

Using this definition of the class of admissible functions, from Theorem 2.1 we obtain the following result.

Corollary 3.1 *Let Ω be a subset of $\mathbb{C}$ and $\phi \in \Phi_p[\Omega, M]$. If we suppose that*

$$\left|z\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z)\right)'\right| \le pM, \quad z \in \mathbb{U}, \tag{3.3}$$

and the function q is given by (3.1), then

$$\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta} f(z); z\right) \in \Omega, \quad z \in \mathbb{U},$$

implies

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z) \prec Mz.$$

For the special case when $\Omega = q(\mathbb{U}) = \{w \in \mathbb{C} : |w| < M\}$, Corollary 3.1 reduces to the following corollary.

Corollary 3.2 *Let $\phi \in \Phi_p[q(\mathbb{U}), M]$ and suppose that the function q given by (3.1) satisfies condition (3.3). Then*

$$\left|\phi\left(\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+1,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+2,\delta,\zeta} f(z), \mathcal{N}_{p,\lambda,\mu,\eta}^{k+3,\delta,\zeta} f(z); z\right)\right| < M, \quad z \in \mathbb{U},$$

implies

$$\mathcal{N}_{p,\lambda,\mu,\eta}^{k,\delta,\zeta} f(z) \prec Mz.$$

Let $\phi(\alpha_1, \beta_1, \gamma_1, \varepsilon_1; z) = \alpha_1 + \beta_1$ and $\Omega = h(\mathbb{U})$, where $h(z) = 2Mz$. We will show that $\phi \in \Phi_p[h(\mathbb{U}), M]$ by proving that condition (3.2) is satisfied. Thus,

$$\left|\phi(\alpha_1, \beta_1, L, N; z)\right| = \left|Me^{i\theta} + Me^{i\theta}\right| = 2M,$$

where $\operatorname{Re}(Le^{-i\theta}) \ge p(p-1)M$, $\operatorname{Re}(Ne^{-i\theta}) \ge 0$ for all $\theta \in [0, 2\pi]$ and $p \ge 2$.

Suppose that A and B are two complex-valued functions defined on $\mathbb{U}$ that satisfy $\operatorname{Re} A(z) > 0$, $\operatorname{Re} B(z) > 0$ for all $z \in \mathbb{U}$. Let $\phi(\alpha_1, \beta_1, \gamma_1, \varepsilon_1; z) = 1 + A(z)\alpha_1 + B(z)\beta_1$ and $\Omega = h(\mathbb{U})$, where $h(z) = z$. We will show that $\phi \in \Phi_p[h(\mathbb{U}), M]$ by proving that condition (3.2) is satisfied. Since

$$\left|\phi(\alpha_1, \beta_1, L, N; z)\right| = \left|1 + A(z)Me^{i\theta} + B(z)Me^{i\theta}\right| \ge 1 + M \operatorname{Re}\left[A(z) + B(z)\right] > 1, \quad z \in \mathbb{U}$$

where $\operatorname{Re}(Le^{-i\theta}) \ge p(p-1)M$, $\operatorname{Re}(Ne^{-i\theta}) \ge 0$ for all $\theta \in [0, 2\pi]$ and $p \ge 2$.

Let $A : \mathbb{U} \to (1, +\infty)$, $B : \mathbb{U} \to (0, +\infty)$, $\phi(\alpha_1, \beta_1, \gamma_1, \varepsilon_1; z) = \alpha_1 + \beta_1 + A(z)\gamma_1 + B(z)\varepsilon_1$ and $\Omega = h(\mathbb{U})$, where $h(z) = 4Mz$. We will show that $\phi \in \Phi_p[h(\mathbb{U}), M]$ by proving that condition (3.2) is satisfied. Thus,

$$\left|\phi(\alpha_1, \beta_1, L, N; z)\right| = \left|2Me^{i\theta} + A(z)L + B(z)N\right|$$

$$= \left|2M + A(z)Le^{-i\theta} + B(z)Ne^{-i\theta}\right|$$
$$\geq 2M + A(z)\operatorname{Re}\left(Le^{-i\theta}\right) + B(z)\operatorname{Re}\left(Ne^{-i\theta}\right)$$
$$\geq 2M + p(p-1)MA(z) \geq 2M + p(p-1)M \geq 4M, \quad z \in \mathbb{U},$$

where $\operatorname{Re}(Le^{-i\theta}) \geq p(p-1)M$, $\operatorname{Re}(Ne^{-i\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$ and $p \geq 2$, from Corollary 3.2 we have the following special case.

Example 3.1 If $A : \mathbb{U} \to (1, +\infty)$, $B : \mathbb{U} \to (0, +\infty)$ and $f \in \mathcal{A}(p)$ such that

$$\left|z\left(\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right)'\right| \leq pM, \quad z \in \mathbb{U},$$

then

$$\left|\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) + \mathcal{N}^{k+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) + A(z)\mathcal{N}^{k+2,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) + B(z)\mathcal{N}^{k+3,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right| < 4M, \quad z \in \mathbb{U},$$

implies that

$$\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) \prec Mz.$$

Let $A, B : \mathbb{U} \to \mathbb{C}$, with $\operatorname{Re}[A(z) + B(z)] > 0$ for all $z \in \mathbb{U}$, let $C : \mathbb{U} \to (1, +\infty)$, $D : \mathbb{U} \to (0, +\infty)$, $\phi(\alpha_1, \beta_1, \gamma_1, \varepsilon_1; z) = A(z)\alpha_1 + B(z)\beta_1 + C(z)\gamma_1 + D(z)\varepsilon_1$ and $\Omega = h(\mathbb{U})$, where $h(z) = 2Mz$. We will show that $\phi \in \Phi_p[h(\mathbb{U}), M]$ by proving that condition (3.2) is satisfied. Thus,

$$\begin{aligned}\left|\phi(\alpha_1, \beta_1, L, N; z)\right| &= \left|Me^{i\theta}A(z) + Me^{i\theta}B(z) + C(z)L + D(z)N\right| \\ &= \left|M\left[A(z) + B(z)\right] + C(z)Le^{-i\theta} + D(z)Ne^{-i\theta}\right| \\ &\geq M\operatorname{Re}\left[A(z) + B(z)\right] + C(z)\operatorname{Re}\left(Le^{-i\theta}\right) + D(z)\operatorname{Re}\left(Ne^{-i\theta}\right) \\ &\geq p(p-1)MC(z) \geq p(p-1)M \geq 2M, \quad z \in \mathbb{U},\end{aligned}$$

where $\operatorname{Re}(Le^{-i\theta}) \geq p(p-1)M$, $\operatorname{Re}(Ne^{-i\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$ and $p \geq 2$, from Corollary 3.2 we have the following special case.

Example 3.2 Let $A, B : \mathbb{U} \to \mathbb{C}$, with $\operatorname{Re}[A(z) + B(z)] > 0$ for all $z \in \mathbb{U}$, and $C : \mathbb{U} \to (1, +\infty)$, $D : \mathbb{U} \to (0, +\infty)$. If $f \in \mathcal{A}(p)$ such that

$$\left|z\left(\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right)'\right| \leq pM, \quad z \in \mathbb{U},$$

then

$$\left|A(z)\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) + B(z)\mathcal{N}^{k+1,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) + C(z)\mathcal{N}^{k+2,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) + D(z)\mathcal{N}^{k+3,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right| < 2M,$$
$$z \in \mathbb{U},$$

implies that

$$\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) \prec Mz.$$

Let $\phi(\alpha_1,\beta_1,\gamma_1,\varepsilon_1;z) = 1 + \frac{\gamma_1}{\alpha_1} + \frac{\varepsilon_1}{\alpha_1}$ and $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < 3\}$. We will show that $\phi \in \Phi_p[\Omega, M]$ by proving that condition (3.2) is satisfied. Thus,

$$\operatorname{Re}\phi(\alpha_1,\beta_1,L,N;z) = \operatorname{Re}\left(1 + \frac{Le^{-i\theta}}{M} + \frac{Ne^{-i\theta}}{M}\right) \geq 1 + p(p-1) \geq 3, \quad z \in \mathbb{U},$$

where $\operatorname{Re}(Le^{-i\theta}) \geq p(p-1)M$ and $\operatorname{Re}(Ne^{-i\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$ and $p \geq 2$, from Corollary 3.2 we obtain the following.

Example 3.3 If $f \in \mathcal{A}(p)$ such that

$$\left|z\left(\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)\right)'\right| \leq pM, \quad z \in \mathbb{U},$$

then

$$\operatorname{Re}\left(1 + \frac{\mathcal{N}^{k+2,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)} + \frac{\mathcal{N}^{k+3,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}{\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z)}\right) < 3, \quad z \in \mathbb{U},$$

implies that

$$\mathcal{N}^{k,\delta,\zeta}_{p,\lambda,\mu,\eta}f(z) \prec Mz.$$

Remark 3.1 For different choices of k, δ, ζ, p, λ, μ, and η, we will obtain new results for different operators defined in the introduction.

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