Wang et al. *Journal of Inequalities and Applications* (2019) 2019:243
https://doi.org/10.1186/s13660-019-2196-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalizations and applications of Young's integral inequality by higher order derivatives

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Abstract

In the paper, the authors

1. generalize Young's integral inequality via Taylor's theorems in terms of higher order derivatives and their norms, and
2. apply newly-established integral inequalities to estimate several concrete definite integrals, including a definite integral of a function which plays an indispensable role in differential geometry and has a connection with the Lah numbers in combinatorics, the exponential integral, and the logarithmic integral.

MSC: Primary 26D15; secondary 26A42; 26A48; 26A51; 26D05; 26D07; 33B10; 33B20; 41A58

Keywords: Young's integral inequality; Inverse function; Taylor's theorem; Higher order derivative; Lebesgue measure; Norm; Application; Exponential integral; Logarithmic integral; Existence of partitions of unity

1 A concise retrospection

In 1912, Young proved the following integral inequality in which an inverse function is involved.

Theorem 1.1 ([29]) *Let $h(x)$ be a continuous and strictly increasing function on $[0,c]$ with $c>0$. If $h(0)=0$, $a\in[0,c]$, and $b\in[0,h(c)]$, then*

$$\int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x\ge ab, \tag{1.1}$$

where h^{-1} is the inverse function of h. The equality in (1.1) is valid if and only if $b=h(a)$.

For avoiding confusion, we call inequality (1.1) Young's integral inequality, because several other inequalities, such as

$$\sum_{k=1}^{n}\frac{\cos(k\theta)}{k}>-1,\qquad n\ge2,\theta\in[0,\pi]$$

and the weighted arithmetic-geometric mean inequality, are also called Young's inequality.

In [9, Secton 2.7] and [10, Chapter XIV], many extensions, refinements, generalizations, and applications of Young's integral inequality (1.1) in Theorem 1.1 were collected and reviewed.

In 2008, Hoorfar and Qi obtained the following double inequality which refines Young's integral inequality (1.1).

Theorem 1.2 ([4]) *Let $h(x)$ be a differentiable and strictly increasing function on $[0,c]$ for $c>0$, and let h^{-1} be the inverse function of h. If $h(0)=0$, $a\in[0,c]$, $b\in[0,h(c)]$, and $h'(x)$ is strictly monotonic on $[0,c]$, then*

$$\frac{m}{2}\left[a-h^{-1}(b)\right]^2\le\int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab\le\frac{M}{2}\left[a-h^{-1}(b)\right]^2, \tag{1.2}$$

where

$$m=\min\left\{h'(a),h'\left(h^{-1}(b)\right)\right\}\quad\text{and}\quad M=\max\left\{h'(a),h'\left(h^{-1}(b)\right)\right\}.$$

The equalities in (1.2) are valid if and only if $b=h(a)$.

In 2009 and 2010, motivated by Theorem 1.2 and its proof in [4], among other things, Jakšetić and Pečarić extended and generalized Young's integral inequality (1.1) and Hoorfar–Qi's double inequality (1.2) as the following theorems.

Theorem 1.3 ([5, Theorem 2.1] and [6, Theorem 2.3]) *Let $h(x)$ be a differentiable and strictly increasing function on $[0,c]$ for $c>0$, $h(0)=0$, $a\in[0,c]$, $b\in[0,h(c)]$, and h^{-1} be the inverse function of h. Denote*

$$\alpha=\min\left\{a,h^{-1}(b)\right\}\quad\text{and}\quad\beta=\max\left\{a,h^{-1}(b)\right\}. \tag{1.3}$$

1. *If $h'(x)$ is increasing on $[\alpha,\beta]$ and $b<h(a)$, or if $h'(x)$ is decreasing on $[\alpha,\beta]$ and $b>h(a)$, then*

$$\begin{aligned}\left[a-h^{-1}(b)\right]\left[h\left(\frac{a+h^{-1}(b)}{2}\right)-b\right]&\le\int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab\\&\le\frac{1}{2}\left[a-h^{-1}(b)\right]\left[h(a)-b\right].\end{aligned} \tag{1.4}$$

2. *If $h'(x)$ is increasing on $[\alpha,\beta]$ and $b>h(a)$, or if $h'(x)$ is decreasing on $[\alpha,\beta]$ and $b<h(a)$, then inequality (1.4) is reversed.*
3. *The equality in (1.4) is valid if and only if $h(x)=\lambda x$ for $\lambda>0$ or $b=h(a)$.*

Theorem 1.4 ([6, Theorem 2.6]) *Let $h(x)$ be a differentiable and strictly increasing function on $[0,c]$ for $c>0$, and let h^{-1} be the inverse function of h. If $h(0)=0$, $a\in[0,c]$,*

$b \in [0, h(c)]$, and $h'(x)$ is convex on $[\alpha, \beta]$, then

$$\begin{aligned}\frac{[a-h^{-1}(b)]^2}{2} h'\left(\frac{a+2h^{-1}(b)}{3}\right) &\le \int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \\ &\le \frac{[a-h^{-1}(b)]^2}{3}\left[\frac{h'(a)}{2} + h'\big(h^{-1}(b)\big)\right].\end{aligned} \tag{1.5}$$

If $h'(x)$ is concave, then the double inequality (1.5) is reversed.

Theorem 1.5 ([6, Theorem 2.1]) *Let $h(x)$ be a differentiable and strictly increasing function on $[0, c]$ for $c > 0$, and let h^{-1} be the inverse function of h. If $h(0) = 0$, $a \in [0, c]$, $b \in [0, h(c)]$, and $h'(x)$ is almost everywhere continuous with respect to Lebesgue measure on $[\alpha, \beta]$, then the double inequality*

$$C_u \|h'\|_v \le \int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \le C_p \|h'\|_q \tag{1.6}$$

is valid for all u, v and p, q satisfying

1. $\frac{1}{u} + \frac{1}{v} = 1$ *for* $u, v \in (-\infty, 0) \cup (0, 1)$, *or* $(u, v) = (1, -\infty)$, *or* $(u, v) = (-\infty, 1)$;
2. $\frac{1}{p} + \frac{1}{q} = 1$ *for* $1 < p, q < \infty$, *or* $(p, q) = (+\infty, 1)$, *or* $(p, q) = (1, +\infty)$;

where

$$C_r = \begin{cases} [\frac{|a-h^{-1}(b)|^{r+1}}{r+1}]^{1/r}, & r \ne 0, \pm\infty; \\ |a - h^{-1}(b)|, & r = +\infty; \\ 0, & r = -\infty \end{cases}$$

and

$$\|h'\|_r = \begin{cases} [\int_\alpha^\beta [h'(t)]^r \,\mathrm{d}t]^{1/r}, & r \ne 0, \pm\infty; \\ \sup\{h'(t), t \in [\alpha, \beta]\}, & r = +\infty; \\ \inf\{h'(t), t \in [\alpha, \beta]\}, & r = -\infty. \end{cases}$$

We note that Hoorfar–Qi's double inequality (1.2) has been applied and employed in the paper [7] and in the Undergraduate Texts in Mathematics [8].

In this paper, by virtue of Taylor's theorems with different remainders, we establish some integral inequalities of Hoorfar–Qi's type in terms of higher order derivatives and their norms, demonstrate that these newly-established integral inequalities generalize Young's integral inequality (1.1), Hoorfar–Qi's integral inequality (1.2), and Jakšetić–Pečarić's integral inequalities (1.4), (1.5), and (1.6), and apply these integral inequalities to estimate several concrete definite integrals, including a definite integral of $e^{-1/x}$ which plays an indispensable role in differential geometry and has a connection with the Lah numbers in combinatorics, the exponential integral $\mathrm{Ei}(x)$, and the logarithmic integral $\mathrm{li}(x)$.

2 Lemmas

For proving our main result, we need Taylor's theorems below.

Lemma 2.1 ([2, p. 113, Theorem 5.19]) *Let $f(x)$ be a function having finite nth derivative $f^{(n)}(x)$ everywhere in an open interval (μ,ν) and assume that $f^{(n-1)}(x)$ is continuous on the closed interval $[\mu,\nu]$. Then, for a fixed point $x_0\in[\mu,\nu]$ and every $x\in[\mu,\nu]$ with $x\ne x_0$, there exists a point x_1 interior to the interval joining x and x_0 such that*

$$f(x)=f(x_0)+\sum_{k=1}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n)}(x_1)}{n!}(x-x_0)^n. \tag{2.1}$$

Lemma 2.2 ([1, p. 279, Theorem 7.6] and [11, p. 6, 1.4.37]) *If $f(x)\in C^{n+1}[\mu,\nu]$ and $x_0\in[\mu,\nu]$, then*

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,\mathrm{d}t. \tag{2.2}$$

For proving our main results, we also need Hölder's integral inequality and its reversed version, Čebyšev's integral inequality, discrete and integral versions of Jensen's inequality, and Hermite–Hadamard's integral inequality.

Lemma 2.3 ([10, Chapter V] and [23–25]) *Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>0$ and $p\ne1$, let f and g be real functions on $[\mu,\nu]$, and let $|f|^p$ and $|g|^q$ be integrable on $[\mu,\nu]$.*

1. *If $p>1$, then*

$$\int_{\mu}^{\nu}\left|f(x)g(x)\right|\mathrm{d}x\le\left[\int_{\mu}^{\nu}\left|f(x)\right|^p\mathrm{d}x\right]^{1/p}\left[\int_{\mu}^{\nu}\left|g(x)\right|^q\mathrm{d}x\right]^{1/q}. \tag{2.3}$$

   The equality in (2.3) holds if and only if $A|f(x)|^p=B|g(x)|^q$ almost everywhere for two constants A and B.

2. *If $p<1$ and $p\ne0$, then inequality (2.3) is reversed.*

Lemma 2.4 ([10, Chapter IX] and [19]) *Let $f,g:[\mu,\nu]\to\mathbb{R}$ be integrable functions satisfying that they are both increasing or both decreasing. Then*

$$\int_{\mu}^{\nu}f(x)\,\mathrm{d}x\int_{\mu}^{\nu}g(x)\,\mathrm{d}x\le(\nu-\mu)\int_{\mu}^{\nu}f(x)g(x)\,\mathrm{d}x. \tag{2.4}$$

If one of the functions f or g is nonincreasing and the other nondecreasing, then the inequality in (2.4) is reversed.

Lemma 2.5 ([9, Sect. 1.4] and [10, Chapter I]) *If f is a convex function on an interval $I\subseteq\mathbb{R}$ and if $n\ge2$ and $x_k\in I$ for $1\le k\le n$, then*

$$f\left(\frac{1}{\sum_{k=1}^{n}p_k}\sum_{k=1}^{n}p_kx_k\right)\le\frac{1}{\sum_{k=1}^{n}p_k}\sum_{k=1}^{n}p_kf(x_k), \tag{2.5}$$

where $p_k>0$ for $1\le k\le n$. If f is concave, inequality (2.5) is reversed.

Let ϕ be a convex function on $[\mu,\nu]$, $f\in L_1(\mu,\nu)$, and σ be a nonnegative measure. Then

$$\phi\left(\frac{\int_\mu^\nu f(x)\,\mathrm{d}\sigma}{\int_\mu^\nu \mathrm{d}\sigma}\right)\le\frac{\int_\mu^\nu \phi(f(x))\,\mathrm{d}\sigma}{\int_\mu^\nu \mathrm{d}\sigma}. \tag{2.6}$$

If ϕ is a concave function, then inequality (2.6) is reversed.

These five lemmas are general knowledge in mathematics and they will continue to play important roles in this paper.

Lemma 2.6 ([12, 26–28]) *Let $f(x)$ and $g(x)$ be nonnegative and convex functions on $[\mu,\nu]$. Then*

$$\begin{aligned}&2f\left(\frac{\mu+\nu}{2}\right)g\left(\frac{\mu+\nu}{2}\right)-\frac{1}{6}M(\mu,\nu)-\frac{1}{3}N(\mu,\nu)\\&\quad\le\frac{1}{\nu-\mu}\int_\mu^\nu f(x)g(x)\,\mathrm{d}x\le\frac{1}{3}M(\mu,\nu)+\frac{1}{6}N(\mu,\nu),\end{aligned} \tag{2.7}$$

where

$$M(\mu,\nu)=f(\mu)g(\mu)+f(\nu)g(\nu)\quad\text{and}\quad N(\mu,\nu)=f(\mu)g(\nu)+f(\nu)g(\mu).$$

3 Main results and proofs

Now we are in a position to state and prove our main results.

Theorem 3.1 *Let $h(0)=0$ and $h(x)$ be strictly increasing on $[0,c]$ for $c>0$, let $h^{(n)}(x)$ for $n\ge 0$ be continuous on $[0,c]$, let $h^{(n+1)}(x)$ be finite and strictly monotonic on $(0,c)$, and let h^{-1} be the inverse function of h. For $a\in[0,c]$ and $b\in[0,h(c)]$,*

1. *if $b<h(a)$, then*

$$\begin{aligned}&\sum_{k=1}^{n}h^{(k)}\big(h^{-1}(b)\big)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!}+m_n(a,b)\frac{[a-h^{-1}(b)]^{n+2}}{(n+2)!}\\&\quad\le\int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab\\&\quad\le\sum_{k=1}^{n}h^{(k)}\big(h^{-1}(b)\big)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!}+M_n(a,b)\frac{[a-h^{-1}(b)]^{n+2}}{(n+2)!},\end{aligned} \tag{3.1}$$

where

$$m_n(a,b)=\min\big\{h^{(n+1)}\big(h^{-1}(b)\big),h^{(n+1)}(a)\big\}$$

and

$$M_n(a,b)=\max\big\{h^{(n+1)}\big(h^{-1}(b)\big),h^{(n+1)}(a)\big\};$$

2. *if $b>h(a)$, then*
   (a) *when $n=2\ell$ for $\ell\ge 0$, the double inequality (3.1) is valid;*

(b) *when* $n = 2\ell + 1$ *for* $\ell \ge 0$, *we have*

$$\begin{aligned}
&\sum_{k=1}^{n} h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} - M_n(a,b)\frac{[a-h^{-1}(b)]^{n+2}}{(n+2)!} \\
&\quad\le \int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \\
&\quad\le \sum_{k=1}^{n} h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} - m_n(a,b)\frac{[a-h^{-1}(b)]^{n+2}}{(n+2)!};
\end{aligned} \tag{3.2}$$

3. *if, and only if,* $b = h(a)$, *those equalities in* (3.1) *and* (3.2) *hold.*

Proof Employing several basic properties of definite integrals, such as substitution of variables and integration by parts, reveals

$$\begin{aligned}
&\int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x \\
&\quad= \int_0^a h(x)\,\mathrm{d}x + \int_0^{h^{-1}(b)} y\,\mathrm{d}h(y) \\
&\quad= \int_0^a h(x)\,\mathrm{d}x + yh(y)|_0^{h^{-1}(b)} - \int_0^{h^{-1}(b)} h(y)\,\mathrm{d}y = bh^{-1}(b) + \int_{h^{-1}(b)}^a h(y)\,\mathrm{d}y \\
&\quad= ab + \int_{h^{-1}(b)}^a \bigl[h(y)-b\bigr]\,\mathrm{d}y = ab + \int_{h^{-1}(b)}^a \bigl[h(y)-h\bigl(h^{-1}(b)\bigr)\bigr]\,\mathrm{d}y.
\end{aligned} \tag{3.3}$$

By virtue of formula (2.1), we have

$$\begin{aligned}
h(y) - h\bigl(h^{-1}(b)\bigr) &= \sum_{k=1}^{n} \frac{h^{(k)}(h^{-1}(b))}{k!}\bigl[y-h^{-1}(b)\bigr]^k \\
&\quad+ \frac{h^{(n+1)}(\xi)}{(n+1)!}\bigl[y-h^{-1}(b)\bigr]^{n+1},
\end{aligned}$$

where ξ is a point interior to the interval joining y and $h^{-1}(b)$. As a result,

$$\begin{aligned}
&\int_{h^{-1}(b)}^a \bigl[h(y)-h\bigl(h^{-1}(b)\bigr)\bigr]\,\mathrm{d}y \\
&\quad= \sum_{k=1}^{n} \frac{h^{(k)}(h^{-1}(b))}{k!}\int_{h^{-1}(b)}^a \bigl[y-h^{-1}(b)\bigr]^k\,\mathrm{d}y \\
&\qquad+ \frac{1}{(n+1)!}\int_{h^{-1}(b)}^a h^{(n+1)}(\xi)\bigl[y-h^{-1}(b)\bigr]^{n+1}\,\mathrm{d}y \\
&\quad= \sum_{k=1}^{n} h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} + \int_{h^{-1}(b)}^a h^{(n+1)}(\xi)\frac{[y-h^{-1}(b)]^{n+1}}{(n+1)!}\,\mathrm{d}y.
\end{aligned}$$

When $h^{-1}(b) < a$, if $h^{(n+1)}(x)$ is strictly increasing on $(0,c)$, then

$$h^{(n+1)}\bigl(h^{-1}(b)\bigr) < h^{(n+1)}(\xi) < h^{(n+1)}(y) \le h^{(n+1)}(a);$$

if $h^{(n+1)}(x)$ is strictly decreasing on $(0,c)$, then

$$h^{(n+1)}(a) \le h^{(n+1)}(y) < h^{(n+1)}(\xi) < h^{(n+1)}\bigl(h^{-1}(b)\bigr).$$

Consequently, the double inequality $m_n(a,b) < h^{(n+1)}(\xi) < M_n(a,b)$ is valid. Thus, it follows that

$$\begin{aligned} m_n(a,b)\frac{[a-h^{-1}(b)]^{n+2}}{(n+2)!} &\le \int_{h^{-1}(b)}^{a} h^{(n+1)}(\xi)\frac{[y-h^{-1}(b)]^{n+1}}{(n+1)!}\,\mathrm{d}y \\ &\le M_n(a,b)\frac{[a-h^{-1}(b)]^{n+2}}{(n+2)!}. \end{aligned}$$

Hence, the double inequality (3.1) is proved.

When $h^{-1}(b) > a$, if $h^{(n+1)}(x)$ is increasing on $(0,c)$, then

$$h^{(n+1)}\bigl(h^{-1}(b)\bigr) \ge h^{(n+1)}(\xi) \ge h^{(n+1)}(y) \ge h^{(n+1)}(a);$$

if $h^{(n+1)}(x)$ is decreasing on $(0,c)$, then

$$h^{(n+1)}(a) \ge h^{(n+1)}(y) \ge h^{(n+1)}(\xi) \ge h^{(n+1)}\bigl(h^{-1}(b)\bigr).$$

Consequently, the double inequality $m_n(a,b) < h^{(n+1)}(\xi) < M_n(a,b)$ is still valid. Hence, since

$$\begin{aligned} &\int_{h^{-1}(b)}^{a} h^{(n+1)}(\xi)\frac{[y-h^{-1}(b)]^{n+1}}{(n+1)!}\,\mathrm{d}y \\ &\quad= \frac{(-1)^n}{(n+1)!}\int_{a}^{h^{-1}(b)} h^{(n+1)}(\xi)\bigl[h^{-1}(b)-y\bigr]^{n+1}\,\mathrm{d}y, \end{aligned}$$

we acquire

$$\begin{aligned} \frac{(-1)^n m_n(a,b)}{(n+2)!}\bigl[h^{-1}(b)-a\bigr]^{n+2} &\le \int_{h^{-1}(b)}^{a} h^{(n+1)}(\xi)\frac{[y-h^{-1}(b)]^{n+1}}{(n+1)!}\,\mathrm{d}y \\ &\le \frac{(-1)^n M_n(a,b)}{(n+2)!}\bigl[h^{-1}(b)-a\bigr]^{n+2}, \quad n=2\ell, \end{aligned}$$

and

$$\begin{aligned} \frac{(-1)^n m_n(a,b)}{(n+2)!}\bigl[h^{-1}(b)-a\bigr]^{n+2} &\ge \int_{h^{-1}(b)}^{a} h^{(n+1)}(\xi)\frac{[y-h^{-1}(b)]^{n+1}}{(n+1)!}\,\mathrm{d}y \\ &\ge \frac{(-1)^n M_n(a,b)}{(n+2)!}\bigl[h^{-1}(b)-a\bigr]^{n+2}, \quad n=2\ell+1 \end{aligned}$$

for $\ell \ge 0$. The double inequality (3.2) is thus proved. The proof of Theorem 3.1 is complete. □

Remark 3.1 Taking $n = 0$ in Theorem 3.1 leads to the above Theorem 1.2 in the paper [4].

Theorem 3.2 *Let $n \geq 0$ and $h(x) \in C^{n+1}[0,c]$ such that $h(0) = 0$, $h^{(n+1)}(x) \geq 0$ on $[\alpha,\beta]$, and $h(x)$ is strictly increasing on $[0,c]$ for $c > 0$, let h^{-1} be the inverse function of h, and let $a \in [0,c]$ and $b \in [0,h(c)]$. Then*

1. *when $b > h(a)$ and $n = 2\ell$ for $\ell \geq 0$ or when $b < h(a)$, we have*

$$\begin{aligned}\frac{C_{u,n}}{(n+1)!}\|h^{(n+1)}\|_v \leq{}& \int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \\ &- \sum_{k=1}^n h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} \leq \frac{C_{p,n}}{(n+1)!}\|h^{(n+1)}\|_q;\end{aligned}$$

2. *when $b > h(a)$ and $n = 2\ell+1$ for $\ell \geq 0$, we have*

$$\begin{aligned}-\frac{C_{p,n}}{(n+1)!}\|h^{(n+1)}\|_q \leq{}& \int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \\ &- \sum_{k=1}^n h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} \leq -\frac{C_{u,n}}{(n+1)!}\|h^{(n+1)}\|_v;\end{aligned}$$

where α, β are defined as in (1.3),

$$C_{r,n} = \begin{cases} \left[\frac{|a-h^{-1}(b)|^{r(n+1)+1}}{r(n+1)+1}\right]^{1/r}, & r \neq 0, \pm\infty; \\ |a-h^{-1}(b)|^{n+1}, & r = +\infty; \\ 0, & r = -\infty, \end{cases}$$

$$\|h^{(n+1)}\|_r = \begin{cases} \left[\int_\alpha^\beta [h^{(n+1)}(t)]^r\,\mathrm{d}t\right]^{1/r}, & r \neq 0, \pm\infty; \\ \sup\{h^{(n+1)}(t), t \in [\alpha,\beta]\}, & r = +\infty; \\ \inf\{h^{(n+1)}(t), t \in [\alpha,\beta]\}, & r = -\infty, \end{cases}$$

and u, v, p, q satisfy

1. *$u < 1$ and $u \neq 0$ with $\frac{1}{u} + \frac{1}{v} = 1$, or $(u,v) = (-\infty,1)$, or $(u,v) = (1,-\infty)$;*
2. *$1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, or $(p,q) = (+\infty,1)$, or $(p,q) = (1,+\infty)$.*

Proof Applying formula (2.2) to the last term in (3.3) yields

$$\begin{aligned}&\int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \\ &= \sum_{k=1}^n h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} + \int_{h^{-1}(b)}^a \frac{1}{n!}\int_{h^{-1}(b)}^x (x-t)^n h^{(n+1)}(t)\,\mathrm{d}t\,\mathrm{d}x \\ &= \sum_{k=1}^n h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} + \int_{h^{-1}(b)}^a \frac{1}{n!}\int_t^a (x-t)^n h^{(n+1)}(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_{k=1}^n h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} + \int_{h^{-1}(b)}^a \frac{(a-t)^{n+1}h^{(n+1)}(t)}{(n+1)!}\,\mathrm{d}t. \qquad (3.4)\end{aligned}$$

It is not difficult to verify that

$$\int_{h^{-1}(b)}^{a} (a-t)^{n+1} h^{(n+1)}(t)\,\mathrm{d}t = \begin{cases} \int_\alpha^\beta |a-t|^{n+1} h^{(n+1)}(t)\,\mathrm{d}t, & b < h(a); \\ (-1)^n \int_\alpha^\beta |a-t|^{n+1} h^{(n+1)}(t)\,\mathrm{d}t, & b > h(a). \end{cases}$$

Since

$$\begin{aligned} \int_\alpha^\beta |a-t|^{n+1} h^{(n+1)}(t)\,\mathrm{d}t &\le \left|a - f^{-1}(b)\right|^{n+1} \int_\alpha^\beta h^{(n+1)}(t)\,\mathrm{d}t \\ &= \left|a - f^{-1}(b)\right|^{n+1} \left\|h^{(n+1)}\right\|_1, \\ \int_\alpha^\beta |a-t|^{n+1} h^{(n+1)}(t)\,\mathrm{d}t &\le \sup\left\{h^{(n+1)}(t), t \in [\alpha,\beta]\right\} \int_\alpha^\beta |a-t|^{n+1}\,\mathrm{d}t \\ &= \frac{(\beta-\alpha)^{n+2}}{n+2} \left\|h^{(n+1)}\right\|_{+\infty} = \frac{|a-h^{-1}(b)|^{n+2}}{n+2} \left\|h^{(n+1)}\right\|_{+\infty}, \end{aligned}$$

and, by Hölder's integral inequality (2.3),

$$\begin{aligned} &\int_\alpha^\beta |a-t|^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \\ &\quad\le \left(\int_\alpha^\beta |a-t|^{p(n+1)}\,\mathrm{d}t\right)^{1/p} \left(\int_\alpha^\beta \left[h^{(n+1)}(t)\right]^q\,\mathrm{d}t\right)^{1/q} \\ &\quad= \left[\frac{(\beta-\alpha)^{p(n+1)+1}}{p(n+1)+1}\right]^{1/p} \left\|h^{(n+1)}\right\|_q = \left[\frac{|a-h^{-1}(b)|^{p(n+1)+1}}{p(n+1)+1}\right]^{1/p} \left\|h^{(n+1)}\right\|_q \end{aligned}$$

for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, it follows that

1. when $b > h(a)$ and $n = 2\ell$ for $\ell \ge 0$ or when $b < h(a)$, we have

$$\begin{aligned} &\int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \\ &\quad\le \sum_{k=1}^{n} h^{(k)}\left(h^{-1}(b)\right) \frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} + \frac{C_{p,n}}{(n+1)!} \left\|h^{(n+1)}\right\|_q; \end{aligned}$$

2. when $b > h(a)$ and $n = 2\ell + 1$ for $\ell \ge 0$, we have

$$\begin{aligned} &\int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab \\ &\quad\ge \sum_{k=1}^{n} h^{(k)}\left(h^{-1}(b)\right) \frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} - \frac{C_{p,n}}{(n+1)!} \left\|h^{(n+1)}\right\|_q; \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p, q < \infty$, or $(p,q) = (+\infty, 1)$, or $(p,q) = (1, +\infty)$.

On the other hand, since

$$\int_\alpha^\beta |a-t|^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \ge 0 \int_\alpha^\beta h^{(n+1)}(t)\,\mathrm{d}t = 0\left\|h^{(n+1)}\right\|_1,$$

$$\begin{aligned}\int_\alpha^\beta |a-t|^{n+1}h^{(n+1)}(t)\,\mathrm{d}t &\ge \inf\{h^{(n+1)}(t), t\in[\alpha,\beta]\}\int_\alpha^\beta |a-t|^{n+1}\,\mathrm{d}t\\ &= \frac{(\beta-\alpha)^{n+2}}{n+2}\|h^{(n+1)}\|_{-\infty} = \frac{|a-h^{-1}(b)|^{n+2}}{n+2}\|h^{(n+1)}\|_{-\infty},\end{aligned}$$

and, by the reversed version of Hölder's integral inequality (2.3),

$$\begin{aligned}&\int_\alpha^\beta |a-t|^{n+1}h^{(n+1)}(t)\,\mathrm{d}t\\ &\ge \left(\int_\alpha^\beta |a-t|^{p(n+1)}\,\mathrm{d}t\right)^{1/p}\left(\int_\alpha^\beta \left[h^{(n+1)}(t)\right]^q\,\mathrm{d}t\right)^{1/q}\\ &= \left[\frac{(\beta-\alpha)^{p(n+1)+1}}{p(n+1)+1}\right]^{1/p}\|h^{(n+1)}\|_q = \left[\frac{|a-h^{-1}(b)|^{p(n+1)+1}}{p(n+1)+1}\right]^{1/p}\|h^{(n+1)}\|_q\end{aligned}$$

for $p<1$, $p\neq 0$, and $\frac{1}{p}+\frac{1}{q}=1$, it follows that

1. when $b>h(a)$ and $n=2\ell$ for $\ell\ge 0$ or when $b<h(a)$, we have

$$\begin{aligned}&\int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab\\ &\ge \sum_{k=1}^n h^{(k)}\big(h^{-1}(b)\big)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!}+\frac{C_{p,n}}{(n+1)!}\|h^{(n+1)}\|_q;\end{aligned}$$

2. when $b>h(a)$ and $n=2\ell+1$ for $\ell\ge 0$, we have

$$\begin{aligned}&\int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab\\ &\le \sum_{k=1}^n h^{(k)}\big(h^{-1}(b)\big)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!}-\frac{C_{p,n}}{(n+1)!}\|h^{(n+1)}\|_q;\end{aligned}$$

where $p<1$ and $p\neq 0$ with $\frac{1}{p}+\frac{1}{q}=1$, or $(p,q)=(-\infty,1)$, or $(p,q)=(1,-\infty)$. The proof of Theorem 3.2 is complete. □

Remark 3.2 Taking $n=0$ in Theorem 3.2 leads to [6, Theorem 2.1] mentioned above.

Theorem 3.3 *Let $n\ge 0$ and $h(x)\in C^{n+1}[0,c]$ such that $h(0)=0$ and $h(x)$ is strictly increasing on $[0,c]$ for $c>0$, let h^{-1} be the inverse function of h, let $a\in[0,c]$ and $b\in[0,h(c)]$, and let $\ell\ge 0$ be an integer. Then*

1. *when*
   (a) *either $h(a)>b$ and $h^{(n+1)}(x)$ is increasing on $[\alpha,\beta]$;*
   (b) *or $h(a)<b$, $h^{(n+1)}(x)$ is increasing on $[\alpha,\beta]$, and $n=2\ell+1$;*
   (c) *or $h(a)<b$, $h^{(n+1)}(x)$ is decreasing on $[\alpha,\beta]$, and $n=2\ell$;*

   the inequality

$$\begin{aligned}&\int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab-\sum_{k=1}^n h^{(k)}\big(h^{-1}(b)\big)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!}\\ &\le \frac{[a-h^{-1}(b)]^{n+1}}{(n+2)!}\big[h^{(n)}(a)-h^{(n)}\big(h^{-1}(b)\big)\big] \qquad (3.5)\end{aligned}$$

   is valid;

2. *when*
   (a) *either* $h(a) > b$ *and* $h^{(n+1)}(x)$ *is decreasing on* $[\alpha, \beta]$;
   (b) *or* $h(a) < b$, $h^{(n+1)}(x)$ *is increasing on* $[\alpha, \beta]$, *and* $n = 2\ell$;
   (c) *or* $h(a) < b$, $h^{(n+1)}(x)$ *is decreasing on* $[\alpha, \beta]$, *and* $n = 2\ell + 1$;
   inequality (3.5) *is reversed*;

where α, β *are defined as in* (1.3).

Proof When $a > h^{-1}(b)$ and $h^{(n+1)}(x)$ is increasing, applying Lemma 2.4 gives

$$\begin{aligned}
&\int_{h^{-1}(b)}^{a} (a-t)^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad \le \frac{1}{a-f^{-1}(b)} \int_{h^{-1}(b)}^{a} (a-t)^{n+1}\,\mathrm{d}t \int_{h^{-1}(b)}^{a} h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad = \frac{[a-h^{-1}(b)]^{n+1}}{n+2} \bigl[h^{(n)}(a) - h^{(n)}\bigl(h^{-1}(b)\bigr)\bigr];
\end{aligned}$$

when $a > h^{-1}(b)$ and $h^{(n+1)}(x)$ is decreasing, the above inequality is reversed.

When $a < h^{-1}(b)$, $h^{(n+1)}(x)$ is increasing, and $n = 2\ell$ for $\ell \ge 0$, applying Lemma 2.4 reveals

$$\begin{aligned}
&\int_{h^{-1}(b)}^{a} (a-t)^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad = (-1)^n \int_{a}^{h^{-1}(b)} (t-a)^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad \ge (-1)^n \frac{[h^{-1}(b)-a]^{n+1}}{n+2} \bigl[h^{(n)}\bigl(h^{-1}(b)\bigr) - h^{(n)}(a)\bigr] \\
&\quad = \frac{[a-h^{-1}(b)]^{n+1}}{n+2} \bigl[h^{(n)}(a) - h^{(n)}\bigl(h^{-1}(b)\bigr)\bigr];
\end{aligned} \tag{3.6}$$

when $a < h^{-1}(b)$, $h^{(n+1)}(x)$ is increasing, and $n = 2\ell + 1$ for $\ell \ge 0$, inequality (3.6) is reversed.

By similar argument, we obtain that

1. when $a < h^{-1}(b)$, $h^{(n+1)}(x)$ is decreasing, and $n = 2\ell$ for $\ell \ge 0$, inequality (3.6) is reversed;
2. when $a < h^{-1}(b)$, $h^{(n+1)}(x)$ is decreasing, and $n = 2\ell + 1$ for $\ell \ge 0$, inequality (3.6) is valid.

Substituting these inequalities into (3.4) and simplifying lead to inequality (3.5) and its reversed version for all cases. The proof of Theorem 3.3 is complete. □

Remark 3.3 Taking $n = 0$ in (3.5) derives the right inequality in (1.4).

Theorem 3.4 *Let* $h(x) \in C^{n+1}[0, c]$ *such that* $h(0) = 0$ *and* $h(x)$ *is strictly increasing on* $[0, c]$ *for* $c > 0$, *let* h^{-1} *be the inverse function of* h, *and let* $a \in [0, c]$ *and* $b \in [0, h(c)]$. *If* $h^{(n+1)}(x)$ *is convex on* $[\alpha, \beta]$, *where* α, β *are defined as in* (1.3), *then*

1. *when $h(a) > b$ or when $h(a) < b$ and $n = 2\ell$, we have*

$$\begin{aligned}
&\frac{[a-h^{-1}(b)]^{n+2}}{n+2} h^{(n+1)}\left(\frac{a+(n+2)h^{-1}(b)}{n+3}\right) \\
&\quad \le \int_0^a h(x)\,\mathrm{d}x + \int_0^b h^{-1}(x)\,\mathrm{d}x - ab - \sum_{k=1}^{n} h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!} \\
&\quad \le \bigl[a-h^{-1}(b)\bigr]^{n+2}\frac{h^{(n+1)}(a)+(n+2)h^{(n+1)}(h^{-1}(b))}{(n+3)!};
\end{aligned} \tag{3.7}$$

2. *when $h(a) < b$ and $n = 2\ell + 1$, the double inequality* (3.7) *is reversed*;

where $\ell \ge 0$ is an integer. If $h^{(n+1)}(x)$ is concave on $[\alpha, \beta]$, all the above inequalities are reversed for all corresponding cases.

Proof By substitution of variables, we have

$$\begin{aligned}
&\int_{h^{-1}(b)}^{a} (a-t)^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad = \bigl[a-h^{-1}(b)\bigr]^{n+2}\int_0^1 (1-s)^{n+1} h^{(n+1)}\bigl(sa+(1-s)h^{-1}(b)\bigr)\,\mathrm{d}s.
\end{aligned}$$

Applying inequality (2.5) to the convex function $h^{(n+1)}(x)$ yields that

1. when $a > h^{-1}(b)$, we have

$$\begin{aligned}
&\int_{h^{-1}(b)}^{a} (a-t)^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad \le \bigl[a-h^{-1}(b)\bigr]^{n+2}\int_0^1 (1-s)^{n+1}\bigl[sh^{(n+1)}(a)+(1-s)h^{(n+1)}\bigl(h^{-1}(b)\bigr)\bigr]\,\mathrm{d}s \\
&\quad = \bigl[a-h^{-1}(b)\bigr]^{n+2}\frac{h^{(n+1)}(a)+(n+2)h^{(n+1)}(h^{-1}(b))}{(n+2)(n+3)};
\end{aligned}$$

2. when $a < h^{-1}(b)$ and $n = 2\ell$, the above inequality is still valid;
3. when $a < h^{-1}(b)$ and $n = 2\ell + 1$, the above inequality is reversed.

Substituting these inequalities into (3.4) and simplifying lead to the right inequality in (3.7) and its reversed version for all cases.

Applying inequality (2.6) to the convex function $h^{(n+1)}(x)$ shows that

1. when $a > h^{-1}(b)$, we have

$$\begin{aligned}
&\int_{h^{-1}(b)}^{a} (a-t)^{n+1} h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad \ge \left[\int_{h^{-1}(b)}^{a} (a-t)^{n+1}\,\mathrm{d}t\right] h^{(n+1)}\left(\frac{\int_{h^{-1}(b)}^{a}(a-t)^{n+1}t\,\mathrm{d}t}{\int_{h^{-1}(b)}^{a}(a-t)^{n+1}\,\mathrm{d}t}\right) \\
&\quad = \frac{[a-h^{-1}(b)]^{n+2}}{n+2} h^{(n+1)}\left(\frac{a+(n+2)h^{-1}(b)}{n+3}\right);
\end{aligned}$$

2. when $a < h^{-1}(b)$ and $n = 2\ell$, the above inequality is still valid;
3. when $a < h^{-1}(b)$ and $n = 2\ell$, the above inequality is reversed.

Substituting these related inequalities into (3.4) and rearranging result in the left inequality in (3.7) and its reversed version for all cases.

For the concave function $h^{(n+1)}(x)$, one can derive everything similarly. The proof of Theorem 3.4 is complete. □

Theorem 3.5 *Let $n \ge 0$ and $h(x) \in C^{n+1}[0,c]$ such that $h(0)=0$ and $h(x)$ is strictly increasing on $[0,c]$ for $c>0$, let h^{-1} be the inverse function of h, let $a \in [0,c]$ and $b \in [0,h(c)]$, and let $h^{(n+1)}(x)$ be nonnegative and convex on $[\alpha,\beta]$, where α,β are defined as in (1.3). If $h(a)>b$, then*

$$\begin{aligned}
&\frac{[a-h^{-1}(b)]^{n+2}}{(n+1)!}\left[\frac{1}{2^n}h^{(n+1)}\left(\frac{a+h^{-1}(b)}{2}\right)-\frac{2h^{(n+1)}(a)+h^{(n+1)}(h^{-1}(b))}{6}\right]\\
&\quad\le \int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab-\sum_{k=1}^n h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!}\\
&\quad\le \frac{[a-h^{-1}(b)]^{n+2}}{(n+1)!}\frac{h^{(n+1)}(a)+2h^{(n+1)}(h^{-1}(b))}{6}.
\end{aligned}\tag{3.8}$$

If $h(a)<b$ and $n=2\ell$ for $\ell\ge 0$, then

$$\begin{aligned}
&\frac{[h^{-1}(b)-a]^{n+2}}{(n+1)!}\left[\frac{1}{2^n}h^{(n+1)}\left(\frac{a+h^{-1}(b)}{2}\right)-\frac{2h^{(n+1)}(a)+h^{(n+1)}(h^{-1}(b))}{6}\right]\\
&\quad\le \int_0^a h(x)\,\mathrm{d}x+\int_0^b h^{-1}(x)\,\mathrm{d}x-ab-\sum_{k=1}^n h^{(k)}\bigl(h^{-1}(b)\bigr)\frac{[a-h^{-1}(b)]^{k+1}}{(k+1)!}\\
&\quad\le \frac{[h^{-1}(b)-a]^{n+2}}{(n+1)!}\frac{h^{(n+1)}(a)+2h^{(n+1)}(h^{-1}(b))}{6}.
\end{aligned}\tag{3.9}$$

If $a<h^{-1}(b)$ and $n=2\ell+1$ for $\ell\ge 0$, the double inequality (3.9) is reversed.

Proof When $a>h^{-1}(b)$, employing the double inequality (2.7) gives

$$\begin{aligned}
&2\left[\frac{a-h^{-1}(b)}{2}\right]^{n+1}h^{(n+1)}\left(\frac{a+h^{-1}(b)}{2}\right)\\
&\qquad-\frac{[a-h^{-1}(b)]^{n+1}h^{(n+1)}(h^{-1}(b))}{6}-\frac{[a-h^{-1}(b)]^{n+1}h^{(n+1)}(a)}{3}\\
&\quad\le \frac{1}{a-h^{-1}(b)}\int_{h^{-1}(b)}^a (a-t)^{n+1}h^{(n+1)}(t)\,\mathrm{d}t\\
&\quad\le \frac{[a-h^{-1}(b)]^{n+1}h^{(n+1)}(h^{-1}(b))}{3}+\frac{[a-h^{-1}(b)]^{n+1}h^{(n+1)}(a)}{6}.
\end{aligned}$$

When $a<h^{-1}(b)$, it is clear that

$$\int_{h^{-1}(b)}^a (a-t)^{n+1}h^{(n+1)}(t)\,\mathrm{d}t=(-1)^n\int_a^{h^{-1}(b)}(t-a)^{n+1}h^{(n+1)}(t)\,\mathrm{d}t.$$

If $a < h^{-1}(b)$ and $n = 2\ell$, utilizing the double inequality (2.7) gives

$$\begin{aligned}
&2\left[\frac{h^{-1}(b)-a}{2}\right]^{n+1} h^{(n+1)}\left(\frac{a+h^{-1}(b)}{2}\right) \\
&\qquad - \frac{[h^{-1}(b)-a]^{n+1}h^{(n+1)}(h^{-1}(b))}{6} - \frac{[h^{-1}(b)-a]^{n+1}h^{(n+1)}(a)}{3} \\
&\quad \le \frac{1}{h^{-1}(b)-a}\int_a^{h^{-1}(b)} (t-a)^{n+1}h^{(n+1)}(t)\,\mathrm{d}t \\
&\quad \le \frac{[h^{-1}(b)-a]^{n+1}h^{(n+1)}(h^{-1}(b))}{3} + \frac{[h^{-1}(b)-a]^{n+1}h^{(n+1)}(a)}{6}.
\end{aligned}$$

If $a < h^{-1}(b)$ and $n = 2\ell + 1$, the above double inequality is reversed.

Substituting these related inequalities into (3.4) and rearranging conclude the double inequalities (3.8) and (3.9). The proof of Theorem 3.5 is complete. □

4 Applications and examples

As applications of Theorems 3.1 to 3.5, we now estimate several concrete definite integrals, including a definite integral of $e^{-1/x}$, the exponential integral $\mathrm{Ei}(x)$, and the logarithmic integral $\mathrm{li}(x)$.

Example 4.1 By taking $h(x) = \sqrt[4]{x^4+1} - 1$, $a = 3$, and $b = 2$ in Theorem 1.2, it was obtained in [4] that

$$\begin{aligned}
9 + \frac{4\sqrt[4]{125}}{27}\left[3 - 2\sqrt[4]{5}\right]^2 &= 9.000042866\ldots \\
&< \int_0^3 \sqrt[4]{x^4+1}\,\mathrm{d}x + \int_1^3 \sqrt[4]{x^4-1}\,\mathrm{d}x \\
&< 9 + \frac{27}{2\sqrt[4]{82^3}}\left(3 - 2\sqrt[4]{5}\right)^2 = 9.000042871\ldots.
\end{aligned}$$

The difference between the upper and lower bounds is $0.00000000490079353\ldots$.

In [5, 6], by Theorems 1.3 and 1.4, the above double inequality was improved as

$$9.00004286805\ldots < \int_0^3 \sqrt[4]{x^4+1}\,\mathrm{d}x + \int_1^3 \sqrt[4]{x^4-1}\,\mathrm{d}x < 9.000042868057\ldots.$$

It is easy to see that $h^{-1}(x) = [(x+1)^4 - 1]^{1/4}$. Straightforward computation gives

$$h'(x) = \frac{x^3}{(x^4+1)^{3/4}}, \qquad h''(x) = \frac{3x^2}{(x^4+1)^{7/4}}, \qquad h'''(x) = \frac{x(6-15x^4)}{(x^4+1)^{11/4}}.$$

Since $h^{-1}(2) = 80^{1/4} = 2.990\ldots < 3$, then $(\alpha, \beta) = (80^{1/4}, 3)$. The third derivative $h'''(x)$ has two real nonzero zeros $\pm(\frac{2}{5})^{1/4} = \pm 0.79527\ldots$. This implies that the third derivative $h'''(x)$ is negative and $h''(x)$ is concave on the closed interval $[\alpha, \beta]$. Substituting these data into the reversed version of inequality (3.7) yields

$$\begin{aligned}
&\frac{(3-80^{1/4})^3}{3}h''\left(\frac{3+380^{1/4}}{4}\right)\\
&\quad\ge \int_0^3 \sqrt[4]{x^4+1}\,\mathrm{d}x-3+\int_1^3 \sqrt[4]{x^4-1}\,\mathrm{d}x-6-h'\left(80^{1/4}\right)\frac{(3-80^{1/4})^2}{2!}\\
&\quad\ge \left(3-80^{1/4}\right)^3\frac{h''(3)+3h''(80^{1/4})}{4!},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{(3-80^{1/4})^3}{3}\frac{3072(\sqrt[4]{95}\sqrt{2}+3)^2}{[(\sqrt[4]{95}\sqrt{2}+3)^4+256]^{7/4}}+9+\frac{8\times 5^{3/4}}{27}\frac{(3-80^{1/4})^2}{2!}\\
&\quad= 9.0000428983186013\ldots\ge \int_0^3 \sqrt[4]{x^4+1}\,\mathrm{d}x+\int_1^3 \sqrt[4]{x^4-1}\,\mathrm{d}x\\
&\quad\ge \frac{(3-80^{1/4})^3}{4!}\left(\frac{27}{82^{7/4}}+3\times\frac{4\sqrt{5}}{729}\right)+9+\frac{8\times 5^{3/4}}{27}\frac{(3-80^{1/4})^2}{2!}\\
&\quad= 9.0000428680640760\ldots.
\end{aligned}$$

This lower estimation is better than all before.

Example 4.2 Let $h(x)=e^{-1/x}$ for $x>0$ and $h(0)=0$. Then $h^{-1}(x)=-\frac{1}{\ln x}$ for $x\in(0,1)$ and $h^{-1}(0)=0$. The function $h(x)=e^{-1/x}$ plays an indispensable role in the proof of the existence of partitions of unity in differential geometry [30] and is a generating function of the Lah numbers in combinatorics [13–18, 20, 22].

Direct computation gives

$$h'(x)=\frac{e^{-1/x}}{x^2}\quad\text{and}\quad h''(x)=\frac{e^{-1/x}(1-2x)}{x^4}.$$

This means that the function $h'(x)$ is increasing on $(0,\frac{1}{2}]$ and decreasing on $[\frac{1}{2},\infty)$.

Choosing $a=b=\frac{1}{2}$ means that $a<h^{-1}(\frac{1}{2})=\frac{1}{\ln 2}=1.44\ldots$ and $(\alpha,\beta)=(\frac{1}{2},\frac{1}{\ln 2})$. Then it follows from the double inequality (1.4) that

$$\begin{aligned}
&\frac{1}{4}+\left(\frac{1}{2}-\frac{1}{\ln 2}\right)\left[\frac{1}{e^{\frac{1}{2}(\frac{1}{2}+\frac{1}{\ln 2})}}-\frac{1}{2}\right]\\
&\quad= 0.364469045537996606\ldots\\
&\quad\le \int_0^{1/2}\frac{1}{e^{1/x}}\,\mathrm{d}x-\int_0^{1/2}\frac{1}{\ln x}\,\mathrm{d}x=\mathrm{Ei}(-2)-\mathrm{li}\left(\frac{1}{2}\right)+\frac{1}{2e^2}\\
&\quad\le \frac{1}{4}+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{\ln 2}\right)\left(\frac{1}{e^2}-\frac{1}{2}\right)=0.421883810040011829\ldots,
\end{aligned}$$

where

$$\mathrm{Ei}(x)=-\int_{-x}^{\infty}\frac{e^{-t}}{t}\,\mathrm{d}t\quad\text{and}\quad \mathrm{li}(x)=\int_0^x\frac{\mathrm{d}t}{\ln t}$$

are respectively called the exponential integral and the logarithmic integral.

Example 4.3 Let $h(x) = e^{x^2} - 1$ for $x \ge 0$. Then $h^{-1}(x) = \sqrt{\ln(1+x)}$ for $x \ge 0$.

In [21, Sect. 1.4], it was computed that

$$h^{(n)}(x) = \left(e^{x^2}\right)^{(n)} = e^{x^2} \frac{n!}{(2x)^n} \sum_{k=0}^{n} \binom{k}{n-k} \frac{(2x)^{2k}}{k!}, \quad n \ge 1.$$

This implies that the function $h(x)$ is absolutely monotonic, that is, $h^{(n)}(x) \ge 0$ for all $n \ge 0$, on $(0, \infty)$. For more information on absolutely monotonic functions, please refer to [3] and closely related references therein.

Choosing $a = x$ and $b = x^2$ for $x > 0$ implies that $h^{-1}(x^2) = \ln^{1/2}(1+x^2) < x$ for all $x > 0$. Applying these data to the double inequality (3.8), we arrive at

$$\begin{aligned}
&\frac{[x - \ln^{1/2}(1+x^2)]^{n+2}}{(n+1)!} \left[\frac{1}{2^n} h^{(n+1)}\left(\frac{x + \ln^{1/2}(1+x^2)}{2}\right) \right. \\
&\quad \left. - \frac{2h^{(n+1)}(x) + h^{(n+1)}(\ln^{1/2}(1+x^2))}{6}\right] \\
&\le \int_0^x e^{t^2}\,\mathrm{d}t + \int_0^{x^2} \sqrt{\ln(1+t)}\,\mathrm{d}t - x - x^3 \\
&\quad - \sum_{k=1}^{n} h^{(k)}\left(\ln^{1/2}(1+x^2)\right) \frac{[x - \ln^{1/2}(1+x^2)]^{k+1}}{(k+1)!} \\
&\le \frac{[x - \ln^{1/2}(1+x^2)]^{n+2}}{(n+1)!} \frac{h^{(n+1)}(x) + 2h^{(n+1)}(\ln^{1/2}(1+x^2))}{6}
\end{aligned}$$

for $n \ge 0$ and $x > 0$. In particular, letting $n = 1$ derives

$$\begin{aligned}
&\frac{[x - \ln^{1/2}(1+x^2)]^{3}}{2} \left[\frac{1}{2} h''\left(\frac{x + \ln^{1/2}(1+x^2)}{2}\right) \right. \\
&\quad \left. - \frac{2h''(x) + h''(\ln^{1/2}(1+x^2))}{6}\right] \\
&\le \int_0^x e^{t^2}\,\mathrm{d}t + \int_0^{x^2} \sqrt{\ln(1+t)}\,\mathrm{d}t - x - x^3 \\
&\quad - h'\left(\ln^{1/2}(1+x^2)\right) \frac{[x - \ln^{1/2}(1+x^2)]^{2}}{2} \\
&\le \frac{[x - \ln^{1/2}(1+x^2)]^{3}}{2} \frac{h''(x) + 2h''(\ln^{1/2}(1+x^2))}{6}.
\end{aligned}$$

Further taking $x = 1$ results in

$$\begin{aligned}
&\frac{(1 - \ln^{1/2} 2)^3}{2} \left[\frac{1}{2} h''\left(\frac{1 + \ln^{1/2} 2}{2}\right) - \frac{2h''(1) + h''(\ln^{1/2} 2)}{6}\right] + h'\left(\ln^{1/2} 2\right) \frac{(1 - \ln^{1/2} 2)^2}{2} + 2 \\
&\le \int_0^1 e^{t^2}\,\mathrm{d}t + \int_0^1 \sqrt{\ln(1+t)}\,\mathrm{d}t \\
&\le \frac{(1 - \ln^{1/2} 2)^3}{2} \frac{h''(1) + 2h''(\ln^{1/2} 2)}{6} + h'\left(\ln^{1/2} 2\right) \frac{(1 - \ln^{1/2} 2)^2}{2} + 2,
\end{aligned}$$

which can be numerically computed as

$$2.044751320\ldots \le \int_0^1 e^{t^2}\,\mathrm{d}t + \int_0^1 \sqrt{\ln(1+t)}\,\mathrm{d}t \le 2.060536019\ldots.$$

Acknowledgements
The authors thank Di-San Chen (Lijiang College of Guangxi Normal University), Jing-Feng Tian (North China Electric Power University), and Bo-Yan Xi (Inner Mongolia University for Nationalities) for their supplying electronic versions of several references in this paper. The authors appreciate anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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Received: 14 May 2019 Accepted: 30 August 2019 Published online: 11 September 2019

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