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Convergence and superconvergence of variational discretization for parabolic bilinear optimization problems

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Abstract

In this paper, we investigate a variational discretization approximation of parabolic bilinear optimal control problems with control constraints. For the state and co-state variables, triangular linear finite element and difference methods are used for space and time discretization, respectively, superconvergence in H^1-norm between the numerical solutions and elliptic projections are derived. Although the control variable is not discrete directly, convergence of second order in L^2-norm is obtained. These theoretical results are confirmed by two numerical examples.

MSC: 49J20; 65M60

Keywords: Superconvergence; Variational discretization; Bilinear optimal control

1 Introduction

It is well known that optimal control and optimization problems are approximated by many numerical methods, such as standard finite element methods (FEMs), mixed FEMs, space-time FEMs, finite volume element methods, spectral methods, multigrid methods etc.; see e.g., [5, 8, 10, 16, 17, 24–26, 31]. There is no doubt that FEMs occupy the most important position in these methods.

For a control constrained elliptic optimal control problem (OCP), the regularity of the control variable is lower than the regularity of the state or co-state variable. Hence, most of the researchers use piecewise constant function and piecewise linear function to approximate the control variable and the state or co-state variable, respectively. If the mesh size is h, the convergent order in L^2-norm for the control or in H^1-norm for the state and co-state is just $\mathcal{O}(h)$; see e.g., [2, 9, 12, 18]. When we use these techniques to deal with control constrained parabolic OCP, the similar convergent order is $\mathcal{O}(h+k)$. In order to boost the accuracy and efficiency, superconvergence and adaptive algorithm of FEMs have become research focus. The convergent order will be improved to $\mathcal{O}(h^{\frac{3}{2}})$ or $\mathcal{O}(h^{\frac{3}{2}}+k)$ by superconvergence analysis. Some superconvergence results of FEMs for linear and semilinear elliptic or parabolic OCPs can be found in [4, 6, 15, 27–29]. Adaptive FEMs that approximate elliptic and parabolic OCPs have been investigated in [1, 11, 19, 32] and [3], respectively.

Hinze presents a variational discretization (VD) concept for control constrained optimization problems in [13]. It cannot only save some computation cost but also improve the convergent order to $\mathcal{O}(h^2)$. Recent years, VD are used to solve different kinds of constrained OCPs, for example, VD approximation of a convection dominated diffusion OCP with control constraints and linear parabolic OCPs with pointwise state constraints are investigated in [14] and [7], respectively.

In this paper, we consider VD approximation for constrained parabolic bilinear OCPs. The main purpose is to analyze the convergence and superconvergence. We are interested in the following control constrained parabolic bilinear OCP:

$$\min_{u\in K} \frac{1}{2}\int_0^T \left(\left\|y(t,x)-y_d(t,x)\right\|^2+\alpha\left\|u(t,x)\right\|^2\right)dt, \tag{1}$$

$$y_t(t,x)-\operatorname{div}\bigl(A(x)\nabla y(t,x)\bigr)+u(t,x)y(t,x)=f(t,x), \quad t\in J, x\in\Omega, \tag{2}$$

$$y(t,x)=0, \quad t\in J, x\in\partial\Omega, \tag{3}$$

$$y(0,x)=y_0(x), \quad x\in\Omega, \tag{4}$$

where $\alpha>0$ represents the weight of the cost of the control, $\Omega\in\mathbb{R}^2$ is a convex bounded open set with smooth boundary $\partial\Omega$ and $J=[0,T]$ $(0<T<+\infty)$. The symmetric and positive definite matrix $A(x)=(a_{ij}(x))_{2\times 2}\in[W^{1,\infty}(\bar{\Omega})]^{2\times 2}$. Moreover, we assume that $f(t,x)\in C(J;L^2(\Omega))$, $y_0(x)\in H_0^1(\Omega)$, and the set of admissible controls K is defined by

$$K=\bigl\{v(t,x)\in L^\infty\bigl(J;L^2(\Omega)\bigr): a\le v(t,x)\le b, \text{ a.e. in } \Omega, t\in J\bigr\},$$

where $0\le a<b$ are real numbers.

In this paper, we adopt the notation $L^s(J;W^{m,q}(\Omega))$ for the Banach space of all L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))}=(\int_0^T\|v\|^s_{W^{m,q}(\Omega)}\,dt)^{\frac{1}{s}}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$, where $W^{m,q}(\Omega)$ is Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and semi-norm $|\cdot|_{W^{m,q}(\Omega)}$. We set $H_0^1(\Omega)\equiv\{v\in H^1(\Omega): v|_{\partial\Omega}=0\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. Similarly, one can define $H^l(J;W^{m,q}(\Omega))$ and $C^k(J;W^{m,q}(\Omega))$ (see e.g. [22]). In addition, c or C is a generic positive constant.

The plan of our paper is as follows. In Sect. 2, we present VD approximation scheme for the model problem (1)–(4). In Sect. 3, we introduce some important intermediate variables and their error estimates. Convergence of the control variable is derived in Sect. 4. Superconvergence of the state and the co-state are established in Sect. 5. In Sect. 6, we present two numerical examples to illustrate our theoretical results.

2 VD approximation for parabolic bilinear OCP

In this section, we construct VD approximation for (1)–(4). We set $L^p(J;W^{m,q}(\Omega))$ and $\|\cdot\|_{L^p(J;W^{m,q}(\Omega))}$ by $L^p(W^{m,q})$ and $\|\cdot\|_{L^p(W^{m,q})}$, respectively. Let $W=H_0^1(\Omega)$ and $U=L^2(\Omega)$. Moreover, we denote $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|_m$ and $\|\cdot\|$, respectively. Let

$$a(v,w)=\int_\Omega(A\nabla v)\cdot\nabla w, \quad \forall v,w\in W,$$

$$(f_1,f_2)=\int_\Omega f_1\cdot f_2, \quad \forall f_1,f_2\in U.$$

According to the assumptions on A, we have

$$a(v,v)\ge c\|v\|_1^2, \qquad \bigl|a(v,w)\bigr|\le C\|v\|_1\|w\|_1, \quad \forall v,w\in W.$$

We recast (1)–(4) as the following weak formulation:

$$\min_{u\in K} \frac{1}{2}\int_0^T \left(\|y-y_d\|^2+\alpha\|u\|^2\right)dt, \tag{5}$$

$$(y_t,w)+a(y,w)+(uy,w)=(f,w), \quad \forall w\in W, t\in J, \tag{6}$$

$$y(x,0)=y_0(x), \quad \forall x\in \Omega. \tag{7}$$

It follows from (see e.g. [21]) that the problem (5)–(7) has at least one solution (y,u), and that if the pair $(y,u)\in (H^2(L^2)\cap L^2(H^1))\times K$ is a solution of the formulation (5)–(7), then there is a co-state $p\in H^2(L^2)\cap L^2(H^1)$ such that the triplet (y,p,u) satisfies the following optimality conditions:

$$(y_t,w)+a(y,w)+(uy,w)=(f,w), \quad \forall w\in W, t\in J, \tag{8}$$

$$y(0,x)=y_0(x), \quad \forall x\in \Omega, \tag{9}$$

$$-(p_t,q)+a(q,p)+(up,q)=(y-y_d,q), \quad \forall q\in W, t\in J, \tag{10}$$

$$p(T,x)=0, \quad \forall x\in \Omega, \tag{11}$$

$$(\alpha u-yp,v-u)\geq 0, \quad \forall v\in K, t\in J. \tag{12}$$

As in Ref. [27], we can easily prove the following lemma.

Lemma 2.1 *Let (y,p,u) be the solution of (8)–(12). Then*

$$u=\min\left(\max\left(a,\frac{yp}{\alpha}\right),b\right). \tag{13}$$

Let $\mathcal{T}^h$ be regular triangulations of Ω, such that $\bar{\Omega}=\bigcup_{\tau\in\mathcal{T}^h}\bar{\tau}$ and $h=\max_{\tau\in\mathcal{T}^h}\{h_\tau\}$, where h_τ is the diameter of the triangle element τ. Furthermore, we set

$$W_h=\left\{v_h\in C(\bar{\Omega}): v_h|_\tau\in\mathbb{P}_1, \forall\tau\in\mathcal{T}^h, v_h|_{\partial\Omega}=0\right\},$$

where $\mathbb{P}_1$ denotes the space of polynomials no more than order 1.

Let $0=t_0<t_1<\cdots<t_N=T$, $k_n=t_n-t_{n-1}$, $n=1,2,\ldots,N$, $k=\max_{1\leq n\leq N}\{k_n\}$. Set $\varphi^n=\varphi(x,t_n)$ and

$$d_t\varphi^n=\frac{\varphi^n-\varphi^{n-1}}{k_n}, \quad n=1,2,\ldots,N.$$

Moreover, we define for $1\leq p<\infty$ the discrete time-dependent norms

$$|||\varphi|||_{l^p(J;W^{m,q}(\Omega))}:=\left(\sum_{n=1-l}^{N-l}k_n\|\varphi^n\|^p_{W^{m,q}(\Omega)}\right)^{\frac{1}{p}},$$

where $l=0$ for the control u and the state y and $l=1$ for the co-state p, with the standard modification for $p=\infty$. For convenience, we denote $|||\cdot|||_{l^s(J;W^{m,q}(\Omega))}$ by $|||\cdot|||_{l^s(W^{m,q})}$ and let

$$l^p\left(H^s\right):=\left\{f: |||f|||_{l^p(H^s)}<\infty\right\}, \quad 1\leq p\leq\infty.$$

Then a possible VD approximation of (1)–(4) is as follows:

$$\min_{u_h^n \in K} \frac{1}{2} \sum_{n=1}^{N} k_n \left(\left\| y_h^n - y_d^n \right\|^2 + \alpha \left\| u_h^n \right\|^2 \right), \tag{14}$$

$$\left(d_t y_h^n, w_h \right) + a\left(y_h^n, w_h \right) + \left(u_h^n y_h^n, w_h \right) = \left(f^n, w_h \right), \tag{15}$$

$$\forall w_h \in W_h, n = 1, 2, \ldots, N,$$

$$y_h^0(x) = y_0^h(x), \quad \forall x \in \Omega, \tag{16}$$

where $y_0^h(x) = R_h(y_0(x))$ and R_h is an elliptic projection operator which will be specified later.

For $n = 1, 2, \ldots, N$, the OCP (14)–(16) again has a solution (y_h^n, u_h^n) and that if $(y_h^n, u_h^n) \in W_h \times K$ is a solution of (14)–(16), then there is a co-state $p_h^{n-1} \in W_h$, such that the triplet $(y_h^n, p_h^{n-1}, u_h^n) \in W_h \times W_h \times K$, satisfies the following optimality conditions:

$$\left(d_t y_h^n, w_h \right) + a\left(y_h^n, w_h \right) + \left(u_h^n y_h^n, w_h \right) = \left(f^n, w_h \right), \quad \forall w_h \in W_h, \tag{17}$$

$$y_h^0(x) = y_0^h(x), \quad \forall x \in \Omega, \tag{18}$$

$$-\left(d_t p_h^n, q_h \right) + a\left(q_h, p_h^{n-1} \right) + \left(u_h^n p_h^{n-1}, q_h \right) = \left(y_h^n - y_d^n, q_h \right), \quad \forall q_h \in W_h, \tag{19}$$

$$p_h^N(x) = 0, \quad \forall x \in \Omega, \tag{20}$$

$$\left(\alpha u_h^n - y_h^n p_h^{n-1}, v^n - u_h^n \right) \geq 0, \quad \forall v \in K. \tag{21}$$

Similar to (13), the variational inequality (21) can be equivalently rewritten as follows.

Lemma 2.2 *Let (y_h, p_h, u_h) be the solution of (17)–(21). Then, for $n = 1, 2, \ldots, N$, we have*

$$u_h^n = \min\left(\max\left(a, \frac{y_h^n p_h^{n-1}}{\alpha} \right), b \right). \tag{22}$$

Remark 2.1 It should be pointed out that we minimize over the infinite dimensional set K instead of minimizing over a finite dimensional subset of K in (21). Then we just need to solve the discrete equations (17)–(20) and obtain u_h from (22).

3 Error estimates of intermediate variables

Some useful intermediate variables and their important error estimates will be introduced in this section. For any control function $v \in K$ and $w_h, q_h \in W_h$, let $y_h^n(v), p_h^n(v) \in W_h$ for $n = 1, 2, \ldots, N$ satisfy the following system:

$$\left(d_t y_h^n(v), w_h \right) + a\left(y_h^n(v), w_h \right) + \left(v^n y_h^n(v), w_h \right) = \left(f^n, w_h \right), \tag{23}$$

$$y_h^0(v) = y_0^h(x), \quad \forall x \in \Omega, \tag{24}$$

$$-\left(d_t p_h^n(v), q_h \right) + a\left(q_h, p_h^{n-1}(v) \right) + \left(v^n p_h^{n-1}(v), q_h \right) = \left(y_h^n(v) - y_d^n, q_h \right), \tag{25}$$

$$p_h^N(v) = 0, \quad \forall x \in \Omega. \tag{26}$$

If (y_h, p_h, u_h) be the solutions of and (17)–(21), then $(y_h, p_h) = (y_h(u_h), p_h(u_h))$.

We introduce the elliptic projection operator $R_h : W \to W_h$, which satisfies: for any $\phi \in W$,

$$a(R_h\phi - \phi, w_h) = 0, \quad \forall \phi \in W, w_h \in W_h. \tag{27}$$

It has the following property (see e.g., [4]):

$$\|R_h\phi - \phi\|_s \le Ch^{2-s}\|\phi\|_2, \quad \forall \phi \in H^2(\Omega), s = 0, 1. \tag{28}$$

Lemma 3.1 *Let (y, p, u) be the solution of (8)–(12) and $(y_h(u), p_h(u))$ be the discrete solution of (23)–(26) with $v = u$. Suppose that $u \in l^2(H^1)$ and $y, p \in l^2(H^2) \cap H^2(L^2) \cap H^1(H^2)$, we have*

$$\left\|y_h(u) - y\right\|_{l^2(L^2)} + \left\|p_h(u) - p\right\|_{l^2(L^2)} \le C\left(h^2 + k\right). \tag{29}$$

Proof Set $v = u$ in (23), then from Eq. (8) and the elliptic projection operator R_h. For $n = 1, 2, \dots, N$ and $\forall w_h \in W_h$, we derive

$$\begin{aligned}
&\left(d_t y_h^n(u) - d_t R_h y^n, w_h\right) + a\left(y_h^n(u) - R_h y^n, w_h\right) + \left(u^n\left(y_h^n(u) - R_h y^n\right), w_h\right) \\
&\quad = -\left(d_t R_h y^n, w_h\right) - a\left(y^n, w_h\right) - \left(u^n R_h y^n, w_h\right) + \left(f^n, w_h\right) \\
&\quad = -\left(d_t R_h y^n - d_t y^n, w_h\right) - \left(d_t y^n - y_t^n, w_h\right) - \left(u^n\left(R_h y^n - y^n\right), w_h\right).
\end{aligned} \tag{30}$$

We note that

$$\begin{aligned}
&\left(d_t y_h^n(u) - d_t R_h y^n, y_h^n(u) - R_h y^n\right) \\
&\quad \ge \frac{1}{k_n}\left(\left\|y_h^n(u) - R_h y^n\right\|^2 - \left\|y_h^n(u) - R_h y^n\right\|\left\|y_h^{n-1}(u) - R_h y^{n-1}\right\|\right)
\end{aligned} \tag{31}$$

and

$$a\left(y_h^n(u) - R_h y^n, y_h^n(u) - R_h y^n\right) \ge \left(u^n\left(y_h^n(u) - R_h y^n\right), R_h y^n - y_h^n(u)\right). \tag{32}$$

By choosing $w_h = y_h^n(u) - R_h y^n$ in (30) and using (31)–(32) and Hölder's inequality, and multiplying both sides of (30) by k_n and summing n from 1 to N^* ($1 \le N^* \le N$), we get

$$\begin{aligned}
&\left\|y_h^{N^*}(u) - R_h y^{N^*}\right\| \\
&\quad \le \sum_{n=1}^{N^*}\left\|(R_h - I)\left(y^n - y^{n-1}\right)\right\| + \sum_{n=1}^{N^*}\left\|y^n - y^{n-1} - k_n y_t^n\right\| \\
&\qquad + \sum_{n=1}^{N^*} k_n\left\|u^n\left(R_h y^n - y^n\right)\right\| \\
&\quad \le \sum_{n=1}^{N^*} Ch^2\left\|y^n - y^{n-1}\right\|_2 + \sum_{n=1}^{N^*}\int_{t_{n-1}}^{t_n}\left\|(t_{n-1} - t)y_{tt}\right\| dt \\
&\qquad + C\sum_{n=1}^{N^*} k_n\left\|R_h y^n - y^n\right\|
\end{aligned}$$

$$\begin{aligned}
&\le Ch^2 \sum_{n=1}^{N^*} \int_{t_{n-1}}^{t_n} \|y_t\|_2 \, dt + k \sum_{n=1}^{N^*} \int_{t_{n-1}}^{t_n} \|y_{tt}\| \, dt + Ch^2 \sum_{n=1}^{N^*} k_n \|y^n\|_2 \\
&\le Ch^2 \int_0^{t_{N^*}} \|y_t\|_2 \, dt + k \int_0^{t_{N^*}} \|y_{tt}\| \, dt + Ch^2 |||y|||_{l^2(H^2)} \\
&\le C\big(h^2 \|y_t\|_{L^2(H^2)} + k\|y_{tt}\|_{L^2(L^2)} + h^2 |||y|||_{l^2(H^2)}\big).
\end{aligned} \tag{33}$$

Hence

$$\big|||y_h(u) - R_h y|||_{l^\infty(L^2)} \le C\big(h^2 + k\big). \tag{34}$$

It follows from (28) that

$$|||R_h y - y|||^2_{l^2(L^2)} = \sum_{n=1}^{N} k_n \|R_h y^n - y^n\|^2 \le Ch^4 |||y|||^2_{l^2(H^2)}. \tag{35}$$

According to (34)–(35) and the embedding theorem, we obtain

$$|||y_h(u) - y|||_{l^2(L^2)} \le C\big(h^2 + k\big). \tag{36}$$

Similarly, we can derive

$$|||p_h(u) - p|||_{l^2(L^2)} \le C\big(h^2 + k\big). \tag{37}$$

Therefore, (29) follows from (36) and (37). □

4 Convergence analysis

In this section, we will derive the convergence analysis for the control variable. For ease of exposition, we set

$$\begin{aligned}
J(u) &= \frac{1}{2} \int_0^T \big(\|y - y_d\|^2 + \alpha \|u\|^2\big) \, dt, \\
J_h(u_h) &= \frac{1}{2} \int_0^T \big(\|y_h - y_d\|^2 + \alpha \|u_h\|^2\big) \, dt.
\end{aligned}$$

It can be shown that

$$\begin{aligned}
\big(J'(u), v\big) &= \int_0^T (\alpha u - yp, v) \, dt, \\
\big(J'_{hk}(u_h), v\big) &= \sum_{n=1}^{N} k_n \big(\alpha u_h^n - y_h^n(u_h) p_h^{n-1}(u_h), v\big).
\end{aligned}$$

In many applications, the objective functional $J(\cdot)$ is uniform convex near the solution u (see, e.g., [23]) that is closely related to the second order sufficient conditions of the control problem. It is assumed in many studies on numerical methods of the problem (see, e.g., [2]). Hence, if h and k are small enough, we can assume that $J_{hk}(\cdot)$ is uniform convex, namely, there is a positive constant c, such that

$$c|||u - v|||^2_{l^2(L^2)} \le \big(J'_{hk}(u) - J'_{hk}(v), u - v\big), \quad \forall u, v \in K. \tag{38}$$

Theorem 4.1 *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (8)–(12) and (17)–(21), respectively. Assume that $y_h(u), p \in l^\infty(L^\infty)$ and all the conditions in Lemma 3.1 are valid. Then we have*

$$|||u-u_h|||_{l^2(L^2)} \le C\big(h^2+k\big). \tag{39}$$

Proof Set $v=u_h$ and $v=u$ in (12) and (21), respectively, we obtain

$$(\alpha u, u-u_h) \le (yp, u-u_h), \quad \forall t \in J, \tag{40}$$

and

$$\big(\alpha u_h^n - y_h^n p_h^{n-1}, u^n - u_h^n\big) \ge 0, \quad n=1,2,\ldots,N. \tag{41}$$

From (38) and (40)–(41), we have

$$\begin{aligned} c|||u-u_h|||_{l^2(L^2)}^2 &\le \big(J_{hk}'(u)-J_{hk}'(u_h), u-u_h\big) \\ &= \sum_{n=1}^N k_n\big(\alpha u^n - y_h^n(u)p_h^{n-1}(u), u^n-u_h^n\big) \\ &\quad - \sum_{n=1}^N k_n\big(\alpha u_h^n - y_h^n(u_h)p_h^{n-1}(u_h), u^n-u_h^n\big) \\ &\le \sum_{n=1}^N k_n\big(y^n p^n - y_h^n(u)p^n, u^n-u_h^n\big) \\ &\quad + \sum_{n=1}^N k_n\big(y_h^n(u)p^n - y_h^n(u)p^{n-1}, u^n-u_h^n\big) \\ &\quad + \sum_{n=1}^N k_n\big(y_h^n(u)p^{n-1} - y_h^n(u)p_h^{n-1}(u), u^n-u_h^n\big) \\ &:= I_1+I_2+I_3. \end{aligned} \tag{42}$$

According to Young's inequality with ϵ and Lemma 3.1, I_1 can be estimated as follows:

$$\begin{aligned} I_1 &= \sum_{n=1}^N k_n\big(p^n\big(y^n-y_h^n(u)\big), u^n-u_h^n\big) \\ &\le C(\epsilon)\big|\big|\big|y-y_h(u)\big|\big|\big|_{l^2(L^2)}^2 + \epsilon|||u-u_h|||_{l^2(L^2)}^2 \\ &\le C(\epsilon)\big(h^2+k\big)^2 + \epsilon|||u-u_h|||_{l^2(L^2)}^2. \end{aligned} \tag{43}$$

For the second term I_2, by using Young's inequality with ϵ, we have

$$\begin{aligned} I_2 &= \sum_{n=1}^N k_n\big(y_h^n(u)\big(p^n-p^{n-1}\big), u^n-u_h^n\big) \\ &= C(\epsilon)k^2|||p_t|||_{l^2(L^2)}^2 + \epsilon|||u-u_h|||_{l^2(L^2)}^2. \end{aligned} \tag{44}$$

From Young's inequality with ϵ and Lemma 3.1, we get

$$\begin{aligned} I_3 &= \sum_{n=1}^{N} k_n\big(y_h^n(u)\big(p^{n-1} - p_h^{n-1}(u)\big), u^n - u_h^n\big) \\ &\le C(\epsilon)\big\|\big|p - p_h(u)\big\|\big|^2_{l^2(L^2)} + \epsilon |||u - u_h|||^2_{l^2(L^2)} \\ &\le C(\epsilon)\big(h^2 + k\big)^2 + \epsilon |||u - u_h|||^2_{l^2(L^2)}. \end{aligned} \tag{45}$$

Let ϵ be small enough, then (39) follows from (42)–(45). □

5 Superconvergence analysis

In this section, we will derive superconvergence of the state and co-state variables.

Theorem 5.1 *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (8)–(12) and (17)–(21), respectively. Assume that $y_h \in l^\infty(L^\infty)$ all the conditions in Theorem 4.1 hold, we have*

$$|||R_h y - y_h|||_{l^2(H^1)} + |||R_h p - p_h|||_{l^2(H^1)} \le C\big(h^2 + k\big). \tag{46}$$

Proof From (8) and (17), for any $w_h \in W^h$ and $n = 1, 2, \ldots, N$, we have

$$\begin{aligned} &\big(y_t^n - d_t y_h^n, w_h\big) + a\big(y^n - y_h^n, w_h\big) + \big(u^n\big(y^n - y_h^n\big), w_h\big) \\ &\quad = \big(y_h^n\big(u_h^n - u^n\big), w_h\big). \end{aligned} \tag{47}$$

According to the definition of R_h, we get

$$\begin{aligned} &\big(d_t R_h y^n - d_t y_h^n, w_h\big) + a\big(R_h y^n - y_h^n, w_h\big) + \big(u^n\big(R_h y^n - y_h^n\big), w_h\big) \\ &\quad = \big(d_t R_h y^n - d_t y^n + d_t y^n - y_t^n + u^n\big(R_h y^n - y^n\big) + y_h^n\big(u_h^n - u^n\big), w_h\big). \end{aligned} \tag{48}$$

Note that

$$\begin{aligned} &\big(d_t R_h y^n - d_t y_h^n, R_h y^n - y_h^n\big) \\ &\quad \ge \frac{1}{2k_n}\big(\big\|R_h y^n - y_h^n\big\|^2 - \big\|R_h y^{n-1} - y_h^{n-1}\big\|^2\big) \end{aligned} \tag{49}$$

and

$$\begin{aligned} \big(d_t R_h y^n - d_t y^n, R_h y^n - y_h^n\big) &\le \big\|d_t R_h y^n - d_t y^n\big\| \big\|R_h y^n - y_h^n\big\| \\ &\le Ch^2 \big\|d_t y^n\big\|_2 \big\|R_h y^n - y_h^n\big\| \\ &\le Ch^2 k_n^{-1} \int_{t_{n-1}}^{t_n} \|y_t\|_2 \, dt \big\|R_h y^n - y_h^n)\big\| \\ &\le Ch^2 k_n^{-\frac{1}{2}} \|y_t\|_{L^2(t_{n-1}, t_n; H^2(\Omega))} \big\|R_h y^n - y_h^n\big\|. \end{aligned} \tag{50}$$

In addition

$$
\begin{aligned}
\left(d_t y^n - y_t^n, R_h y^n - y_h^n\right) &= k_n^{-1}\left(y^n - y^{n-1} - k_n y_t^n, R_h y^n - y_h^n\right) \\
&\le k_n^{-1}\left\|y^n - y^{n-1} - k_n y_t^n\right\|\left\|R_h y^n - y_h^n\right\| \\
&= k_n^{-1}\left\|\int_{t_{n-1}}^{t_n}(t_{n-1}-s)(y_{tt})(s)\,ds\right\|\left\|R_h y^n - y_h^n\right\| \\
&\le Ck_n^{\frac{1}{2}}\left\|y_{tt}(\nu)\right\|_{L^2(t_{n-1},t_n;L^2(\Omega))}\left\|R_h y^n - y_h^n\right\|. \qquad (51)
\end{aligned}
$$

By choosing $w_h = R_h y^n - y_h^n$ in (48) and using (49)–(51) and Young's inequality with ϵ, then multiplying both sides of (48) by $2k_n$ and summing n from 1 to N, we get

$$
\begin{aligned}
&\left\|R_h y^N - y_h^N\right\|^2 + c\sum_{n=1}^{N} k_n\left\|R_h y^n - y_h^n\right\|_1^2 \\
&\quad\le C(\epsilon)\left(h^4\left\|y_t\right\|_{L^2(H^2)}^2 + k^2\left\|y_{tt}\right\|_{L^2(L^2)}^2 + h^4|||y|||_{l^2(H^2)}^2 + |||u_h - u|||_{l^2(L^2)}^2\right) \\
&\qquad + \epsilon\sum_{n=1}^{N} k\left\|R_h y^n - y_h^n\right\|^2. \qquad (52)
\end{aligned}
$$

From (39) and (52), we obtain

$$
|||R_h y - y_h|||_{l^2(H^1)} \le C\left(h^2 + k\right). \qquad (53)
$$

Similarly, we can prove

$$
|||R_h p - p_h|||_{l^2(H^1)} \le C\left(h^2 + k\right). \qquad (54)
$$

Hence, (46) follows from (53)–(54). □

6 Numerical experiments

For an acceptable error *Tol*, we present the following VD approximation algorithm in which we have omitted the subscript h just for ease of exposition.

Algorithm 6.1 (VD approximation algorithm)

Step 1. Initialize u_0.

Step 2. Solve the following equations:

$$
\begin{cases}
(\frac{y_n^i - y_n^{i-1}}{k}, w) + a(y_n^i, w) + (u_n^i y_n^i, w) = (f^i, w), \\
y_n^i, y_n^{i-1} \in W_h, \quad \forall w \in W_h, \\
(\frac{p_n^{i-1} - p_n^i}{k}, q) + a(q, p_n^{i-1}) + (u_n^i p_n^{i-1}, q) = (y_n^i - y_d^i, q), \\
up_n^i, p_n^{i-1} \in W_h, \quad \forall q \in W_h, \\
u_{n+1} = \min(\max(a, \frac{y_n p_n}{\alpha}), b).
\end{cases} \qquad (55)
$$

Step 3. Calculate the iterative error: $E_{n+1} = |||u_{n+1} - u_n|||_{l^2(L^2)}$.

Step 4. If $E_{n+1} \le Tol$, stop; else go to Step 2.

Let $\Omega = [0,1] \times [0,1]$, $T = 1$, $\alpha = 1$, $a = 0$, $b = 1$ and $A(x)$ is a unit matrix. We solve the following two examples with AFEPack. The details can be found at [20]. We denote $|||\cdot|||_{l^2(H^1)}$ and $|||\cdot|||_{l^2(L^2)}$ by $|||\cdot|||_1$ and $|||\cdot|||$, respectively. The convergence order rate: $Rate = \frac{\log(e_{i+1})-\log(e_i)}{\log(h_{i+1})-\log(h_i)}$, where e_i and e_{i+1} denote errors when mesh size $h = h_i$ and $h = h_{i+1}$, respectively.

Example 1 The data are as follows:

$$
\begin{aligned}
&p(x,t) = (1-t)\sin(2\pi x_1)\sin(2\pi x_2),\\
&y(t,x) = t\sin(2\pi x_1)\sin(2\pi x_2),\\
&u(t,x) = \min\bigl(\max\bigl(0, y(t,x)p(t,x)\bigr), 1\bigr),\\
&f(t,x) = y_t(t,x) - \operatorname{div}\bigl(A(x)\nabla y(t,x)\bigr) + u(t,x)y(t,x),\\
&y_d(t,x) = y(t,x) + p_t(t,x) + \operatorname{div}\bigl(A^*(x)\nabla p(t,x)\bigr) - u(t,x)p(t,x).
\end{aligned}
$$

The errors based on a sequence of uniformly meshes are shown in Table 1, where we can see $|||u-u_h||| = \mathcal{O}(h^2+k)$, $|||R_hy-y_h|||_1 = \mathcal{O}(h^2+k)$ and $|||R_hp-p_h|||_1 = \mathcal{O}(h^2+k)$. When $h = \frac{1}{80}$, $k = \frac{1}{640}$ and $t = 0.5$, the numerical solution u_h is shown in Fig. 1.

Example 2 The data are as follows:

$$
\begin{aligned}
&p(t,x) = (1-t)x_1(1-x_1)x_2(1-x_2),\\
&y(t,x) = tx_1(1-x_1)x_2(1-x_2),\\
&u(t,x) = \min\bigl(\max\bigl(0, y(t,x)p(t,x)\bigr), 1\bigr),
\end{aligned}
$$

Table 1 Numerical results of Example 1

h	k	$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$	Rate	$\lvert\lvert\lvert R_hy-y_h\rvert\rvert\rvert_1$	Rate	$\lvert\lvert\lvert R_hp-p_h\rvert\rvert\rvert_1$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	1.46324e–02	–	3.35880e–02	–	7.31505e–02	–
$\frac{1}{20}$	$\frac{1}{40}$	3.91388e–03	1.90	8.18124e–03	2.04	1.89129e–02	1.95
$\frac{1}{40}$	$\frac{1}{160}$	9.11840e–04	2.10	1.91874e–03	2.09	4.64444e–03	2.03
$\frac{1}{80}$	$\frac{1}{640}$	2.01991e–04	2.17	3.97933e–04	2.27	1.03486e–03	2.17

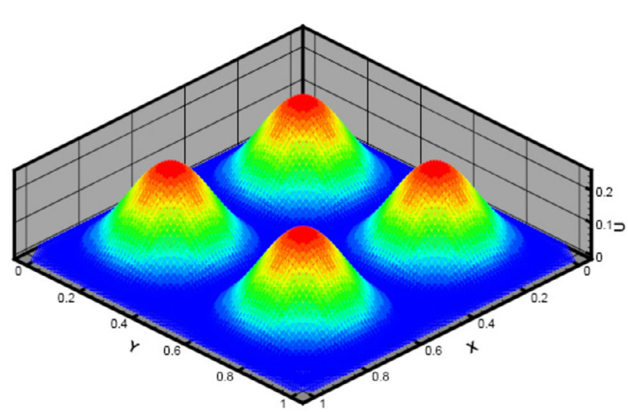

Figure 1 The numerical solution u_h at $t = 0.5$, Example 1

Table 2 Numerical results of Example 2

h	k	$\|\|\|u-u_h\|\|\|$	Rate	$\|\|\|R_hy-y_h\|\|\|_1$	Rate	$\|\|\|R_hp-p_h\|\|\|_1$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	2.45170e–05	–	5.45644e–04	–	3.56804e–03	–
$\frac{1}{20}$	$\frac{1}{40}$	6.45526e–06	1.93	1.29880e–04	2.07	9.02173e–04	1.98
$\frac{1}{40}$	$\frac{1}{160}$	1.72058e–06	1.91	3.19495e–05	2.02	2.26222e–04	2.00
$\frac{1}{80}$	$\frac{1}{640}$	4.3535e–07	1.98	7.85015e–06	2.03	5.64788e–05	2.00

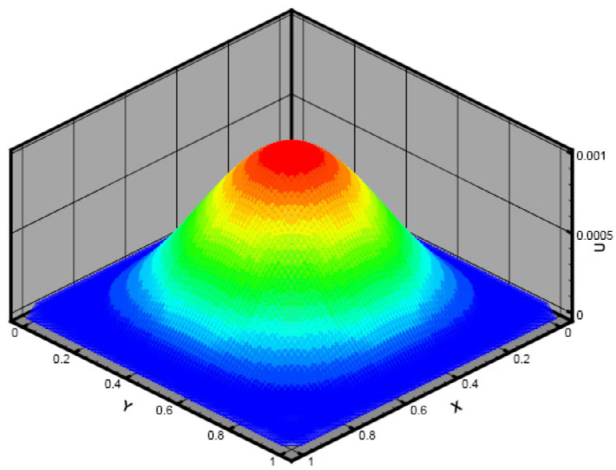

Figure 2 The numerical solution u_h at $t = 0.5$, Example 2

$$f(t,x) = y_t(t,x) - \operatorname{div}\big(A(x)\nabla y(t,x)\big) + u(t,x)y(t,x),$$
$$y_d(t,x) = y(t,x) + p_t(t,x) + \operatorname{div}\big(A^*(x)\nabla p(t,x)\big) - u(t,x)p(t,x).$$

The errors $|||u-u_h|||$, $|||R_hy-y_h|||_1$ and $|||R_hp-p_h|||_1$ on a sequence of uniformly meshes are shown in Table 2. When $h = \frac{1}{80}$, $k = \frac{1}{640}$ and $t = 0.5$, we plot the profile of u_h in Fig. 2.

From the numerical results in Example 1 and Example 2, we see that $|||u-u_h|||$, $|||R_hy - y_h|||_1$ and $|||R_hp-p_h|||_1$ are the second order convergent. Our numerical results and theoretical results are consistent.

7 Conclusions

Although there has been extensive research on convergence and superconvergence of FEMs for various parabolic OCPs, mostly focused on linear or semilinear parabolic cases (see, e.g., [6, 10, 16, 26, 30]), the results on convergence and superconvergence are $\mathcal{O}(h+k)$ and $\mathcal{O}(h^{\frac{3}{2}}+k)$, respectively. Recent years, VD are used to deal with different OCPs in [7, 13, 14]. While there is little work on bilinear OCPs. Hence, our results on convergence and superconvergence of VD for bilinear parabolic OCPs are new.

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Abbreviations
FEMs, finite element methods; OCP, optimal control problem; VD, variational discretization.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have participated in the sequence alignment and drafted the manuscript. They have approved the final manuscript.

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References
1. Becker, R., Kapp, H., Rannacher, R.: Adaptive finite element methods for optimal control of partial differential equations: basic concept. SIAM J. Control Optim. **39**(1), 113–132 (2000)
2. Casas, E., Tröltzsch, F.: Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory. SIAM J. Control Optim. **13**(2), 406–431 (2002)
3. Chen, T., Xiao, J., Wang, H.: Multi-mesh adaptive finite element algorithms for constrained optimal control problems governed by semi-linear parabolic equations. Acta Math. Appl. Sin. Engl. Ser. **30**(2), 411–428 (2014)
4. Chen, Y., Dai, Y.: Superconvergence for optimal control problems governed by semi-linear elliptic equations. J. Sci. Comput. **39**, 206–221 (2009)
5. Chen, Y., Huang, F.: Galerkin spectral approximation of elliptic optimal control problems with H^1-norm state constraint. J. Sci. Comput. **67**, 65–83 (2016)
6. Dai, Y., Chen, Y.: Superconvergence for general convex optimal control problems governed by semilinear parabolic equations. ISRN Appl. Math. **2014**, 1–12 (2014)
7. Deckelnick, K., Hinze, M.: Variational discretization of parabolic control problems in the presence of pointwise state constraints. J. Comput. Math. **29**(1), 1–16 (2011)
8. Engel, M., Griebel, M.: A multigrid method for constrained optimal control problems. J. Comput. Appl. Math. **235**, 4368–4388 (2011)
9. Falk, R.: Approximation of a class of optimal control problems with order of convergence estimates. J. Math. Anal. Appl. **44**, 28–47 (1973)
10. Fu, H., Rui, H.: A priori error estimates for optimal control problems governed by transient advection-diffusion equations. J. Sci. Comput. **38**, 290–315 (2009)
11. Ge, L., Liu, W., Yang, D.: Adaptive finite element approximation for a constrained optimal control problems via multi-meshes. J. Sci. Comput. **41**, 238–255 (2009)
12. Geveci, T.: On the approximation of the solution of an optimal control problem governed by an elliptic equation. RAIRO. Anal. Numér. **13**, 313–328 (1979)
13. Hinze, M.: A variational discretization concept in control constrained optimization: the linear-quadratic case. Comput. Optim. Appl. **30**, 45–61 (2005)
14. Hinze, M., Yan, N., Zhou, Z.: Variational discretization for optimal control governed by convection dominated diffusion equations. J. Comput. Math. **27**(2–3), 237–253 (2009)
15. Hou, C., Chen, Y., Lu, Z.: Superconvergence property of finite element methods for parabolic optimal control problems. J. Ind. Manag. Optim. **7**(4), 927–945 (2011)
16. Hou, T., Chen, Y.: Superconvergence of fully discrete rectangular mixed finite element methods of parabolic control problems. J. Comput. Appl. Math. **286**, 79–92 (2015)
17. Hou, T., Li, L.: Superconvergence and a posteriori error estimates of splitting positive definite mixed finite element methods for elliptic optimal control problems. Appl. Math. Comput. **273**, 1196–1207 (2016)
18. Kröner, A., Vexler, B.: A priori error estimates for elliptic optimal control problems with a bilinear state equation. J. Comput. Appl. Math. **230**, 781–802 (2009)
19. Li, R., Liu, W., Ma, H., Tang, T.: Adaptive finite element approximation for distributed elliptic optimal control problems. SIAM J. Control Optim. **41**(5), 1321–1349 (2002)
20. Li, R., Liu, W., Yan, N.: A posteriori error estimates of recovery type for distributed convex optimal control problems. J. Sci. Comput. **33**, 155–182 (2007)
21. Lions, J.: Optimal Control of Systems Governed by Partial Differential Equations. Springer, Berlin (1971)
22. Lions, J., Magenes, E.: Non Homogeneous Boundary Value Problems and Applications. Springer, Berlin (1972)
23. Liu, W., Tiba, D.: Error estimates for the finite element approximation of a class of nonlinear optimal control problems. J. Numer. Funct. Optim. **22**, 953–972 (2001)
24. Liu, W., Yan, N.: Adaptive Finite Element Methods for Optimal Control Governed by PDEs. Science Press, Beijing (2008)
25. Luo, X., Chen, Y., Huang, Y.: A priori error estimates of Crank–Nicolson finite volume element method for parabolic optimal control problems. Adv. Appl. Math. Mech. **5**(5), 688–704 (2013)

26. Meidner, D., Vexler, B.: A priori error estimates for space-time finite element discretization of parabolic optimal control problems part II: problems with control constraints. SIAM J. Control Optim. **47**(3), 1301–1329 (2008)
27. Meyer, C., Rösch, A.: Superconvergence properties of optimal control problems. SIAM J. Control Optim. **43**(3), 970–985 (2004)
28. Strouboulis, T., Wang, D., Babuška, I.: Superconvergence of elliptic reconstructions of finite element solutions of parabolic problems in domains with piecewise smooth boundaries. Comput. Methods Appl. Mech. Eng. **241**(3), 128–141 (2012)
29. Tang, Y., Chen, Y.: Superconvergence analysis of fully discrete finite element methods for semilinear parabolic optimal control problems. Front. Math. China **8**(2), 443–464 (2013)
30. Tang, Y., Hua, Y.: Superconvergence of fully discrete finite element for parabolic control problems with integral constraints. East Asian J. Appl. Math. **3**(2), 138–153 (2013)
31. Vallejos, M., Borzì, A.: Multigrid optimization methods for linear and bilinear elliptic optimal control problems. Computing **82**, 31–52 (2008)
32. Vexler, B., Wollner, W.: Adaptive finite element for elliptic optimization problems with control constraints. SIAM J. Control Optim. **47**(1), 509–534 (2008)