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Approximation by a generalized class of Dunkl type Szász operators based on post quantum calculus

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Abstract

The main purpose of this paper is to introduce a generalized class of Dunkl type Szász operators via post quantum calculus on the interval $[\frac{1}{2}, \infty)$. This type of modification allows a better estimation of the error on $[\frac{1}{2}, \infty)$ rather than $[0, \infty)$. We establish Korovkin type result in weighted spaces and also study approximation properties with the help of modulus of continuity of order one, Lipschitz type maximal functions, and Peetre's K -functional. Furthermore, we estimate the degrees of approximations of the operators by modulus of continuity of order two.

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1 Introduction and preliminaries

The first most elegant and easiest proof of Weierstrass approximation theorem was given by S.N Bernstein by introducing positive linear operators [8] known as Bernstein operators. The q -analogue of the Bernstein operators was studied by Lupaş [17] and Phillips [33].

For all $g \in C[0, \infty)$, $x \geq 0$, and $n \in \mathbb{N}$, Szász introduced positive linear operators called Szász operators [38] which are defined by

$$S_n(g; x) = \frac{1}{e(nx)} \sum_{u=0}^{\infty} \frac{(nx)^u}{u!} g\left(\frac{u}{n}\right). \quad (1.1)$$

Recently Szász operators have been studied via Dunkl modification such as the classical Dunkl Szász operators [37], q -Dunkl–Szász operators [14], and (p, q) -Dunkl–Szász operators [7] (see also [1, 6, 32] and [9, 15, 19, 25, 27, 31, 34, 35]). The (p, q) -analogue of Bernstein operators was given in [24] and the Dunkl type modification was studied in [7] (see also [2–5, 18, 20, 21, 26, 28–30, 36, 39]). For some recent work on statistical approximation of positive linear operators, we refer to [12, 22, 23].

The (p, q) -integer $[n]_{p,q}$ is given by $[n]_{p,q} = \frac{p^n - q^n}{p - q}$ for $n = 0, 1, 2, \dots$; for more details on $[n]_{p,q}$ -integers, see [16]. For the exponential function on (p, q) -analogues one has $e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_{p,q}!}$ and $E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_{p,q}!}$.

The q -Hermite type polynomials on q -Dunkl were given in [10] and a recursion formula was obtained by applying a relation. For $\vartheta > \frac{1}{2}$, $x \geq 0$, $0 < q < 1$, and $g \in C[0, \infty)$, İçöz gave a Dunkl generalization of Szász operators via q -calculus [14] as follows:

$$D_{n,q}(f; x) = \frac{1}{e_{\vartheta,q}([n]_q x)} \sum_{u=0}^{\infty} \frac{([n]_q x)^u}{\Theta_{\vartheta,q}(u)} g\left(\frac{1 - q^{2\vartheta\theta_u + u}}{1 - q^n}\right). \quad (1.2)$$

Recently, the (p, q) -approximation of Szász operators on Dunkl analogue has been studied in [7] by using the following exponential function:

$$e_{\vartheta,p,q}(x) = \sum_{u=0}^{\infty} p^{\frac{u(u-1)}{2}} \frac{x^u}{\Theta_{\vartheta,p,q}(u)} \quad (1.3)$$

for $\vartheta > \frac{1}{2}$, $0 < q < p \leq 1$, $x \in [0, \infty)$, and $u \in \mathbb{N}$. The explicit formula for $\Theta_{\vartheta,p,q}(u)$ is given by

$$\begin{aligned} \Theta_{\vartheta,p,q}(u) &= \frac{\prod_{i=0}^{\lfloor \frac{u+1}{2} \rfloor - 1} p^{2\vartheta(-1)^{i+1} + 1} ((p^2)^i p^{2\vartheta+1} - (q^2)^i q^{2\vartheta+1}) \prod_{j=0}^{\lfloor \frac{u}{2} \rfloor - 1} p^{2\vartheta(-1)^j + 1} ((p^2)^j p^2 - (q^2)^j q^2)}{(p - q)^u}, \end{aligned} \quad (1.4)$$

where $\lfloor \frac{u}{2} \rfloor$ denotes the greatest integer functions for $u \in \mathbb{N} \cup \{0\}$. Also

$$\Theta_{\vartheta,p,q}(u+1) = \frac{p^{2\vartheta(-1)^{u+1} + 1} (p^{2\vartheta\theta_{u+1} + u + 1} - q^{2\vartheta\theta_{u+1} + u + 1})}{(p - q)} \Theta_{\vartheta,p,q}(u), \quad u \in \mathbb{N}, \quad (1.5)$$

and

$$\theta_u = \begin{cases} 0 & \text{if } u = 2m, \text{ for } m = 0, 1, 2, 3, \dots, \\ 1 & \text{if } u = 2m + 1, \text{ for } m = 0, 1, 2, 3, \dots \end{cases}$$

2 Auxiliary results

Let $\{\zeta_n(x)\}_{n \geq 1}$ be a sequence of nonnegative continuous functions on $[0, \infty)$ such that

$$\zeta_n(x) = \left(x - \frac{1}{2[n]_{p,q}}\right)_+, \quad n \in \mathbb{N}, \quad (2.1)$$

where

$$\tau_+ = \begin{cases} \tau & \text{if } \tau \geq 0, \\ 0 & \text{if } \tau < 0. \end{cases} \quad (2.2)$$

Moreover, suppose

$$\mathcal{H}_{n,\vartheta}(x) = \frac{e_{\vartheta,p,q}\left(\frac{q}{p}[n]_{p,q}x\right)}{e_{\vartheta,p,q}([n]_{p,q}x)}. \quad (2.3)$$

Let $x \in [0, \infty)$, $g \in C[0, \infty)$, $n \in \mathbb{N}$, $0 < q < p \leq 1$, and $\vartheta > \frac{1}{2}$. We define the new operators by

$$A_{n,p,q}^*(g; x) = \frac{1}{e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u}{\Theta_{\vartheta,p,q}(u)} p^{\frac{u(u-1)}{2}} g\left(\frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)}\right). \quad (2.4)$$

If we put $\zeta_n(x) = x$, then these operators are reduced to the operators studied in [7] and, in addition, if $p = 1$, then we get the operators studied in [14].

Lemma 2.1 Suppose that the operators $A_{n,p,q}^*(\cdot; \cdot)$ are given by (2.4). Then, for all $x \geq \frac{1}{2[n]_{p,q}}$ and $n \in \mathbb{N}$, one obtains

- (1) $A_{n,p,q}^*(1; x) = 1$;
- (2) $A_{n,p,q}^*(t; x) = x - \frac{1}{2[n]_{p,q}}$;
- (3) $x^2 + \frac{1}{[n]_{p,q}}(q^{2\vartheta}[1 - 2\vartheta]_{p,q}\mathcal{H}_{n,\vartheta}(x) - 1)x + \frac{1}{4[n]_{p,q}^2}(1 - 2q^{2\vartheta}[1 - 2\vartheta]_{p,q}\mathcal{H}_{n,\vartheta}(x)) \leq A_{n,p,q}^*(t^2; x) \leq x^2 + \frac{1}{[n]_{p,q}}([1 + 2\vartheta]_{p,q} - 1)x + \frac{1}{4[n]_{p,q}^2}(1 - 2[1 + 2\vartheta]_{p,q})$.

Lemma 2.2 For all $x \geq \frac{1}{2[n]_{p,q}}$ and $n \in \mathbb{N}$, the operators $A_{n,p,q}^*(\cdot; \cdot)$ satisfy

- (1) $A_{n,p,q}^*(t - x; x) = -\frac{1}{2[n]_{p,q}}$;
- (2) $A_{n,p,q}^*((t - x)^2; x) \leq \frac{1}{[n]_{p,q}}[1 + 2\vartheta]_{p,q}x + \frac{1}{4[n]_{p,q}^2}(1 - [1 + 2\vartheta]_{p,q})$.

3 Approximation in weighted spaces

This section deals with the approximation properties of the operators $A_{n,p,q}^*$ in weighted spaces. We evaluate the order of approximation by using the modulus of continuity and Lipschitz class and study some direct theorems. We also obtain the approximation results by modulus of continuity of order two. We denote $C_B(\mathbb{R}^+)$ for the set of all bounded and continuous functions on \mathbb{R}^+ equipped with the norm

$$\|g\|_{C_B} = \sup_{x \geq 0} |g(x)|,$$

where $\mathbb{R}^+ = [0, \infty)$. We suppose $F := \{g : x \in [0, \infty)\}$ such that $\frac{g(x)}{1+x^2}$ is convergent when $x \rightarrow \infty$. Let $B_{\zeta}(\mathbb{R}^+)$ be the set of all functions satisfying $g(x) \leq u_g \zeta(x)$ with $\zeta(x) = 1 + \xi^2(x)$ and $\xi(x) \rightarrow x$ in which u_g is a constant depending on g (see Gadžiev [13]). Moreover, take $C_{\zeta}(\mathbb{R}^+) = B_{\zeta}(\mathbb{R}^+) \cap C(\mathbb{R}^+)$. Note that $B_{\zeta}(\mathbb{R}^+)$ is a normed space with the norm given by

$$\|g\|_{\zeta} = \sup_{x \geq 0} \frac{|g(x)|}{\zeta(x)}.$$

Let $C_{\zeta}^0(\mathbb{R}^+)$ be a subset of $C_{\zeta}(\mathbb{R}^+)$ such that

$$\lim_{x \rightarrow \infty} \frac{g(x)}{\zeta(x)} = u_g.$$

We consider the positive sequences $q = q_n$ and $p = p_n$ with $0 < q_n < 1$ and $q_n < p_n \leq 1$ such that

$$\lim_n p_n \rightarrow 1, \quad \lim_n q_n \rightarrow 1 \quad \text{and} \quad \lim_n p_n^n \rightarrow c, \quad \lim_n q_n^n \rightarrow d, \quad (3.1)$$

where $0 < c, d \leq 1$.

Theorem 3.1 Let the sequences of positive numbers p_n and q_n be such that $0 < q_n < p_n \leq 1$. Then, for all $f \in C[0, \infty) \cap F$, the operators $A_{n,p_n,q_n}^*(\cdot; \cdot)$ are uniformly convergent on each compact subset of $[0, \infty)$.

Proof In the light of Korovkin's theorem, we prove the uniform convergence of a sequence of A_{n,p_n,q_n}^* on $[0, 1]$ as $n \rightarrow \infty$ by

$$A_{n,p_n,q_n}^*(t^i; x) \rightarrow x^i, \quad i = 0, 1, 2.$$

Clearly, from (3.1) and $\frac{1}{[n]_{p_n,q_n}} \rightarrow 0$ ($n \rightarrow \infty$), we have

$$\lim_{n \rightarrow \infty} A_{n,p_n,q_n}^*(t; x) = x, \quad \lim_{n \rightarrow \infty} A_{n,p_n,q_n}^*(t^2; x) = x^2. \quad \square$$

Theorem 3.2 Let $A_{n,p_n,q_n}^* : C_\varsigma(\mathbb{R}^+) \rightarrow B_\varsigma(\mathbb{R}^+)$. Then, for all $g \in C_\varsigma^0(\mathbb{R}^+)$,

$$\lim_{n \rightarrow \infty} \|A_{n,p_n,q_n}^*(g(t); x) - g(x)\|_\varsigma = 0$$

if and only if

$$\lim_{n \rightarrow \infty} \|A_{n,p_n,q_n}^*(\xi^u(t); x) - \xi^u(x)\|_\varsigma = 0, \quad u = 0, 1, 2.$$

Proof Consider $\xi(x) = x$, $\varsigma = 1 + \xi^2(x)$ and

$$\begin{aligned} & \|A_{n,p_n,q_n}^*(t^\ell; x) - x^\ell\|_\varsigma \\ &= \sup_{x \geq 0} \frac{|A_{n,p_n,q_n}^*(t^\ell; x) - x^\ell|}{1 + x^2}. \end{aligned}$$

From Korovkin's theorem, easily we obtain $\lim_{n \rightarrow \infty} \|A_{n,p_n,q_n}^*(t^\ell; x) - x^\ell\|_\varsigma = 0$ for $\ell = 0, 1, 2$. Hence, for any $g \in C_\varsigma^0(\mathbb{R}^+)$, we get

$$\|A_{n,p_n,q_n}^*(g(t); x) - g(x)\|_\varsigma = 0. \quad \square$$

Theorem 3.3 For every $g \in C_\varsigma^0(\mathbb{R}^+)$, we have

$$\lim_{n \rightarrow \infty} \|A_{n,p_n,q_n}^*(g; x) - g\|_\varsigma = 0.$$

Proof We prove this theorem in the light of Theorem 3.2. Take $f(t) = t^\ell$ for $\ell = 0, 1, 2$ in Lemma 2.1. Then Korovkin's theorem allows for every $g(t) \in C_\varsigma^0(\mathbb{R}^+)$ if it satisfies $A_{n,p_n,q_n}^*(t^\ell; x) \rightarrow x^\ell$ uniformly. Then, for $\ell = 0$, Lemma 2.1 gives $A_{n,p_n,q_n}^*(1; x) = 1$, which implies that

$$\lim_{n \rightarrow \infty} \|A_{n,p_n,q_n}^*(1; x) - 1\|_\varsigma = 0. \quad (3.2)$$

If $\ell = 1$

$$\|A_{n,p_n,q_n}^*(t; x) - x\|_\varsigma$$

$$\begin{aligned}
&= \sup_{x \geq 0} \frac{|A_{n,p_n,q_n}^*(t; x) - x|}{1 + x^2} \\
&= \sup_{x \geq 0} \frac{\left| -\frac{1}{2[n]_{p_n,q_n}} \right|}{1 + x^2} \\
&\leq \frac{1}{2[n]_{p_n,q_n}} \sup_{x \geq 0} \frac{1}{1 + x^2},
\end{aligned}$$

then

$$\lim_{n \rightarrow \infty} \|A_{n,p_n,q_n}^*(t; x) - x\|_{\zeta} = 0. \quad (3.3)$$

Similarly, for $\ell = 2$, we have

$$\begin{aligned}
&\|A_{n,p_n,q_n}^*(t^2; x) - x^2\|_{\zeta} \\
&= \sup_{x \geq 0} \frac{|A_{n,p_n,q_n}^*(t^2; x) - x^2|}{1 + x^2} \\
&\leq \frac{1}{[n]_{p,q}} ([1 + 2\vartheta]_{p,q} - 1) \sup_{x \geq 0} \frac{x}{1 + x^2} + \frac{1}{4[n]_{p,q}^2} (1 - 2[1 + 2\vartheta]_{p,q}) \sup_{x \geq 0} \frac{1}{1 + x^2}.
\end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \|A_{n,p_n,q_n}^*(t^2; x) - x^2\|_{\zeta} = 0. \quad (3.4)$$

This completes the proof. \square

4 Rate of convergence

Here, we compute the rate of convergence of our new operators (2.4) with the help of modulus of continuity and Lipschitz type maximal functions.

Let $g \in C[0, \infty]$. The modulus of continuity of g is given by

$$\omega_{\varrho}(g; \delta) = \sup_{|y-x| \leq \delta} |g(y) - g(x)|, \quad x, y \in [0, \varrho] \quad (4.1)$$

for any $\delta > 0$. It is known that $\lim_{\delta \rightarrow 0+} \omega_{\varrho}(g; \delta) = 0$, and one has

$$|g(y) - g(x)| \leq \left(\frac{|y-x|}{\delta} + 1 \right) \omega_{\varrho}(g; \delta). \quad (4.2)$$

Theorem 4.1 *Let $\omega_{\varrho}(g; \delta)$ be defined on the interval $[0, \varrho + 1] \subset [0, \infty)$ with $\varrho > 0$. Then, for every $g \in C_{\zeta}^u$ on $[0, \infty)$, we have*

$$|A_{n,p,q}^*(g; x) - g(x)| \leq \left\{ 1 + \sqrt{[1 + 2\vartheta]_{p,q} \left(x - \frac{1}{4[n]_{p,q}} \right) + \frac{1}{4[n]_{p,q}}} \right\} \omega \left(g; \frac{1}{\sqrt{[n]_{p,q}}} \right).$$

Proof To prove this theorem, we use the Cauchy–Schwarz inequality and apply (4.1) and (4.2). Thus, we have

$$|A_{n,p,q}^*(g; x) - g(x)|$$

$$\begin{aligned}
&\leq \frac{1}{e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u}{\Theta_{\vartheta,p,q}(u)} p^{\frac{u(u-1)}{2}} \\
&\quad \times \left| g\left(\frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)}\right) - g(x) \right| \\
&\leq \frac{1}{e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u}{\Theta_{\vartheta,p,q}(u)} p^{\frac{u(u-1)}{2}} \\
&\quad \times \left\{ 1 + \frac{1}{\delta} \left| \left(\frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)} \right) - x \right| \right\} \omega_{\varrho}(g; \delta) \\
&= \left\{ 1 + \frac{1}{\delta} \left(\frac{1}{e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u}{\Theta_{\vartheta,p,q}(u)} p^{\frac{u(u-1)}{2}} \right. \right. \\
&\quad \times \left. \left| \frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)} - x \right| \right) \right\} \omega_{\varrho}(g; \delta) \\
&\leq \left\{ 1 + \frac{1}{\delta} \left(\frac{1}{e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u}{\Theta_{\vartheta,p,q}(u)} p^{\frac{u(u-1)}{2}} \right. \right. \\
&\quad \times \left. \left. \left(\frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)} - x \right)^2 \right)^{\frac{1}{2}} \right\} \omega_{\varrho}(g; \delta) \\
&= \left\{ 1 + \frac{1}{\delta} (A_{n,p,q}^*(t-x)^2; x)^{\frac{1}{2}} \right\} \omega_{\varrho}(g; \delta) \\
&\leq \left\{ 1 + \frac{1}{\delta} \sqrt{\frac{1}{[n]_{p,q}} [1+2\vartheta]_{p,q} x + \frac{1}{4[n]_{p,q}^2} (1 - [1+2\vartheta]_{p,q})} \right\} \omega_{\varrho}(g; \delta),
\end{aligned}$$

if we choose $\delta = \sqrt{\frac{1}{[n]_{p,q}}}$, then we get our result. \square

We now give the rate of convergence of $A_{n,p,q}^*$ in terms of the elements of the usual Lipschitz class $\text{Lip}_K(\mu)$.

Let $g \in C[0, \infty)$, $K > 0$, and $0 < \mu \leq 1$. The Lipschitz class $\text{Lip}_K(\mu)$ is given by

$$\text{Lip}_K(\mu) = \{g : |g(\varphi_1) - g(\varphi_2)| \leq K|\varphi_1 - \varphi_2|^{\mu} (\varphi_1, \varphi_2 \in [0, \infty))\}. \quad (4.3)$$

Theorem 4.2 Let $A_{n,p,q}^*(\cdot; \cdot)$ be the operator defined in (2.4). Then, for each $g \in \text{Lip}_K(\mu)$ with $K > 0$, $0 < \mu \leq 1$ and satisfying (4.3), we have

$$|A_{n,p,q}^*(g; x) - f(x)| \leq K \left(\frac{1}{[n]_{p,q}} [1+2\vartheta]_{p,q} x + \frac{1}{4[n]_{p,q}^2} (1 - [1+2\vartheta]_{p,q}) \right)^{\frac{\mu}{2}}.$$

Proof We apply Hölder's inequality.

$$\begin{aligned}
|A_{n,p,q}^*(g; x) - g(x)| &\leq |A_{n,p,q}^*(g(t) - g(x); x)| \\
&\leq A_{n,p,q}^*(|g(t) - g(x)|; x) \\
&\leq KA_{n,p,q}^*(|t - x|^{\mu}; x).
\end{aligned}$$

Therefore

$$\begin{aligned}
 & |A_{n,p,q}^*(g; x) - f(x)| \\
 & \leq K \frac{1}{e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u}{\Theta_{\vartheta,p,q}(u)} p^{\frac{u(u-1)}{2}} \\
 & \quad \times \left| \frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)} - x \right|^{\mu} \\
 & \leq K \frac{1}{e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x))} \sum_{u=0}^{\infty} \left(\frac{([n]_{p,q}\zeta_n(x))^u p^{\frac{u(u-1)}{2}}}{\Theta_{\vartheta,p,q}(u)} \right)^{\frac{2-\mu}{2}} \\
 & \quad \times \left(\frac{([n]_{p,q}\zeta_n(x))^u p^{\frac{u(u-1)}{2}}}{\Theta_{\vartheta,p,q}(u)} \right)^{\frac{\mu}{2}} \left| \frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)} - x \right|^{\mu} \\
 & \leq K \left(\frac{1}{(e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x)))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u p^{\frac{u(u-1)}{2}}}{\Theta_{\vartheta,p,q}(u)} \right)^{\frac{2-\mu}{2}} \\
 & \quad \times \left(\frac{1}{(e_{\vartheta,p,q}([n]_{p,q}\zeta_n(x)))} \sum_{u=0}^{\infty} \frac{([n]_{p,q}\zeta_n(x))^u p^{\frac{u(u-1)}{2}}}{\Theta_{\vartheta,p,q}(u)} \right. \\
 & \quad \times \left. \left| \frac{p^{2\vartheta\theta_u+u} - q^{2\vartheta\theta_u+u}}{p^{u-1}(p^n - q^n)} - x \right|^2 \right)^{\frac{\mu}{2}} \\
 & \leq K (A_{n,p,q}^*(t-x)^2; x)^{\frac{\mu}{2}},
 \end{aligned}$$

which proves the theorem. \square

We consider the following space:

$$C_B^2(\mathbb{R}^+) = \{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\}, \quad (4.4)$$

which is equipped with the norm

$$\|g\|_{C_B^2(\mathbb{R}^+)} = \|g\|_{C_B(\mathbb{R}^+)} + \|g'\|_{C_B(\mathbb{R}^+)} + \|g''\|_{C_B(\mathbb{R}^+)}, \quad (4.5)$$

also

$$\|g\|_{C_B(\mathbb{R}^+)} = \sup_{x \in \mathbb{R}^+} |g(x)|. \quad (4.6)$$

Theorem 4.3 *Let us consider the operators $A_{n,p,q}^*(\cdot; \cdot)$ given in (2.4). Then, for any $g \in C_B^2(\mathbb{R}^+)$, we have*

$$|A_{n,p,q}^*(g; x) - g(x)| \leq \frac{1 + [1 + 2\vartheta]_{p,q}}{2[n]_{p,q}} x \|g\|_{C_B^2(\mathbb{R}^+)} + \frac{1}{[n]_{p,q}^2} (1 - [1 + 2\vartheta]_{p,q}) \frac{\|g\|_{C_B^2(\mathbb{R}^+)}}{8}.$$

Proof Suppose that $g \in C_B^2(\mathbb{R}^+)$. It follows from Taylor series expansion that

$$g(t) = g(x) + g'(x)(t-x) + g''(\varphi) \frac{(t-x)^2}{2}, \quad \varphi \in (x, t).$$

Since the operator $A_{n,p,q}^*$ is linear, by operating $A_{n,p,q}^*$ on both sides of the last equality, we have

$$A_{n,p,q}^*(g; x) - g(x) = g'(x)A_{n,p,q}^*((t-x); x) + \frac{g''(\varphi)}{2}A_{n,p,q}^*((t-x)^2; x),$$

which yields

$$\begin{aligned} |A_{n,p,q}^*(g; x) - g(x)| &\leq \left(\frac{1}{[n]_{p,q}} [1 + 2\vartheta]_{p,q} x + \frac{1}{4[n]_{p,q}^2} (1 - [1 + 2\vartheta]_{p,q}) \right) \frac{\|g''\|_{C_B(\mathbb{R}^+)}}{2} \\ &\quad + \frac{1}{2[n]_{p,q}} \|g'\|_{C_B(\mathbb{R}^+)}. \end{aligned}$$

From (4.5), we have

$$\|g'\|_{C_B(\mathbb{R}^+)} \leq \|g\|_{C_B^2(\mathbb{R}^+)} \quad \text{and} \quad \|g''\|_{C_B(\mathbb{R}^+)} \leq \|g\|_{C_B^2(\mathbb{R}^+)}.$$

Consequently,

$$|A_{n,p,q}^*(g; x) - g(x)| \leq \frac{1}{8[n]_{p,q}} \left(4 + 4[1 + 2\vartheta]_{p,q} x + \frac{1}{[n]_{p,q}} (1 - [1 + 2\vartheta]_{p,q}) \right) \|g\|_{C_B^2(\mathbb{R}^+)},$$

which completes the proof. \square

Peetre's K -functional is defined by

$$K_2(g; \delta) = \inf_{C_B^2(\mathbb{R}^+)} \{ (\|g - f\|_{C_B(\mathbb{R}^+)} + \delta \|f\|_{C_B^2(\mathbb{R}^+)}) : f \in \mathcal{W}^2 \}, \quad (4.7)$$

where

$$\mathcal{W}^2 = \{ f \in C_B(\mathbb{R}^+) : f', f'' \in C_B(\mathbb{R}^+) \}. \quad (4.8)$$

Then there exists a constant $M > 0$ such that

$$K_2(g; \delta) \leq M \omega_2(g; \delta^{\frac{1}{2}}) \quad (\delta > 0),$$

where $\omega_2(g; \delta^{\frac{1}{2}})$ (second order modulus of continuity) is given by

$$\omega_2(g; \delta^{\frac{1}{2}}) = \sup_{0 < h < \delta^{\frac{1}{2}}} \sup_{x \in \mathbb{R}^+} |g(x + 2h) - 2g(x + h) + g(x)|. \quad (4.9)$$

Theorem 4.4 For every $g \in C_B(\mathbb{R}^+)$, there exists a positive constant M such that

$$|A_{n,p,q}^*(g; x) - g(x)| \leq 2M \{ \omega_2(g; \sqrt{\Lambda_n(x)}) + \min(1, \Lambda_n(x)) \|g\|_{C_B(\mathbb{R}^+)} \}.$$

Proof We prove this by using Theorem (4.3)

$$|A_{n,p,q}^*(g; x) - g(x)|$$

$$\begin{aligned}
&\leq |A_{n,p,q}^*(g-f;x)| + |A_{n,p,q}^*(f;x) - f(x)| \\
&\quad + |g(x) - f(x)| \\
&\leq \frac{1 + [1 + 2\vartheta]_{p,q}}{2[n]_{p,q}} x \|f\|_{C_B^2(\mathbb{R}^+)} \\
&\quad + \frac{1}{[n]_{p,q}^2} (1 - [1 + 2\vartheta]_{p,q}) \frac{\|f\|_{C_B^2(\mathbb{R}^+)}}{8} + 2\|g - f\|_{C_B(\mathbb{R}^+)} \\
&\leq 2 \left\{ \left(\frac{1}{4[n]_{p,q}} (1 + [1 + 2\vartheta]_{p,q}) x \right. \right. \\
&\quad \left. \left. + \frac{1}{16[n]_{p,q}^2} (1 - [1 + 2\vartheta]_{p,q}) \right) \|f\|_{C_B^2(\mathbb{R}^+)} + \|g - f\|_{C_B(\mathbb{R}^+)} \right\}.
\end{aligned}$$

Considering the infimum over all $f \in C_B^2(\mathbb{R}^+)$ and using (4.7), we obtain

$$|A_{n,p,q}^*(g;x) - g(x)| \leq 2K_2(g; \Lambda_n(x)),$$

where

$$\Lambda_n(x) = \frac{1}{4[n]_{p,q}} (1 + [1 + 2\vartheta]_{p,q}) x + \frac{1}{16[n]_{p,q}^2} (1 - [1 + 2\vartheta]_{p,q}).$$

Now, for an absolute constant $M > 0$ in [11], we use the relation

$$K_2(g; \delta) \leq M \{ \omega_2(g; \sqrt{\delta}) + \min(1, \delta) \|g\| \},$$

which proves our theorem. \square

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