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New refinements of the Hadamard inequality on coordinated convex function

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Abstract

In this paper, new refinements of the Hadamard inequality on coordinated convex function are established. Besides, a simple proof of the Hadamard type for linear functions is also found. Moreover, some examples of the new refinement inequalities are given to check their validity.

Keywords: Convex function; Hermite–Hadamard inequality; Generalized convexity

1 Introduction and preliminaries

Convex functions have many applications in various areas of science such as management science, finance and engineering. Moreover, numerous extensions and generalizations of convexity have been applied in mathematical inequalities and optimization. A function $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$ is called a convex if, for any $m, n \in J$ and $\theta \in [a_1, a_2]$,

$$\psi\bigl(\theta m + (1-\theta)n\bigr) \le \theta\psi(m) + (1-\theta)\psi(n).$$

The most famous inequality in the literature for convex functions is known as Hadamard's inequality. This inequality was proposed in 1893 by Hadamard. The following theorem states the double inequality, introduced by Pečarić and Tong ([1], 1992).

Theorem 1 *Let $\psi : [a_1, a_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function where $a_1 \le a_2$, then*

$$\psi\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \psi(x)\,dx \le \frac{\psi(a_1) + \psi(a_2)}{2}. \tag{1}$$

Hadamard's inequalities play a crucial role in various branches of science, including engineering, economics, astronomy, and mathematics. Thus, due to its great utility in several areas of pure and applied mathematics, much attention has been paid, by many mathematicians, to Hadamard's inequality. Consequently, such inequalities were studied extensively by many authors. Also, numerous generalizations and extensions have been reported in a number of papers [2–4].

Dragomir [5] defined the following mapping which is considered to be naturally connected with Hadamard's result:

$$H:[0,1]\to\mathbb{R},\qquad H(t):=\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi\left(tx+(1-t)\frac{a_1+a_2}{2}\right)dx,$$

where $\psi:[a_1,a_2]\to\mathbb{R}$ is a convex function defined on $[a_1,a_2]$.

Using the above result, we now state the following theorem.

Theorem 2 *Let $\psi:[a_1,a_2]\to\mathbb{R}$ be a convex function. We define $H:[0,1]\to\mathbb{R}$ to get the following inequalities*:

$$\psi\left(\frac{a_1+a_2}{2}\right)\le H(t)\le\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(x)\,dx.$$

ElFarissi [6] established a new generalization of inequality (1) as follows.

Theorem 3 *Assume that $\psi:[a_1,a_2]\to\mathbb{R}$ is a convex function on $[a_1,a_2]$, then for all $\alpha\in[0,1]$ we have*

$$\psi\left(\frac{a_1+a_2}{2}\right)\le m(\alpha)\le\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(x)\,dx\le M(\alpha)\le\frac{\psi(a_1)+\psi(a_2)}{2},\tag{2}$$

where

$$m(\alpha)=:\alpha\psi\left(\frac{ta_2+(2-t)a_1}{2}\right)+(1-\alpha)\psi\left(\frac{(1+\alpha)a_2+(1-\alpha)a_1}{2}\right)$$

and

$$M(\alpha)=:\frac{1}{2}\bigl[\psi\bigl(\alpha a_2+(1-\alpha)a_1\bigr)+\alpha\psi(a_1)+(1-\alpha)\psi(a_2)\bigr].$$

The concept of coordinated convexity was introduced by Dragomir [7]. This is a modification of the convex functions, as given by the following definition.

Definition 1 A function $\psi:A=[a_1,a_2]\times[b_1,b_2]\to\mathbb{R}$ with $a_1<a_2$ and $b_1<b_2$ is called a convex function on coordinates on A if the partial mappings $\psi_x:[b_1,b_2]\to\mathbb{R}$, $\psi_x(v)=\psi(x,v)$, and $\psi_y:[a_1,a_2]\to\mathbb{R}$, $\psi_y(u)=\psi(u,y)$ defined for all $x\in[a_1,a_2]$ and $y\in[b_1,b_2]$, are convex.

Following the above definition, we remark that every convex function, $\psi:[a_1,a_2]\times[b_1,b_2]\to\mathbb{R}$, is convex on coordinates. However, the converse is not true; see ([7]).

Moreover, a formal definition of the coordinate convex function is given now.

Definition 2 A function $\psi:A=[a_1,a_2]\times[b_1,b_2]\to\mathbb{R}$ is said to be convex on coordinates on A if the following inequality holds:

$$\begin{aligned}\psi\bigl(sx+(1-s)y,tu+(1-t)v\bigr)&\le st\psi(x,u)+t(1-s)\psi(y,u)+s(1-t)\psi(x,v)\\&\quad+(1-s)(1-t)\psi(y,v).\end{aligned}$$

The Hadamard inequalities for coordinate convex functions were further modified by Dragomir [7]. These inequalities provide continuous scales of refinements to Hadamard's inequalities.

Theorem 4 *The function $\psi : A = [a_1, a_2] \times [b_1, b_2] \to \mathbb{R}$ is convex on coordinates on A. Thus the following inequalities hold*:

$$\begin{aligned}\psi\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right) &\leq \frac{1}{(a_2-a_1)(b_2-b_1)} \int_{a_1}^{a_2} \int_{b_1}^{b_2} \psi(x,y)\,dx\,dy \\ &\leq \frac{\psi(a_1,b_1)+\psi(a_1,b_2)+\psi(a_2,b_1)+\psi(a_2,b_2)}{4}. \end{aligned} \tag{3}$$

Many improvements and generalizations of the above result have been extensively investigated by a number of researchers. For several recent results, see [8, 9] and the references therein. While the Hadamard inequalities were proved by considering convexity on coordinates, this paper is aimed at proving the inequality using the definition of convexity. This can be obtained by a change of variables.

2 Simple proof of Hadamard type for linear functions

In order to prove the Hadamard inequality, the following lemma is considered.

Lemma 1 *Let ψ be an integrable function on A. Then the following result holds*:

$$\frac{1}{(a_2-a_1)(b_2-b_1)} \int_{a_1}^{a_2} \int_{b_1}^{b_2} \psi(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 \psi\big(\theta_1 a_2 + (1-\theta_1)a_1, \theta_2 b_1 + (1-\theta_2)b_2\big)\,d\theta_1\,d\theta_2 \tag{4}$$

$$= \int_0^1 \int_0^1 \psi\big(\theta_1 a_1 + (1-\theta_1 a)a_2, \theta_2 b_2 + (1-\theta_2)b_1\big)\,d\theta_1\,d\theta_2. \tag{5}$$

Proof We let $x = \theta_1 a_2 + (1-\theta_1)a_1$ and $y = \theta_2 b_1 + (1-\theta_2)b_2$ to prove (4). Meanwhile, (5) can be proved by denoting $x = \theta_1 a_1 + (1-\theta_1)a_2$ and $y = \theta_2 b_2 + (1-\theta_2)b_1$, which is required. □

Now, a simple proof of Hermite–Hadamard's inequality for linear functions can be given as follows.

Proof We let $x = \theta_1 a_2 + (1-\theta_1)a_1$, and $y = \theta_1 a_1 + (1-\theta_1)a_2$. Since ψ is a convex function, we have, for all $\theta_1, \theta_2 \in [0,1]$,

$$\psi\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$$

$$= \psi\left(\frac{\theta_1 a_2 + (1-\theta_1)a_1 + \theta_1 a_1 + (1-\theta_1)a_2}{2}, \frac{\theta_2 b_1 + (1-\theta_2)b_2 + \theta_2 b_2 + (1-\theta_2)b_1}{2}\right)$$

$$\leq \left(\frac{\psi(\theta_1 a_2 + (1-\theta_1)a_1, 0) + \psi(\theta_1 a_1 + (1-\theta_1)a_2, 0)}{2}, \frac{\psi(0, \theta_2 b_2 + (1-\theta_2)b_2) + \psi(0, \theta_2 b_2 + (1-\theta_2)b_1)}{2} \right)$$
$$\leq \frac{\psi(a_1,b_1) + \psi(a_1,b_2) + \psi(a_2,b_1) + \psi(a_2,b_2)}{4}.$$

Thus, one can write

$$\psi\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$$
$$\leq \left(\frac{\psi(\theta_1 a_2 + (1-\theta_1)a_1, 0) + \psi(\theta_1 a_1 + (1-\theta_1)a_2, 0)}{2}, \frac{\psi(0, \theta_2 b_1 + (1-\theta_2)b_2) + \psi(0, \theta_2 b_2 + (1-\theta_2)b_1)}{2} \right)$$
$$\leq \frac{\psi(a_1,b_1) + \psi(a_1,b_2) + \psi(a_2,b_1) + \psi(a_2,b_2)}{4}. \tag{6}$$

Using Lemma 1 and integrating inequality (6) over $[0,1]^2$, we obtain (3). □

Since the aim of this article is to establish refinements inequalities of (3), the new inequality related to both sides of Hadamard's result can be obtained.

3 The refinements of Hadamard's type for convex functions

The Hadamard' inequalities for convex functions on coordinates can be refined as follows.

Theorem 5 *Assume that $\psi : A \to \mathbb{R}$ is a convex function on A. Then, for all $\lambda \in [0,1]$, we have*

$$\begin{aligned}\psi\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right) &\leq s(\theta_1,\theta_2) \\ &\leq \frac{1}{(a_2-a_1)(b_2-b_1)} \int_{a_1}^{a_2} \int_{b_1}^{b_2} \psi(x,y)\,dx\,dy \\ &\leq S(\theta_1,\theta_2) \\ &\leq \frac{1}{4}\big[\psi(a_1,b_1) + \psi(a_1,b_2) + \psi(a_2,b_1) + \psi(a_2,b_2)\big],\end{aligned} \tag{7}$$

where

$$\begin{aligned}s(\theta_1,\theta_2) &= \theta_1\theta_2\psi\left(\frac{\theta_1 a_2 + (2-\theta_1)a_1}{2}, \frac{\theta_2 b_2 + (2-\theta_2)b_1}{2}\right) \\ &\quad + (1-\theta_1)(1-\theta_2)\psi\left(\frac{(1+\theta_1)a_2 + (1-\theta_1)a_1}{2}, \frac{(1+\theta_2)b_2 + (1-\theta_2)b_1}{2}\right)\end{aligned}$$

and

$$S(\theta_1,\theta_2) = \frac{\theta_1\theta_2}{2}\big(\psi(a_1,b_1) + \psi\big(a_1, N(\theta_2)\big) + \psi\big(N(\theta_1), b_1\big)\psi\big(N(\theta_1), N(\theta_2)\big) + \psi(a_2,b_2)\big).$$

Proof Let ψ be a convex function on A. Then, by applying (1) on the subinterval $[a_1, N(\theta_1)]$ and $[b_1, N(\theta_2)]$, with $N(\theta_1) = \theta_1 a_2 + (1-\theta_1)a_1$, $N(\theta_2) = \theta_2 b_2 + (1-\theta_2)b_1$, and $\theta_1, \theta_2 \neq 0$, we obtain

$$\begin{aligned}
\psi\left(\frac{a_1 + N(\theta_1)}{2}, \frac{b_1 + N(\theta_2)}{2}\right) &= \psi\left(\frac{\theta_1 a_2 + (2-\theta_1)a_1}{2}, \frac{\theta_2 b_2 + (2-\theta_2)b_1}{2}\right) \\
&\leq \frac{1}{\theta_1\theta_2(a_2-a_1)(b_2-b_1)} \int_{a_1}^{N(\theta_1)} \int_{b_1}^{N(\theta_2)} \psi(x,y)\,dx\,dy \\
&\leq \frac{\psi(a_1,b_1) + \psi(a_1,N(\theta_2))}{4} \\
&\quad + \frac{\psi(N(\theta_1),b_1) + \psi(N(\theta_1),N(\theta_2))}{4}.
\end{aligned} \tag{8}$$

By applying (1) again on $[N(\theta_1), a_2]$ and $[N(\theta_2), b_2]$, with $\theta_1, \theta_2 \neq 1$, one can obtain the following:

$$\begin{aligned}
&\psi\left(\frac{N(\theta_1) + a_2}{2}, \frac{N(\theta_2) + b_2}{2}\right) \\
&\quad = \psi\left(\frac{(1+\theta_1)a_2 + (1-\theta_1)a_1}{2}, \frac{(1+\theta_2)b_2 + (1-\theta_2)b_1}{2}\right) \\
&\quad \leq \frac{1}{(1-\theta_1)(1-\theta_2)(a_2-a_1)(b_2-b_1)} \int_{N(\theta_1)}^{a_2} \int_{N(\theta_2)}^{b_2} \psi(x,y)\,dx\,dy \\
&\quad \leq \frac{\psi(\theta_1 a_1 + (1-\theta_1)a_1, N(\theta_2)) + \psi(N(\theta_1), b_2)}{4} \\
&\qquad + \frac{\psi(a_2, N(\theta_2)) + \psi(a_2,b_2)}{4}.
\end{aligned} \tag{9}$$

Multiplying (8) by $\theta_1\theta_2$ and (9) by $(1-\theta_1)(1-\theta_2)$ yields

$$\begin{aligned}
&\theta_1\theta_2\psi\left(\frac{\theta_1 a_2 + (2-\theta_1)a_1}{2}, \frac{\theta_2 b_2 + (2-\theta_2)b_1}{2}\right) \\
&\quad \leq \frac{1}{(a_2-a_1)(b_2-b_1)} \int_{a_1}^{N(\theta_1)} \int_{b_1}^{N(\theta_2)} \psi(x,y)\,dx\,dy \\
&\quad \leq (\theta_1\theta_2)\left(\frac{\psi(a_1,b_1) + \psi(a_1,N(\theta_2)) + \psi(N(\theta_1),b_1)}{4}\right. \\
&\qquad \left. + \psi\frac{(N(\theta_1),N(\theta_2))}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\theta_1)(1-\theta_2)f\left(\frac{(1+\theta_1)a_2 + (1-\theta_1)a_1}{2}, \frac{(1+\theta_2)b_2 + (1-\theta_2)b_1}{2}\right) \\
&\quad \leq \frac{1}{(a_2-a_1)(b_2-b_1)} \int_{N(\theta_1)}^{a_2} \int_{N(\theta_2)}^{b_2} \psi(x,y)\,dx\,dy \\
&\quad \leq (1-\theta_1)(1-\theta_2)\left(\frac{\psi(N(\theta_1),N(\theta_2)}{4}\right. \\
&\qquad \left. + \frac{\psi(N(\theta_1),b_2) + \psi(a_2,N(\theta_2) + \psi(a_2,b_2)}{4}\right).
\end{aligned}$$

When combining the resulting inequalities, we have

$$s(\theta_1,\theta_2) \le \frac{1}{(a_2-a_1)(b_2-b_1)} \int_{a_1}^{a_2} \int_{b_2}^{b_1} \psi(x,y)\,dx\,dy \le S(\theta_1,\theta_2), \tag{10}$$

where $s(\theta_1,\theta_2)$ and $S(\theta_1,\theta_2)$ are defined in the same way as in Theorem 5.

Since ψ is a convex function, we obtain

$$\begin{aligned}
&\psi\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right) \\
&\quad = \psi\left(\theta_1\theta_2\left(\frac{\theta_1 a_2 + (2-\theta_1)a_1}{2}, \frac{\theta_2 b_2 + (2-\theta_2)b_1}{2}\right)\right) \\
&\qquad + \psi\left((1-\theta_1)(1-\theta_2)\left(\frac{(1+\theta_1)a_2 + (1-\theta_1)a_1}{2}, \frac{(1+\theta_2)b_2 + (1-\theta_2)b_1}{2}\right)\right) \\
&\quad \le \theta_1\theta_2\psi\left(\frac{\theta_1 a_2 + (2-\theta_1)a_1}{2}, \frac{\theta_2 b_2 + (2-\theta_2)b_1}{2}\right) \\
&\qquad + (1-\theta_1)(1-\theta_2)f\left(\frac{(1+\theta_1)a_2 + (1-\theta_1)a_1}{2}, \frac{(1+\theta_2)b_2 + (1-\theta_2)b_1}{2}\right) \\
&\quad \le \theta_1\theta_2\psi\left(\frac{N(\theta_1)+a_1}{2}, \frac{N(\theta_2)+b_1}{2}\right) \\
&\qquad + (1-\theta_1)(1-\theta_2)\psi\left(\frac{N(\theta_1)+a_2}{2}, \frac{N(\theta_2)+b_2}{2}\right) \\
&\quad \le \frac{1}{4}\bigl(\psi(a_1,b_1) + \psi(a_1,b_2) + \psi(a_2,b_1) + \psi(a_2,b_2)\bigr).
\end{aligned} \tag{11}$$

Then, by (10) and (11), we obtain (7). This process completes the proof. □

Corollary 1 *Suppose that $\psi : A \to \mathbb{R}$ is coordinated convex on A. Using Theorem 5, with $\theta_1 = \theta_2 = 1$, we obtain the following inequalities*:

$$s(1,1) \le \frac{1}{(b-a)(d-c)} \int_{a_1}^{a_2} \int_{b_1}^{b_2} \psi(x,y)\,dx\,dy \le S(1,1).$$

The next proposition is immediate.

Proposition 1 *Assume that $\psi : A \to \mathbb{R}$ is a convex on A. Then the following inequalities are obtained*:

$$\begin{aligned}
\psi\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right) &\le \sup_{\theta_1\in[0,1]} s(\theta_1,\delta) \le \frac{1}{(a_2-a_1)(b_2-b_1)} \int_{a_1}^{a_2} \int_{b_1}^{b_2} \psi(x,y)\,dx\,dy \\
&\le \inf_{\theta_1\in[0,1]} S(\theta_1,\theta_2) \\
&\le \frac{1}{4}\bigl[\psi(a_1,b_1) + \psi(a_1,b_2) + \psi(a_2,b_1) + \psi(a_2,b_2)\bigr].
\end{aligned} \tag{12}$$

Remark 1 In particular,

(i) If we choose $\theta_1 = 1/2$ and $\theta_2 = 1/3$ in Theorem 5, then we obtain

$$s\left(\frac{1}{2},\frac{1}{3}\right) = \frac{1}{6}\left(\psi\left(\frac{a_2+3a_1}{4}, \frac{b_2+5b_1}{6}\right) + 2\psi\left(\frac{3a_2+a_1}{4}, \frac{2b_2+b_1}{3}\right)\right)$$

and

$$\begin{aligned} S\left(\frac{1}{2},\frac{1}{3}\right) &= \frac{1}{24}\left(\psi(a_1,b_1)+\psi\left(\frac{b_2+2b_1}{3}\right)\right.\\ &\quad+\psi\left(\frac{a_2+a_1}{2},b_1\right)+3\psi\left(\frac{a_2+a_1}{2},\frac{b_2+2b_1}{3}\right)\\ &\quad\left.+2\left(\psi\left(\frac{a_2+a_1}{2},b_2\right)+\psi\left(a_2+\frac{b_2+2b_1}{3}+\psi(a_2,b_2)\right)\right)\right). \end{aligned}$$

(ii) If we choose $\theta_1 = \theta_2 = 1/2$, then one can obtain

$$s\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{16}\bigl(\psi(2\theta_1 a_2+3a_1,2\delta b_2+3b_1)+\psi(3a_2+a_1,3b_2+b_1)\bigr)$$

and

$$\begin{aligned} S\left(\frac{1}{2},\frac{1}{2}\right) &= \frac{1}{16}\left(\psi(a_1,b_1)+\psi\left(a_1,\frac{b_1+b_2}{2}\right)+\psi\left(\frac{a_1+a_2}{2},b_1\right)\right.\\ &\quad+\psi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)+\psi\left(\frac{a_1+a_2}{2},b_2\right)\\ &\quad\left.+\psi\left(a_2,\frac{b_1+b_2}{2}\right)+\psi(a_2,b_2)\right). \end{aligned}$$

Example 1 Assume that $\theta_1 = \frac{a_1}{a_1+a_2}$ and $\theta_2 = \frac{b_1}{b_1+b_2}$, where $0 \le a_1, b_1 \le a_2, b_2$. Then we have

$$\begin{aligned} s\left(\frac{a_1}{a_1+a_2},\frac{b_1}{b_1+b_2}\right) &= \frac{a_1 b_1}{(a_1+a_2)(b_1+b_2)}\psi\left(\frac{a_1(3a_2+a_1)}{2(a_1+a_2)},\frac{b_1(3b_2+b_1)}{2(b_1+b_2)}\right)\\ &\quad+\frac{a_2 b_2}{(a_1+a_2)(b_1+b_2)}\psi\left(\frac{a_2(3a_1+a_2)}{2(a_1+a_2)},\frac{b_2(3b_1+b_2)}{2(b_1+b_2)}\right) \end{aligned}$$

and

$$\begin{aligned} &S\left(\frac{a_1}{a_1+a_2},\frac{b_1}{b_1+b_2}\right)\\ &\quad= \frac{a_1 b_1}{4(a_1+a_2)(b_1+b_2)}\left(\psi(a_1,b_1)+\psi\left(a_1,\frac{2b_1b_2}{b_1+d}\right)\right.\\ &\qquad+\frac{a_1 b_1}{4(a_1+a_2)(b_1+b_2)}\left(\psi\left(\frac{2a_1a_2}{a_1+a_2},b_1\right)+\psi\left(\frac{2a_1a_2}{a_1+a_2},\frac{2b_1b_2}{b_1+b_2}\right)\right)\\ &\qquad+\frac{a_2 b_2}{4(a_1+a_2)(b_1+b_2)}\left(\psi\left(\frac{2a_1a_2}{a_1+a_2},\frac{2b_1b_2}{b_1+b_2}\right)+\psi\left(\frac{2a_1a_2}{a_1+a_2},d\right)\right)\\ &\qquad+\frac{a_2 b_2}{4(a_1+a_2)(b_1+b_2)}\left(\psi\left(a_2,\frac{2b_1b_2}{b_1+b_2}\right)+\psi(a_2,b_2)\right). \end{aligned}$$

(i) If we choose $\psi(x,y) = x^2 + y^2$ with $a_1, b_1 = 0$ and $a_2, b_2 = 1$, then we obtain

$$\begin{aligned} \psi\left(\frac{1}{2},\frac{1}{2}\right) &= \frac{1}{2} \le s(0,1) = \frac{1}{2}\\ &\le \int_0^1\int_0^1 x^2+y^2\,dx\,dy = \frac{2}{3} \end{aligned}$$

$$\begin{aligned}
&\le S(0,1) = 1 \\
&\le \frac{1}{4}\big[\psi(a_1,b_1) + \psi(a_1,b_2) + \psi(a_2,b_1) + \psi(a_2,b_2)\big] = 1.
\end{aligned}$$

(ii) If we choose $\psi(x,y) = x^2y^2$ with $a_1, b_1 = 0$ and $a_2, b_2 = 1$, then we obtain

$$\begin{aligned}
\psi\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{16} &\le s(0,0) = \frac{1}{16} \\
&\le \int_0^1\int_0^1 x^2y^2\,dx\,dy = \frac{1}{9} \\
&\le S(0,0) = \frac{1}{4} \\
&\le \frac{1}{4}\big[\psi(a_1,b_1) + \psi(a_1,b_2) + \psi(a_2,b_1) + \psi(a_2,b_2)\big] = \frac{1}{4}.
\end{aligned}$$

4 Conclusions

This paper has presented some new refinements of the Hadamard inequality on coordinate convex functions. In addition, Hadamard's inequality for linear functions has been proved using the definition of convexity. In order to ascertain the validity of the new refinement inequalities, some examples are also presented. The obtained results in this paper would be useful for generalization of inequalities that were proved in previous work.

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