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A note on type 2 q-Bernoulli and type 2 q-Euler polynomials

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Abstract

As is well known, power sums of consecutive nonnegative integers can be expressed in terms of Bernoulli polynomials. Also, it is well known that alternating power sums of consecutive nonnegative integers can be represented by Euler polynomials. In this paper, we show that power sums of consecutive positive odd q-integers can be expressed by means of type 2 q-Bernoulli polynomials. Also, we show that alternating power sums of consecutive positive odd q-integers can be represented by virtue of type 2 q-Euler polynomials. The type 2 q-Bernoulli polynomials and type 2 q-Euler polynomials are introduced respectively as the bosonic p-adic q-integrals on $\mathbb{Z}_p$ and the fermionic p-adic q-integrals on $\mathbb{Z}_p$. Along the way, we will obtain Witt type formulas and explicit expressions for those two newly introduced polynomials.

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1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ be the p-adic norm which is normalized as $|p|_p=\frac{1}{p}$. Let q be an indeterminate such that $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-\frac{1}{p-1}}$, and $[x]_q=\frac{1-q^x}{1-q}$. Note that $[x]_{-q}=\frac{1-(-q)^x}{1+q}$.

The bosonic p-adic q-integrals on $\mathbb{Z}_p$ are defined by Kim as

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}f(x)q^x \quad \text{(see [7])}, \tag{1.1}$$

where f is a uniformly differentiable function on $\mathbb{Z}_p$.

In [1, 2], Carlitz considered the q-Bernoulli numbers which are given by the recurrence relation:

$$\beta_{0,q}=1,\quad q(q\beta_q+1)^n-\beta_{n,q}=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{1.2}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.

In [6, 7], Kim gave the following Witt type formula:

$$\int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x) = \beta_{n,q} \quad (n \geq 0). \tag{1.3}$$

Carlitz also defined the q-Bernoulli polynomials by

$$\beta_{n,q}(x) = \left(q^x \beta_q + [x]_q\right)^n = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q} \quad \text{(see [1, 2])}, \tag{1.4}$$

where n is a nonnegative integer.

An integral representation for $\beta_{n,q}(x)$, $(n \geq 0)$ was given by Kim as follows:

$$\int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0), \text{ (see [5, 7])}. \tag{1.5}$$

In [6, 8, 10], Kim introduced the modified q-Bernoulli polynomials as the p-adic q-integral on $\mathbb{Z}_p$ given by

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-y} [x+y]_q^n \, d\mu_q(y) \quad (n \geq 0). \tag{1.6}$$

For $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the modified q-Bernoulli numbers.

From (1.1), we note that

$$B_{0,q} = \frac{q-1}{\log q}, \quad (qB_q + 1)^n - B_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1.7}$$

with the usual convention about replacing B_q^n by $B_{n,q}$.

By (1.6), we easily get

$$\begin{aligned} B_{n,q}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} B_{l,q} \quad (n \geq 0), \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l}{[l]_q} \quad (n \geq 0) \text{ (see [6, 8, 10])}. \end{aligned}$$

It is well known that

$$0^k + 1^k + 2^k + \cdots + (n-1)^k = \frac{1}{k+1}\left(B_{k+1}(n) - B_{k+1}\right) \quad (n \geq 1, k \geq 0). \tag{1.8}$$

Here $B_k(x)$ are the Bernoulli polynomials given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{(see [1–17])}, \tag{1.9}$$

and $B_n = B_n(0)$ are called the Bernoulli numbers.

In [8], Kim proved that the power sums of consecutive nonnegative q-integers are given by

$$[0]_q^k + q[1]_q^k + q^2[2]_q^k + \cdots + q^{n-1}[n-1]_q^k = \frac{1}{k+1}\bigl(B_{k+1,q}(n) - B_{k+1,q}\bigr) \quad (n \geq 1, k \geq 0).$$

Now, we consider the power sums of consecutive odd positive q-integers and ask the following question:

$$q[1]_q^k + q^3[3]_q^k + q^5[5]_q^k + \cdots + q^{2n-1}[2n-1]_q^k = ? \tag{1.10}$$

In addition, we ask the following question:

$$[1]_q^k - q[3]_q^k + q^2[5]_q^k - \cdots + (-1)^{n-1}q^{n-1}[2n-1]_q^k = ? \tag{1.11}$$

We will see that (1.10) can be expressed in terms of type 2 q-Bernoulli polynomials and (1.11) by virtue of type 2 q-Euler polynomials. Here we note that the type 2 q-Bernoulli polynomials are represented by bosonic p-adic q-integrals on $\mathbb{Z}_p$ and the type 2 q-Euler polynomials by fermionic p-adic q-integrals on $\mathbb{Z}_p$.

2 Type 2 q-Bernoulli polynomials and numbers

From (1.1), we have

$$\int_{\mathbb{Z}_p} q^{-x} f(x+1)\, d\mu_q(x) = \int_{\mathbb{Z}_p} q^{-x} f(x)\, d\mu_q(x) + \frac{q-1}{\log q} f'(0). \tag{2.1}$$

By using (2.1) and induction, we get

$$\int_{\mathbb{Z}_p} q^{-x} f(x+n)\, d\mu_q(x) = \int_{\mathbb{Z}_p} q^{-x} f(x)\, d\mu_q(x) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l), \tag{2.2}$$

where n is a positive integer.

In view of (1.6), we consider the generating function of the type 2 q-Bernoulli polynomials given by the following p-adic q-integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p} q^{-y} e^{[2y+x+1]_q t}\, d\mu_q(y) = \sum_{n=0}^{\infty} b_{n,q}(x) \frac{t^n}{n!}. \tag{2.3}$$

From (2.3), we have

$$\int_{\mathbb{Z}_p} q^{-y} [2y+x+1]_q^n\, d\mu_q(y) = b_{n,q}(x) \quad (n \geq 0). \tag{2.4}$$

For $x = 0$, $b_{n,q} = b_{n,q}(0)$ are called the type 2 q-Bernoulli numbers.

By (2.4), we get

$$b_{n,q} = \int_{\mathbb{Z}_p} q^{-y} [2y+1]_q^n\, d\mu_q(y) = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{l}{[2l]_q}. \tag{2.5}$$

By (2.5), we easily get

$$\begin{aligned} b_{n,q} &= \frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^l \frac{l}{[2l]_q} \\ &= \frac{2n}{(1-q)^n}\sum_{l=1}^{n}\binom{n-1}{l-1}(-1)^l q^l (1-q)\sum_{m=0}^{\infty} q^{2lm} \\ &= \frac{-2n}{(1-q)^{n-1}}\sum_{m=0}^{\infty} q^{2m+1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{(2m+1)l} \\ &= -2n\sum_{m=0}^{\infty} q^{2m+1}[2m+1]_q^{n-1}. \end{aligned} \tag{2.6}$$

Theorem 2.1 *For $n \geq 0$, we have*

$$\begin{aligned} b_{n,q} &= \frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^l \frac{l}{[2l]_q} \\ &= -2n\sum_{m=0}^{\infty} q^{2m+1}[2m+1]_q^{n-1}. \end{aligned} \tag{2.7}$$

By (2.7), we can derive the generating function for the type 2 q-Bernoulli numbers as follows:

$$\sum_{n=0}^{\infty} b_{n,q}\frac{t^n}{n!} = -2t\sum_{m=0}^{\infty} q^{2m+1} e^{[2m+1]_q t}. \tag{2.8}$$

From (2.4), we note that

$$b_{n,q}(x) = \sum_{l=0}^{n}\binom{n}{l} q^{lx} b_{l,q}[x]_q^{n-l}, \tag{2.9}$$

and that

$$\left(q^2 b_q + 1 + q\right)^n - b_{n,q} = 2nq \quad (n \geq 0). \tag{2.10}$$

From (2.4), we easily get

$$\begin{aligned} b_{n,q}(x) &= \frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{(x+1)l}\frac{l}{[2l]_q} \\ &= -2n\sum_{m=0}^{\infty} q^{2m+1+x}[2m+1+x]_q^{n-1} \quad (n \geq 0). \end{aligned} \tag{2.11}$$

From (2.2), we note that

$$\begin{aligned} b_{m,q}(2n) - b_{m,q} &= \int_{\mathbb{Z}_p} q^{-x}[2x+1+2n]_q^m \, d\mu_q(x) - \int_{\mathbb{Z}_p} q^{-x}[2x+1]_q^m \, d\mu_q(x) \\ &= 2m \sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q^{m-1}. \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.2 *For $m \geq 0$ and $n \in \mathbb{N}$, we have*

$$\frac{1}{m+1}\left(b_{m+1,q}(2n) - b_{m+1,q}\right) = 2\sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q^m. \tag{2.13}$$

From (2.13), we note that

$$\begin{aligned} \sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q^m &= \frac{1}{2m+2}\left(\sum_{l=0}^{m+1}\binom{m+1}{l} q^{2nl} b_{l,q}[2n]_q^{m+1-l} - b_{m+1,q}\right) \\ &= \frac{1}{2m+2}\left(\sum_{l=0}^{m}\binom{m+1}{l} q^{2nl} b_{l,q}[2n]_q^{m+1-l} \right. \\ &\quad \left. + (q-1)\bigl[2n(m+1)\bigr]_q b_{m+1,q}\right). \end{aligned} \tag{2.14}$$

By (2.14), we get the following corollary.

Corollary 2.3 *For $n \in \mathbb{N}$ and $m \geq 0$, we have*

$$\begin{aligned} \sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q^m &= \frac{1}{2m+2}\sum_{l=0}^{m}\binom{m+1}{l} q^{2nl}[2n]_q^{m+1-l} b_{l,q} \\ &\quad + \frac{(q-1)[2n(m+1)]_q b_{m+1,q}}{2m+2}. \end{aligned} \tag{2.15}$$

Example Here we check formula (2.15) for $m = 1$. First, we observe that

$$\begin{aligned} [2n+1]_q^2 - 1 &= \sum_{l=0}^{n-1}\left([2l+3]_q^2 - [2l+1]_q^2\right) \\ &= \sum_{l=0}^{n-1}\left([2l+1]_q + q^{2l+1}[2]_q - [2l+1]_q\right) \\ &\quad \times \left([2l+1]_q + q^{2l+1}[2]_q + [2l+1]_q\right) \\ &= 2[2]_q \sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q + \frac{[2]_q^2 q^2 [4n]_q}{[4]_q}. \end{aligned} \tag{2.16}$$

By (2.16), we get

$$\sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q = \frac{1}{2[2]_q}\left([2n+1]_q^2 - 1 - [2]_q^2 q^2 \frac{[4n]_q}{[4]_q}\right)$$
$$= \frac{1}{2(q+1)}\left\{\left(\frac{1-q^{2n+1}}{1-q}\right)^2 - 1 - q^2(q+1)^2 \frac{1-q^{4n}}{1-q^4}\right\}$$
$$= \frac{q(1-q^{2n})}{(1-q)(1-q^2)} - \frac{q^2(1-q^{4n})}{(1-q)(1-q^4)}. \tag{2.17}$$

We now show that (2.17) agrees with the result in (2.15). For this, we first note the following from (2.7):

$$b_{0,q} = \frac{q-1}{\log q}, \qquad b_{1,q} = -\frac{1}{\log q} - \frac{2q}{1-q^2},$$
$$b_{2,q} = \frac{2}{1-q}\left(-\frac{1}{2\log q} - \frac{2q}{1-q^2} + \frac{2q^2}{1-q^4}\right).$$

Then, from (2.15), we have

$$\sum_{l=0}^{n-1} q^{2l+1}[2l+1]_q = \frac{1}{4}\left([2n]_q^2 b_{0,q} + 2q^{2n}[2n]_q b_{1,q} + (q-1)[4n]_q b_{2,q}\right)$$
$$= \frac{1}{4}\left\{\left(\frac{1-q^{2n}}{1-q}\right)^2 \frac{q-1}{\log q} + 2q^{2n}\frac{1-q^{2n}}{1-q}\left(-\frac{1}{\log q} - \frac{2q}{1-q^2}\right)\right.$$
$$\left. + (q-1)\frac{1-q^{4n}}{1-q}\frac{2}{1-q}\left(-\frac{1}{2\log q} - \frac{2q}{1-q^2} + \frac{2q^2}{1-q^4}\right)\right\}$$
$$= \frac{q(1-q^{2n})}{(1-q)(1-q^2)} - \frac{q^2(1-q^{4n})}{(1-q)(1-q^4)}.$$

3 Type 2 q-Euler polynomials and numbers

It is known that the fermionic p-adic q-integrals on $\mathbb{Z}_p$ are defined by Kim as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \quad \text{(see [11, 14, 17])}, \tag{3.1}$$

where $[x]_{-q} = \frac{1-(-q)^x}{1+q}$.

From (3.1), we note that

$$qI_{-q}(f_1) = q\int_{\mathbb{Z}_p} f(x+1)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{q}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x+1)(-q)^x$$
$$= -\lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=1}^{p^N} f(x)(-q)^x = -I_{-q}(f) + [2]_q f(0). \tag{3.2}$$

By (3.2), we get

$$qI_{-q}(f_1) = -I_{-q}(f) + [2]_q f(0) \tag{3.3}$$

and

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \tag{3.4}$$

where $f_n(x) = f(x+n)$, with $n \in \mathbb{N}$.

As is known, Carlitz considered q-Euler numbers given by the recurrence relation

$$E_{0,q} = 1, \quad q(qE_q + 1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{3.5}$$

with the usual convention about replacing E_q^l by $E_{l,q}$ (see [1, 2]).

In [11], Kim obtained the Witt type formula for Carlitz's q-Euler numbers which is represented by the fermionic p-adic q-integrals on $\mathbb{Z}_p$

$$\int_{\mathbb{Z}_p} [x]_q^n \, d\mu_{-q}(x) = E_{n,q} \quad (n \geq 0). \tag{3.6}$$

From (3.6), we note that

$$E_{n,q} = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{l+1}} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [m]_q^n. \tag{3.7}$$

By (3.7), we readily see that the generating function for Carlitz's q-Euler numbers is given by

$$F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m]_q t}. \tag{3.8}$$

It is known that

$$q^n E_{m,q}(n) + E_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m \quad \text{(see [1, 2])}, \tag{3.9}$$

where $n \in \mathbb{N}$ with $n \equiv 1 (\text{mod } 2)$. Note that equation (3.9) is an alternating sum of powers of consecutive positive q-integers.

Now, we consider an alternating sum of powers of consecutive positive odd q-integers which are given by

$$\sum_{l=0}^{n-1} (-1)^l q^l [2l+1]_q^m = [1]_q^m - q[3]_q^m + q^2[5]_q^m - \cdots + (-1)^{n-1} q^{n-1} [2n-1]_q^m. \tag{3.10}$$

Let us define the type 2 q-Euler polynomials which are given by

$$\int_{\mathbb{Z}_p} [2y+x+1]_q^m \, d\mu_{-q}(y) = \mathcal{E}_{m,q}(x) \quad (m \geq 0). \tag{3.11}$$

When $x = 0$, $\mathcal{E}_{n,q} = \mathcal{E}_{n,q}(0)$, $(n \geq 0)$ are called the type 2 q-Euler numbers.

From (3.11), we note that

$$\begin{aligned}\mathcal{E}_{m,q}(x) &= \frac{[2]_q}{(1-q)^m}\sum_{l=0}^{m}\binom{m}{l}(-1)^l q^{l(1+x)}\frac{1}{1+q^{2l+1}} \\ &= [2]_q\sum_{k=0}^{\infty}(-1)^k q^k[2k+1+x]_q^m \quad (m \geq 0).\end{aligned} \tag{3.12}$$

By (3.12), we get the following generating function for the q-Euler polynomials:

$$\sum_{m=0}^{\infty}\mathcal{E}_{m,q}(x)\frac{t^m}{m!} = [2]_q\sum_{k=0}^{\infty}(-1)^k q^k e^{[2k+1+x]_q t}. \tag{3.13}$$

Theorem 3.1 *For $m \geq 0$, we have*

$$\begin{aligned}\mathcal{E}_{m,q}(x) &= \frac{[2]_q}{(1-q)^m}\sum_{l=0}^{m}\binom{m}{l}(-1)^l q^{l(1+x)}\frac{1}{1+q^{2l+1}} \\ &= [2]_q\sum_{k=0}^{\infty}(-1)^k q^k[2k+1+x]_q^m.\end{aligned} \tag{3.14}$$

From (3.11), we have

$$\begin{aligned}\mathcal{E}_{n,q}(x) &= \int_{\mathbb{Z}_p}[2y+1+x]_q^n\,d\mu_{-q}(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\int_{\mathbb{Z}_p}[2y+1]_q^l\,d\mu_{-q}(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\mathcal{E}_{l,q} \quad (n \geq 0).\end{aligned} \tag{3.15}$$

Also, by (3.3), we get

$$q\mathcal{E}_{m,q}(2) + \mathcal{E}_{m,q} = [2]_q \quad (n \geq 0). \tag{3.16}$$

Therefore, by (3.15) and (3.16), we obtain the following theorem.

Theorem 3.2 *For $m \geq 0$, we have*

$$\mathcal{E}_{m,q}(x) = \sum_{l=0}^{m}\binom{m}{l}[x]_q^{m-l}q^{lx}\mathcal{E}_{l,q}. \tag{3.17}$$

In particular,

$$q\mathcal{E}_{m,q}(2) = [2]_q - \mathcal{E}_{m,q} \quad (m \geq 0). \tag{3.18}$$

Let n be a positive integer with $n \equiv 1 (\mathrm{mod}\ 2)$. From (3.4), we have

$$q^n \int_{\mathbb{Z}_p} [2y+2n+1]_q^m \, d\mu_{-q}(y) + \int_{\mathbb{Z}_p} [2y+1]_q^m \, d\mu_{-q}(y) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l [2l+1]_q^m. \tag{3.19}$$

By (3.11) and (3.19), we get

$$q^n \mathcal{E}_{m,q}(2n) + \mathcal{E}_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l [2l+1]_q^m. \tag{3.20}$$

Therefore, by (3.20), we obtain the following theorem.

Theorem 3.3 *For $n \in \mathbb{N}$ with $n \equiv 1 (\mathrm{mod}\ 2)$ and $m \geq 0$, we have*

$$q^n \mathcal{E}_{m,q}(2n) + \mathcal{E}_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l [2l+1]_q^m. \tag{3.21}$$

4 Conclusions

In an introductory calculus class, the following formulas are proved by mathematical induction and used in Riemann sum evaluations of some definite integrals:

$$\sum_{k=0}^{n} k = 1+2+\cdots+n = \frac{n(n+1)}{2} = \binom{n+1}{2},$$
$$\sum_{k=0}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{k=0}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \binom{n+1}{2}^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

The problem of finding formulas for power sums of consecutive nonnegative integers has captivated mathematicians for many centuries. Even since generalized formulas for the power sums, $S_k(n) = \sum_{l=0}^{n} l^k$, were established, the various representations and number-theoretic properties have been studied by Faulhaber. In this paper, we studied the q-analogues of Faulhaber's well-known formula expressing the power sums in terms of Bernoulli polynomials. Indeed, we showed that power sums of consecutive positive odd q-integers can be expressed by means of type 2 q-Bernoulli polynomials. Also, we showed that alternating power sums of consecutive positive odd q-integers can be represented by virtue of type 2 q-Euler polynomials. The type 2 q-Bernoulli polynomials and type 2 q-Euler polynomials were introduced respectively as the bosonic p-adic q-integrals on $\mathbb{Z}_p$ and the fermionic p-adic q-integrals on $\mathbb{Z}_p$. Along the way, we also obtained Witt type formulas and explicit expressions for those two newly introduced polynomials.

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