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Riesz transforms on the Hardy space associated with generalized Schrödinger operators

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Abstract

Let $\mathcal{L} = -\Delta + \mu$ be a generalized Schrödinger operator, where the measure μ is a nonnegative Radon measure. In this paper, we establish the molecular characterization of the Hardy type space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ associated with $\mathcal{L}$. As applications, we obtain the $H^1_{\mathcal{L}}$-boundedness of Riesz transforms and the imaginary power related to $\mathcal{L}$.

MSC: Primary 42B35; 47B38; 42B35

Keywords: Generalized Schrödinger operator; Hardy type space; Riesz transform; Molecular decomposition

1 Introduction

Consider the generalized Schrödinger operator

$$\mathcal{L} = -\Delta + \mu \quad \text{in } \mathbb{R}^n, n \geq 3, \tag{1.1}$$

where μ is a nonnegative Radon measure on $\mathbb{R}^n$. Throughout this paper we assume that μ satisfies the following conditions: there exist positive constants C_0, C_1, and δ such that

$$\mu\big(B(x,r)\big) \leq C_0\left(\frac{r}{R}\right)^{n-2+\delta} \mu\big(B(x,R)\big) \tag{1.2}$$

and

$$\mu\big(B(x,2r)\big) \leq C_1\big\{\mu\big(B(x,r)\big) + r^{n-2}\big\} \tag{1.3}$$

for all $x \in \mathbb{R}^n$ and $0 < r < R$, where $B(x,r)$ denotes the open ball centered at x with radius r. Condition (1.2) may be regarded as scale-invariant Kato-condition, and (1.3) says that the measure μ is doubling on balls satisfying $\mu(B(x,r)) \geq cr^{n-2}$.

Hardy spaces are widely used various fields of analysis and partial differential equations. Let Δ be the Laplace operator on $\mathbb{R}^n$. It is well known that $H^1(\mathbb{R}^n)$ can be characterized by the maximal function $\sup_{t>0} |e^{-t\Delta} f(x)|$. See Stein [14]. In a sense, $H^1(\mathbb{R}^n)$ can be seen as the Hardy space associated with the operator $-\Delta$. Let $\mathcal{L}$ be a general differential operator, such as second order elliptic self-adjoint operators in divergence form, degenerate Schrödinger operators with nonnegative potential, Schrödinger operators with nonnegative potential, and so on. In recent years, the Hardy spaces associated with $\mathcal{L}$ have become one of hot issues in harmonic analysis, see [2, 4–10] and the references therein.

Let $\mathcal{L}$ be a generalized Schrödinger operator. Denote by $\{T_t\}_{t>0} := \{e^{-t\mathcal{L}}\}_{t>0}$ the heat semigroup generated by $-\mathcal{L}$. The kernel of $\{T_t\}$ is denoted by $K_t^{\mathcal{L}}(\cdot,\cdot)$, that is,

$$T_t f(x) = \int_{\mathbb{R}^n} K_t^{\mathcal{L}}(x,y) f(y)\, d\mu(y).$$

The maximal function associated with $\{T_t\}$ is defined as

$$\mathcal{M}_{\mathcal{L}}(f)(x) := \sup_{t>0} \left| e^{-t\mathcal{L}} f(x) \right| \in L^1(\mathbb{R}^n).$$

In [15], Wu and Yan introduced the following Hardy type space associated with $\mathcal{L}$.

Definition 1.1 A Hardy type space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ related to $\mathcal{L}$ is defined as the set of all functions in $f \in L^1(\mathbb{R}^n)$ satisfying $\mathcal{M}_{\mathcal{L}}(f) \in L^1(\mathbb{R}^n)$. The norm of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is defined as $\|f\|_{H^1_{\mathcal{L}}} := \|\mathcal{M}_{\mathcal{L}}(f)\|_{L^1}$.

Let $\mathcal{L} = -\Delta$. $H^1_{\mathcal{L}}(\mathbb{R}^n)$ goes back to the classical Hardy space $H^1(\mathbb{R}^n)$. For a linear operator T, one of the methods to derive the H^1-boundedness is the so-called "atomic-molecular" method. In recent years, several authors used this method to investigate the boundedness on Hardy spaces associated with operators, see [3, 11, 13]. In Sect. 3.1, via a class of $(1,q)$-type atoms associated with $\mathcal{L}$, we obtain the corresponding atomic characterization of $H^1_{\mathcal{L}}(\mathbb{R}^n)$, see Sect. 3.1. Further, in Sect. 3.2, we introduce the (p,q,ε)-moleculars associated with $\mathcal{L}$ and establish the molecular decomposition of $H^1_{\mathcal{L}}(\mathbb{R}^n)$, see Theorem 3.6. In Sect. 4, let $R_{\mathcal{L}}$ and $\mathcal{L}^{i\gamma}$ denote the Riesz transforms and the imaginary power associated with $\mathcal{L}$, i.e.,

$$\begin{cases} R_{\mathcal{L}} := \nabla(-\Delta+\mu)^{-1/2}; \\ \mathcal{L}^{\gamma} := (-\Delta+\mu)^{i\gamma}. \end{cases}$$

By the aid of the regularities of the integral kernels, we can apply Theorems 3.3 & 3.6 to derive the $H^1_{\mathcal{L}}$-boundedness of $R_{\mathcal{L}}$ and $\mathcal{L}^{i\gamma}$, see Theorems 4.4 & 4.6, respectively.

Throughout this article, we will use c and C to denote the positive constants, which are independent of the main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \le B_1/B_2 \le C$.

2 Preliminaries

2.1 Generalized Schrödinger operators

Let μ be a Radon measure satisfying conditions (1.2) & (1.3). The auxiliary function $m(x,\mu)$ is defined by

$$\frac{1}{m(x,\mu)} =: \sup\left\{ r > 0 : \frac{\mu(B(x,r))}{r^{n-2}} \le C_1 \right\}.$$

We begin by recalling some basic properties of the function $m(x,\mu)$.

Lemma 2.1 ([12, Proposition 1.8 & Remark 1.9]) *Suppose that* μ *satisfies* (1.2) *&* (1.3). *Then*

(i) $0 < m(x,\mu) < \infty$ *for every* $x \in \mathbb{R}^n$.

(ii) *If* $r = m(x,\mu)^{-1}$, *then* $r^{n-2} \leq \mu(B(x,r)) \leq C_1 r^{n-2}$.

(iii) *If* $|x-y| \leq Cm(x,\mu)^{-1}$, *then* $m(x,\mu) \approx m(y,\mu)$.

(iv) *There exist constants* $c, C > 0$ *such that, for* $x, y \in \mathbb{R}^n$,

$$\frac{cm(y,\mu)}{\{1+|x-y|m(y,\mu)\}^{k_0/(1+k_0)}} \leq m(x,\mu) \leq Cm(y,\mu)\bigl\{1+|x-y|m(y,\mu)\bigr\}^{k_0}$$

with $k_0 = C_2/\delta > 0$ *and* $C_2 = \log_2(C_1 + 2^{n-2})$.

With the modified Agmon metric $ds^2 = m(x,\mu)\{dx_1^2 + \cdots + dx_n^2\}$, the distance function $d(x,y,\mu)$ is given by

$$d(x,y,\mu) = \inf_{\gamma} \int_0^1 m\bigl(\gamma(\tau),\mu\bigr)\bigl|\gamma'(\tau)\bigr|\,d\tau, \tag{2.1}$$

where $\gamma : [0,1] \to \mathbb{R}^n$ is absolutely continuous and $\gamma(0) = x$, $\gamma(1) = y$.

A parabolic-type distance function associated with $m(x,\mu)$ is defined by

$$d_\mu(x,y,t) = \inf_{\gamma} \int_0^1 m\bigl(\tilde{\gamma}(\tau),\mu\bigr)\max\bigl\{\bigl|(\tilde{\gamma})'(\tau)\bigr|, \bigl|(\gamma_{n+1})'(\tau)\bigr|\bigr\}\,d\tau, \tag{2.2}$$

where $\gamma(\tau) = (\gamma_1(\tau),\dots,\gamma_n(\tau)) = (\tilde{\gamma}(\tau),\gamma_{n+1}(\tau)) : [0,1] \to \mathbb{R}^n \times \mathbb{R}_+$ is absolutely continuous with $\gamma(0) = (x,0)$, $\gamma(1) = (y,\sqrt{t})$.

Lemma 2.2 ([15, Lemma 2.2]) *For the distance function* $d(x,y,\mu)$ *in* (2.1), *we have*

(i) *For every* $x, y, z \in \mathbb{R}^n$,

$$d(x,y,\mu) \leq d(x,z,\mu) + d(z,y,\mu).$$

(ii) *There are two positive constants* c *and* C *such that, for any* $x, y \in \mathbb{R}^n$,

$$c\bigl\{\bigl\{1+|x-y|m(x,\mu)\bigr\}^{1/(k_0+1)} - 1\bigr\} \leq d(x,y,\mu) \leq C\bigl\{1+|x-y|m(x,\mu)\bigr\}^{k_0+1}.$$

Lemma 2.3 ([15, Lemma 2.3]) *For the distance function* $d_\mu(x,y,t)$, *there exist two positive constants* c *and* C *such that, for any* $x, y \in \mathbb{R}^n$, $x \neq y$, *and* $t > 0$,

$$\begin{cases} d_\mu(x,y,t) \geq c\{\{1+\max\{|x-y|,\sqrt{t}\}m(x,\mu)\}^{1/(k_0+1)} - 1\}; \\ d_\mu(x,y,t) \leq C\{1+\max\{|x-y|,\sqrt{t}\}m(x,\mu)\}^{k_0+1}. \end{cases}$$

It follows from (1.2), (1.3), and Lemma 2.1 that there exists a constant $C > 0$ such that, for every $x \in \mathbb{R}^n$,

$$\mu\bigl(B(x,r)\bigr) \leq \begin{cases} C(rm(x,\mu))^{\delta} r^{n-2}, & r < m(x,\mu)^{-1}; \\ C(rm(x,\mu))^{C_2} m(x,\mu)^{2-n}, & r < m(x,\mu)^{-1}, \end{cases} \tag{2.3}$$

see [15, (2.1)]. The above estimate implies the following.

Lemma 2.4 ([15, (2.2)]) *For every nonnegative Schwarz function ω,*

$$\int_{\mathbb{R}^n} t^{-n/2}\omega\left(\frac{x-y}{\sqrt{t}}\right) d\mu(y) \le \begin{cases} Ct^{-1}(\sqrt{t}m(x,\mu))^{\delta}, & t < m(x,\mu)^{-2}; \\ Ct^{-1}(\sqrt{t}m(x,\mu))^{C_2-n+2}, & t \ge m(x,\mu)^{-2}. \end{cases} \tag{2.4}$$

2.2 Function spaces associated with $\mathcal{L}$

In order to characterize $H^1_{\mathcal{L}}(\mathbb{R}^n)$, Wu and Yan [15] introduced the following $H^1_{\mathcal{L}}$-atoms. For $j \in \mathbb{Z}$, define the sets $\mathcal{B}_j$ as

$$\mathcal{B}_j = \left\{x : 2^{j/2} \le m(x,\mu) < 2^{(j+1)/2}\right\}.$$

Since $0 < m(x,\mu) < \infty$, we have $\mathbb{R}^n = \bigcup_{j\in\mathbb{Z}} \mathcal{B}_j$.

Definition 2.5 A function a is a $(1,\infty)$-atom for $H^1_{\mathcal{L}}(\mathbb{R}^n)$ associated with a ball $B(x_0,r)$ if

(i) $\operatorname{supp} a \subset B(x_0,r)$;
(ii) $\|a\|_{L^\infty} \le |B(x_0,r)|^{-1}$;
(iii) if $x_0 \in \mathcal{B}_j$, then $r \le 2^{1-j/2}$;
(iv) if $x_0 \in \mathcal{B}_j$ and $r \le 2^{-1-j/2}$, then $\int a(x)\,dx = 0$.

The atomic norm of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is defined by $\|f\|_{\mathcal{L}\text{-atom}} := \inf\{\sum_j |\lambda_j|\}$, where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$, where $\{a_j\}$ is a sequence of $(1,\infty)$-atoms and $\{\lambda_j\}$ is a sequence of scalars.

One of the main results of [15] is the following proposition.

Proposition 2.6 ([15, Theorem 1.2]) *Assume that μ is a nonnegative Radon measure on $\mathbb{R}^n$ satisfying (1.2) & (1.3) for some $\delta > 0$. Then the norms $\|f\|_{H^1_{\mathcal{L}}}$ and $\|f\|_{\mathcal{L}\text{-atom}}$ are equivalent, that is, there exists a constant $C > 0$ such that*

$$\frac{1}{C}\|f\|_{H^1_{\mathcal{L}}} \le \|f\|_{H^1_{\mathcal{L}}\text{-atom}} \le C\|f\|_{H^1_{\mathcal{L}}}.$$

At the end of this section, we state some regularity estimates for the kernel $K^{\mathcal{L}}_t(\cdot,\cdot)$.

Proposition 2.7 ([15, Lemma 3.7])

(i) *There exist positive constants C and c depending only on n and constants C_0, C_1 and δ in (1.2) & (1.3) such that*

$$0 \le K^{\mathcal{L}}_t(x,y) \le Ch_t(x-y)e^{-cd_\mu(x,y,t)}.$$

(ii) *For every $0 < \delta' < \delta_0 = \min\{\alpha,\delta,\nu\}$, there exists a constant C such that, for every $N' > 0$, there exists a constant $C > 0$ such that, for $|h| < \sqrt{t}$, we have*

$$\left|K^{\mathcal{L}}_t(x+h,y) - K^{\mathcal{L}}_t(x,y)\right| \le C_{N'}\left(\frac{|h|}{\sqrt{t}}\right)^{\delta'} \frac{1}{t^{n/2}} e^{-c|x-y|^2/t} \frac{C_N}{\{1+\sqrt{t}m(x,\mu)+\sqrt{t}m(y,\mu)\}^{N'}}.$$

3 Molecular characterization of $H^1_{\mathcal{L}}(\mathbb{R}^n)$

3.1 The $(1,q)$-atom decomposition

Now we introduce a new type of atoms.

Definition 3.1 A function a is a $(1,q)$-atom of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ if

(i) $\operatorname{supp} a \subset B(x_0,r)$;

(ii) $\|a\|_q \le |B(x_0,r)|^{1/q-1}$;

(iii) if $r \le \rho(x_0)$, then $\int a(x)\,dx = 0$.

Theorem 3.2 *Any $(1,\infty)$-atom is a $(1,q)$-atom.*

Proof In fact, by Hölder's inequality,

$$\|a\|_q \le \|a\|_\infty \big|B(x_0,r)\big|^{1/q} \le \big|B(x_0,r)\big|^{1/q-1}. \qquad \square$$

Theorem 3.3 *Let $\mathcal{L} = -\triangle + \mu$ be a generalized Schrödinger operator, where $\mu \ne 0$ is a nonnegative Radon measure on $\mathbb{R}^n$ satisfying (1.2) & (1.3) for some $\delta > 0$. Then $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ if and only if $f = \sum_j \lambda_j a_j$, where $\{a_j\}$ are $(1,q)$-atoms and $\{\lambda_j\}$ are scalars.*

Proof Because an $(1,\infty)$-atom is also an $(1,q)$-type atom, we only need to prove that there exists a constant c such that, for any $(1,q)$-atom a, $\|\mathcal{M}_{\mathcal{L}}(a)\|_1 \le c$. Suppose that a is a $(1,q)$-atom supported in $B(x_0,r)$. We write $\|\mathcal{M}_{\mathcal{L}}(a)\|_1 \le I_1 + I_2$, where

$$\begin{cases} I_1 := \int_{B(x_0,4r)} |\mathcal{M}_{\mathcal{L}} a(x)|\,dx; \\ I_2 := \int_{B^c(x_0,4r)} |\mathcal{M}_{\mathcal{L}} a(x)|\,dx. \end{cases}$$

By Hölder's inequality and the L^q-boundedness of $\mathcal{M}_{\mathcal{L}}$, we can get

$$I_1 \le \|\mathcal{M}_{\mathcal{L}} a\|_q \big|B(x_0,r)\big|^{1-1/q} \le C\|a\|_q \big|B(x_0,r)\big|^{1-1/q} \le C.$$

The estimation of I_2 is divided into two cases.

Case 1: $1/m(x_0,\mu) \le r \le 1/4m(x_0,\mu)$. For this case, by (i) of Lemma 2.7, we have

$$\begin{aligned} \mathcal{M}_{\mathcal{L}}(a)(x) &\le c \sup_{t>0} \int_{B(x_0,r)} t^{-n/2} e^{-|x-y|^2/t} \big(1 + m(x,\mu)\sqrt{t}\big)^{-N} \big|a(y)\big|\,dy \\ &\le c \sup_{t>0} \int_{B(x_0,r)} t^{-n/2} e^{-|x-y|^2/t} \big(1 + |x-y|^2/\sqrt{t}\big)^{-n-N} \big(1 + m(x,\mu)\sqrt{t}\big)^{-N} \big|a(y)\big|\,dy. \end{aligned}$$

If $y \in B(x_0,r)$ and $|x-x_0| > 4r$, then $|y-x_0| \le |x-x_0|/4$ and $|y-x| \ge 3|x-x_0|/4$. We can apply Lemma 2.1 to obtain

$$\begin{aligned} \big|\mathcal{M}_{\mathcal{L}}(a)(x)\big| &\le c \sup_{t>0} \frac{1}{t^{n/2}} \int_{B(x_0,r)} \big(|x-x_0|/\sqrt{t}\big)^{-n-N} \big(m(x,\mu)\sqrt{t}\big)^{-N} \big|a(y)\big|\,dy \\ &\le c|x-x_0|^{-n-N} \big[m(x,\mu)\big]^{-N} \int_{B(x_0,r)} \big|a(y)\big|\,dy \\ &\le c|x-x_0|^{-n-N/(k_0+1)} \big[m(x_0,\mu)\big]^{-N/(k_0+1)}, \end{aligned}$$

which gives

$$\int_{|x-x_0|>4r} |\mathcal{M}_{\mathcal{L}}a(x)|\,dx \le c\int_{|x-x_0|>4r} |x-x_0|^{-n-N/(k_0+1)}\big[m(x_0,\mu)\big]^{-N/(k_0+1)}\,dx$$
$$\le c|x-x_0|^{-n-N/(k_0+1)}r^{-N/(k_0+1)} \le C,$$

where in the last inequality we have used the fact that $1 \le rm(x_0,\mu) \le 4$.

Case2: $r < 1/m(x_0,\mu)$. By Proposition 2.7 and the symmetry of $K_t^{\mathcal{L}}(\cdot,\cdot)$, we have

$$\big|K_t^{\mathcal{L}}(x,y+h) - K_t^{\mathcal{L}}(x,y)\big| \le C_N\big(|h|/\sqrt{t}\big)^{\delta'} t^{-n/2}e^{-|x-y|^2/ct}\big\{1+\sqrt{t}m(x,\mu)+\sqrt{t}m(y,\mu)\big\}^{-N}.$$

Notice that $|y-x_0| < r$, $|x-x_0| > 4r \Rightarrow |x-y| \ge 3|x-x_0|/4$. By the canceling condition of a, we can get

$$\begin{aligned}\big|\mathcal{M}_{\mathcal{L}}a(x)\big| &\le \sup_{t>0}\left|\int_{B(x_0,r)}\big[K_t^{\mathcal{L}}(x,y)-K_t^{\mathcal{L}}(x,x_0)\big]a(y)\,dy\right| \\ &\le c\sup_{t>0}\int_{B(x_0,r)} t^{-n/2}e^{-|x-y|^2/ct}\big(|y-x_0|/\sqrt{t}\big)^{\delta'}\big|a(y)\big|\,dy \\ &\le c\sup_{t>0}\left\{t^{-n/2}\int_{B(x_0,r)}\big(1+|x-y|/\sqrt{t}\big)^{-n-\delta'}|y-x_0|^{\delta'}/t^{\delta'/2}\big|a(y)\big|\,dy\right\} \\ &\le cr^{\delta'}|x-x_0|^{-n-\delta'},\end{aligned}$$

which gives

$$\int_{|x-x_0|\ge 4r}\big|\mathcal{M}_{\mathcal{L}}a(x)\big|\,dx \le c\int_{|x-x_0|\ge 4r} r^{\delta'}|x-x_0|^{-n-\delta'}\,dx \le C. \qquad \square$$

3.2 Molecular characterization of $H^1_{\mathcal{L}}(\mathbb{R}^n)$

Now we introduce the molecular of $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

Definition 3.4 Let $1 \le q \le \infty$, $\varepsilon > 0$, $b = 1-1/q+\varepsilon$. An L^q-function M is called a $(1,q,\varepsilon)$-molecular centered at x_0 if

(i) $|x|^{nb}M(x) \in L^q(\mathbb{R}^n)_j$;

(ii) $\|M\|_q^{\varepsilon/b}\||x-x_0|^{nb}M(\cdot)\|_q^{1-\varepsilon/b} \le 1$;

(iii) if $x_0 \in B_k$ and $\|M\|_q^{\{n(1/q-1)\}^{-1}} \le m(x_0,\mu)^{-1}$, $\int M(x)\,dx = 0$.

Lemma 3.5 *If a is a $(1,q)$-atom supported on $B(x_0,r)$, a is also a $(1,q,\varepsilon)$-molecular centered at x_0.*

Proof Recall that $\|a\|_q \le |B(x_0,r)|^{1/q-1}$. It is easy to see that

$$\int_{\mathbb{R}^n}\big||x-x_0|^{nb}a(x)\big|^q\,dx \le \big|B(x_0,r)\big|^{bq+1-q},$$

which indicates that $|\cdot-x_0|^{nb}a \in L^q(\mathbb{R}^n)$ with $\||\cdot-x_0|^{nb}a\|_q \le |B(x_0,r)|^q$. Moreover, for $b = 1-1/q+\varepsilon$,

$$\|a\|_q^{\varepsilon/b}\big\||\cdot-x_0|^{nb}a(\cdot)\big\|_q^{1-\varepsilon/b} \le \big|B(x_0,r)\big|^{(1/q-1)(\varepsilon/b)}\big|B(x_0,r)\big|^{\varepsilon(1-\varepsilon/b)} \le 1.$$

We only need to verify the canceling condition, i.e., $\|a\|_q^{1/\{n(1/q-1)\}} \leq m(x_0,\mu)^{-1}$. Denote by ω_n the volume of the unit ball in $\mathbb{R}^n$. It is clear that $\omega_n > 1$ and $\|a\|_q \leq \omega_n^{(1/q-1)} r^{n(1/q-1)} \leq r^{n(1/q-1)}$, equivalently,

$$r \leq \|a\|_q^{1/\{n(1/q-1)\}} \leq m(x_0,\mu)^{-1}.$$

By the canceling condition of $(1,q)$-atoms, we can see that $\int_{\mathbb{R}^n} a(x)\,dx = 0$. So a is a $(1,q,\varepsilon)$-molecular centered at x_0. □

Theorem 3.6 *Let* $1 \leq q \leq \infty$, $\varepsilon > 0$, $b = 1 - 1/q + \varepsilon$. *Then* $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ *if and only if* $f = \sum_j \lambda_j M_j$, *where* $\{M_j\}$ *are* $(1,q,\varepsilon)$*-moleculars and* $\{\lambda_j\}$ *are scalars with* $\inf \sum_j |\lambda_j| \sim \|f\|_{H^1_{\mathcal{L}}}$, *where the infimum is taken over all decompositions.*

Proof We have known that any $(1,q)$-type atom is also a $(1,q,\varepsilon)$-type molecular. By Theorem 3.3, if $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$, then there exist a sequence of $(1,q)$-type atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ such that $f = \sum_j \lambda_j a_j$. This means that f can be represented as a linear combination of $(1,q,\varepsilon)$-moleculars. Conversely, we only need to verify that, for any $(1,q,\varepsilon)$-molecular, $\|M\|_{H^1_{\mathcal{L}}} \leq C$. For simplicity, denote

$$N_{\mathcal{L}}(M) =: \|M\|_q^{\varepsilon/b} \big\| |\cdot - x_0|^{nb} M(\cdot) \big\|_q^{1-\varepsilon/b}.$$

Without loss of generality, we assume that $N_{\mathcal{L}}(M) = 1$ and $q = 2$. Write $\sigma = \|M\|_2^{1/\{n(1/2-1)\}}$. Let

$$\begin{cases} E_0 = \{x : |x - x_0| \leq \sigma\}; \\ E_k = \{x : 2^{k-1}\sigma < |x - x_0| \leq 2^k\sigma\}, \quad k \in N; \\ B_k = \{x : |x - x_0| \leq 2^k\sigma\}, \quad k = 0,1,2,\ldots. \end{cases}$$

Denote by ψ_k the characteristic function $\chi_{E_k}(x)$ and write $M(x) = \sum_k M_k(x)$, where $M_k := M(x)\psi_k(x)$.

Case 1: $\sigma \leq 1/m(x_0,\mu)$. Then $\|M\|_2^{1/\{n(1/2-1)\}} \leq m(x_0,\mu)^{-1}$ and $\int_{\mathbb{R}^n} M(x)\,dx = 0$. The proof is similar to the classical case, and we omit it.

Case 2: $\sigma > 1/m(x_0,\mu)$. For this case, $\||\cdot - x_0|^{n(1/2+\varepsilon)} M(\cdot)\|_2^{1-\varepsilon/b} = \|M\|_2^{-\varepsilon/b}$. Denote by σ the term $\|M\|_2^{1/\{n(1/2-1)\}}$. Then $\||\cdot - x_0|^{n(1/2+\varepsilon)} M(\cdot)\|_2 = \sigma^{n\varepsilon}$ and

$$\frac{1}{|B_0|} \int_{\mathbb{R}^n} \big|M_0(x)\big|^2\,dx \leq \frac{1}{\sigma^n} \|M\|_2^2 = \frac{1}{\sigma^{2n}},$$

which implies that $\|M_0\|_2 \leq |B_0|^{-1/2}$.

For the term M_k, we have

$$\begin{aligned} \frac{1}{|B_k|} \int_{\mathbb{R}^n} \big|M_k(x)\big|^2\,dx &\leq \frac{1}{(2^{k-1}\sigma)^n} \big\| |\cdot - x_0|^{nb} M_k(\cdot) \big\|_2^2 \big(2^{k-1}\sigma\big)^{-n(1+2\varepsilon)} \\ &\leq \big(2^{k-1}\sigma\big)^{-2n-2\varepsilon n} \sigma^{2n\varepsilon} \\ &\leq C_{n,\varepsilon} \big(2^k\sigma\big)^{-2n} 2^{-2k\varepsilon}, \end{aligned}$$

that is, $\|M\|_2 \leq C|B_k|^{-1/2}2^{-k\varepsilon n}$. Let $a_k(x) = \lambda_k^{-1}M_k(x)$, $k = 0,1,2,\ldots$, where $\lambda_k = 2^{-2k\varepsilon n}$ and a_k, $k \in \mathbb{Z}_+$, are $(1,2)$-atoms. Hence $M(x) = \sum_k \lambda_k a_k(x) = \sum_k M_k(x)$ and $\sum_k |\lambda_k| = C\sum_k 2^{-2k\varepsilon n} < \infty$. Repeating the procedure of [3, Theorem 4], we can prove that $M \in H^1_{\mathcal{L}}(\mathbb{R}^n)$. We omit the details, and this completes the proof of Theorem 3.6. □

4 Operators on the Hardy type space $H^1_{\mathcal{L}}(\mathbb{R}^n)$

4.1 The $H^1_{\mathcal{L}}$-boundedness of $\mathcal{L}^{i\gamma}$

Let $q_t(\cdot,\cdot)$ denote the kernel of $e^{-t\mathcal{L}} - e^{-t(-\Delta)}$. We have

$$q_t(x,y) = h_t(x-y) - K_t^{\mathcal{L}}(x-y) = \int_0^t \int_{\mathbb{R}^n} K_s^{\mathcal{L}}(x,t)h_{t-s}(z-y)\,d\mu(z)\,ds.$$

The following estimate was obtained by Wu and Yan [15].

Lemma 4.1 ([15, Lemma 3.6])

(i) *There exist constants C and c such that, for every $x,y \in \mathbb{R}^n$ and $t > 0$,*

$$q_t(x,y) \leq \begin{cases} C(\sqrt{t}m(x,\mu))^{\delta} t^{-n/2}e^{-|x-y|^2/ct}, & \sqrt{t} \leq m(x,\mu)^{-1}; \\ C(\sqrt{t}m(y,\mu))^{\delta} t^{-n/2}e^{-|x-y|^2/ct}, & \sqrt{t} \leq m(y,\mu)^{-1}; \\ h_t(x-y), & \textit{elsewhere}. \end{cases}$$

(ii) *For every $0 < \delta' < \min\{1,\delta\}$ and $C > 0$, there exist constants C' and c such that, for every $h,x,y \in \mathbb{R}^n$, $|h| \leq |x-y|/4$, $|h| \leq Cm(y,\mu)^{-1}$, we have*

$$\bigl|q_t(x,y+h) - q_t(x,y)\bigr| \leq C'\bigl(|h|m(x,\mu)\bigr)^{\delta'} t^{-n/2}e^{-|x-y|^2/ct}.$$

By the functional calculus, we can see that the kernel of $(-\Delta)^{i\gamma} - \mathcal{L}^{i\gamma}$ can be expressed as

$$g(x,y) := \int_0^\infty t^{-i\gamma} q_t(x,y)\frac{dt}{t}. \tag{4.1}$$

Lemma 4.2 *Let $\mathcal{L} = -\Delta + \mu$ be a generalized Schrödinger operator, where $\mu \neq 0$ is a non-negative Radon measure on $\mathbb{R}^n$ satisfying (1.2) & (1.3) for some $\delta > 0$.*

(i) *If $y \in B(x_0,r)$, then*

$$\bigl|g(x,y)\bigr| \leq Cm(x_0,\mu)^{\delta}|x-y|^{\delta-n}. \tag{4.2}$$

(ii) *There exists $0 < \delta' < \delta$ such that*

$$\bigl|g(x,y) - g(x,x_0)\bigr| \leq C|x-y|^{-n}\bigl(|y-x_0|m(x,\mu)\bigr)^{\delta'}.$$

Proof (i). In fact, we can deduce (4.2) from Lemma 4.1. Precisely,

Case 1: $\sqrt{t} \leq 1/m(y,\mu)$. Because $y \in B$, then $|y-x_0| < r < 1/m(x_0,\mu)$, $m(y,\mu) \sim m(x_0,\mu)$. By (i) of Lemma 4.1, we can get

$$\bigl|g(x,y)\bigr| \leq Cm(x_0,\mu)^{\delta}\int_0^\infty t^{-n/2+\delta/2-1}e^{-|x-y|^2/ct}\,dt \leq Cm(x_0,\mu)^{\delta}|x-y|^{\delta-n}.$$

Case 2: $\sqrt{t} > 1/m(y,\mu)$. For this case, $\sqrt{t}m(y,\mu) > 1$. Using Lemma 4.1 again, we can deduce that

$$\left|g(x,y)\right| \le C\int_0^\infty t^{-n/2}e^{-|x-y|^2/ct}\left(\sqrt{t}m(y,\mu)\right)^\delta \frac{dt}{t} \le Cm(x_0,\mu)^\delta|x-y|^{\delta-n}.$$

(ii). It follows from (4.1) that

$$\begin{aligned}\left|g(x,y)-g(x,x_0)\right| &= \left|\int_0^\infty t^{-i\gamma}q_t(x,y)\,dt/t - \int_0^\infty t^{-i\gamma}q_t(x,x_0)\frac{dt}{t}\right| \\ &\le \int_0^\infty t^{-i\gamma}\left|q_t(x,y)-q_t(x,x_0)\right|\frac{dt}{t}.\end{aligned}$$

By (ii) of Lemma 4.1 and a direct computation, we get

$$\begin{aligned}\left|g(x,y)-g(x,x_0)\right| &\le \left|\int_0^\infty t^{-i\gamma}\left[|y-x_0|m(x,\mu)\right]^{\delta'}t^{-n/2}e^{-c|x-y|^2/t}\frac{dt}{t}\right| \\ &\le C\left[|y-x_0|m(x,\mu)\right]^{\delta'}\int_0^\infty t^{-n/2}e^{-c|x-y|^2/t}\frac{dt}{t} \\ &\le C|x-y|^{-n}\left[|y-x_0|m(x,\mu)\right]^{\delta'}.\end{aligned}$$

This completes the proof of Lemma 4.2. □

We recall that an operator T taking $C^\infty(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is called a Calderón–Zygmund operator if

(a) T extends to a bounded operator on $L^2(\mathbb{R}^n,dx)$;

(b) there exists a kernel K such that, for every $f\in L^1_c(\mathbb{R}^n,dx)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy \quad \text{a.e. on } \{\operatorname{supp} f\}^c;$$

(c) the kernel K satisfies

$$\begin{cases} |K(x,y)| \le c/|x-y|^n; \\ |K(x+h,y)-K(x,y)| \le c|h|^\delta/|x-y|^{n+\delta}; \\ |K(x,y+h)-K(x,y)| \le c|h|^\delta/|x-y|^{n+\delta}. \end{cases} \tag{4.3}$$

In [12], Shen proved the following result.

Theorem 4.3 *Let $\mathcal{L}=-\Delta+\mu$ be a generalized Schrödinger operator, where $\mu\neq 0$ is a nonnegative Radon measure on $\mathbb{R}^n$ satisfying (1.2) & (1.3) for some $\delta>0$. Then, for $\gamma\in\mathbb{R}^n$, $\mathcal{L}^{i\gamma}$ is a Calderón–Zygmund operator.*

Now we prove the $H^1_{\mathcal{L}}$-boundedness of $\mathcal{L}^{i\gamma}$.

Theorem 4.4 *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n\ge 3$. Suppose that μ satisfies conditions (1.2) & (1.3) for some $\delta>0$. Then, for $\gamma\in\mathbb{R}^n$, $\mathcal{L}^{i\gamma}$ is bounded on $H^1_{\mathcal{L}}(\mathbb{R}^n)$.*

Proof We only need to prove that, for any $(1,\infty)$ atom a, $\mathcal{L}^{i\gamma}(a)$ is a $(1,q,\varepsilon)$-molecular and $\|\mathcal{L}^{i\gamma}(a)\|_{H^1_{\mathcal{L}}} \le C$. Let a be a $(1,\infty)$ atom supported on $B(x_0,r)$. Then $\|a\|_\infty \le 1/|B(x_0,r)|$. If $r < m(x_0,\mu)^{-1}$, $\int a(x)\,dx = 0$. Set $B^\sharp = B(x_0, 2/m(x_0,\mu))$ and $B^* = B(x_0,2r)$. We divide the proof into three parts.

Part I: $\mathcal{L}^{i\gamma}a \in L^q(\mathbb{R}^n)\&|x|^{nb}L^{i\gamma}(a) \in L^q(\mathbb{R}^n)$.

$$\begin{aligned}\big\||x|^{nb}\mathcal{L}^{i\gamma}a\big\|_q &\le \big\|\chi_{B^*}|\cdot|^{nb}\mathcal{L}^{i\gamma}a\big\|_q + \big\|\chi_{(B^*)^c}|\cdot|^{nb}\mathcal{L}^{i\gamma}a\big\|_q \\ &= \left(\int_{B^*}|x|^{qnb}\big|\mathcal{L}^{i\gamma}(a)\big|^q\,dx\right)^{1/q} + \left(\int_{(B^*)^c}|x|^{qnb}\big|\mathcal{L}^{i\gamma}(a)\big|^q\,dx\right)^{1/q} \\ &\le \big(r+|x_0|\big)^{nb}\big\|\mathcal{L}^{i\gamma}a\big\|_q + \left[\int_{(B^*)^c}|x|^{qnb}\left|\int_{\mathbb{R}^n}K_\gamma^{\mathcal{L}}(x,y)a(y)\,dy\right|^q dx\right]^{1/q}.\end{aligned}$$

By the L^q-boundedness of $\mathcal{L}^{i\gamma}$ and Minkowski's inequality, $\||x|^{nb}\mathcal{L}^{i\gamma}a\|_q \le S_1 + S_2$, where

$$\begin{cases} S_1 := (r+|x_0|)^{nb}\|a\|_q; \\ S_2 := \int_{B(x_0,r)}|a(y)|[\int_{(B^*)^c}|x|^{qnb}|K_\gamma^{\mathcal{L}}(x,y)|^q\,dx]^{1/q}. \end{cases}$$

Because $\|a\|_q \le |B(x_0,r)|^{1/q-1}$, then

$$S_1 = \big(r+|x_0|\big)^{nb}\|a\|_q \le \big(r+|x_0|\big)^{nb}r^{n(1/q-1)}.$$

For the term S_2, recall that

$$K_\gamma^{\mathcal{L}}(x,y) = \int_0^\infty t^{-i\gamma}K_t^{\mathcal{L}}(x,y)\frac{dt}{t}.$$

Proposition 2.7 implies that

$$\begin{aligned}\big|K_\gamma^{\mathcal{L}}(x,y)\big| &\le C_N t^{-n/2}\frac{e^{-c|x-y|^2/t}}{\{1+|x-y|[m(x,\mu)+m(y,\mu)]\}^N}\frac{dt}{t} \\ &\le \frac{C_N}{\{1+|x-y|[m(x,\mu)+m(y,\mu)]\}^N}\frac{1}{|x-y|^n},\end{aligned}$$

which gives

$$S_2 \le C\int_B\big|a(y)\big|\left[\int_{(B^*)^c}|x|^{qnb}\frac{1}{\{1+|x-y|[m(x,\mu)+m(y,\mu)]\}^{qN}}\frac{dx}{|x-y|^{qn}}\right]^{1/q}dy.$$

For $x \in B$ and $y \in (B^*)^c$, we can see that $|x-y| \ge |x-x_0|/2$. Notice that for $y \in B(x_0,r)$, $|x_0-y| < r$ and

$$m(y,\mu)^N \ge \left[\frac{cm(x_0,\mu)}{\{1+|x_0-y|m(x_0,\mu)\}^{k_0/(k_0+1)}}\right]^N.$$

Then, via a direct computation, we have

$$S_2 \le C\int_B\big|a(y)\big|\frac{1}{m(y,\mu)^N}\left\{\int_{(B^*)^c}\frac{|x|^{qnb}\,dx}{|x-x_0|^{(n+N)q}}\right\}^{1/q}dy$$

$$\begin{aligned}
&\le C\int_B \big|a(y)\big|\frac{1}{m(y,\mu)^N}\left\{\left[\int_{(B^*)^c}\frac{1}{|x-x_0|^{(n+N)q-qnb}}\,dx\right]^{1/q}\right.\\
&\quad\left.+\left[\int_{(B^*)^c}\frac{|x_0|^{qnb}}{|x-x_0|^{(n+N)q}}\,dx\right]^{1/q}\right\}dy\\
&\le C\int_B\frac{|a(y)|}{m(y,\mu)^N}\left\{r^{n\varepsilon-N}+|x_0|^{nb}r^{-n-N+n/q}\right\}dy\\
&\le C\frac{\{1+rm(x_0,\mu)\}^{k_0N/(k_0+1)}}{m(x_0,\mu)^N}\left\{r^{n\varepsilon-N}+|x_0|^{nb}r^{-n-N+n/q}\right\}\left(\int_B\big|a(y)\big|\,dy\right)<\infty.
\end{aligned}$$

Part II: $N_{\mathcal{L}}(\mathcal{L}^{i\gamma}a)=\|\mathcal{L}^{i\gamma}a\|_q^{\varepsilon/b}\||\cdot-x_0|^{nb}\mathcal{L}^{i\gamma}a\|_q^{1-\varepsilon/b}\le C$.

Case I: $r\ge\rho(x_0)$. Because $x\in(B^*)^c$ and $y\in B$,

$$\big\|\mathcal{L}^{i\gamma}a\big\|_q\le\left(\int_{B^*}\big|\mathcal{L}^{i\gamma}a(y)\big|^q\,dy\right)^{1/q}+\left(\int_{(B^*)^c}\big|\mathcal{L}^{i\gamma}a(y)\big|^q\,dy\right)^{1/q}.$$

Because $\mathcal{L}^{i\gamma}$ is bounded on $L^q(\mathbb{R}^n)$, $q>1$, then

$$\left(\int_{B^*}\big|\mathcal{L}^{i\gamma}a(y)\big|^q\,dy\right)^{1/q}\le\big\|\mathcal{L}^{i\gamma}a\big\|_q\le C\|a\|_q\le|B|^{1/q-1}.$$

For $y\in B$ and $x\in(B^*)^c$, $|x-y|\ge|x-x_0|/2$. By Theorem 4.3, we can get

$$\begin{aligned}
\left(\int_{(B^*)^c}\big|\mathcal{L}^{i\gamma}a(x)\big|^q\,dx\right)^{1/q}&\le\int_B\big|a(y)\big|\left(\int_{(B^*)^c}\big|K_\gamma^{\mathcal{L}}(x,y)\big|^q\,dx\right)^{1/q}dy\\
&\le\int_B\big|a(y)\big|\left(\int_{(B^*)^c}|x-y|^{-nq}\,dx\right)^{1/q}dy\\
&\le\int_B\big|a(y)\big|r^{n/q-n}\,dy\le Cr^{n/q-n}.
\end{aligned}$$

Because $q>1\&r\ge\rho(x_0)$, the above estimates indicate that

$$\big\|\mathcal{L}^{i\gamma}a(x)\big\|^{1/\{n/q-n\}}\ge\rho(x_0)^{(n/q-n)/\{n/q-n\}}=\rho(x_0),$$

which means that such $\mathcal{L}^{i\gamma}a$ need not satisfy the canceling condition.

On the other hand, we write $\||\cdot-x_0|^{nb}\mathcal{L}^{i\gamma}a\|_q\le I_1+I_2$, where

$$\begin{cases}I_1:=\|\chi_{B^*}|\cdot-x_0|^{nb}\mathcal{L}^{i\gamma}a\|_q;\\ I_2:=\|\chi_{(B^*)^c}|\cdot-x_0|^{nb}\mathcal{L}^{i\gamma}a\|_q.\end{cases}$$

For I_1, by the L^q-boundedness of $\mathcal{L}^{i\gamma}$ and the fact that $\varepsilon-b=1/q-1$, we have

$$I_1\le C|B|^b\big\|\mathcal{L}^{i\gamma}a\big\|_q\le C|B|^\varepsilon.$$

For I_2, because

$$\big|K_\gamma^{\mathcal{L}}(x,y)\big|\le\frac{C_N}{\{1+|x-y|[m(x,\mu)+m(y,\mu)]\}^N}\frac{1}{|x-y|^n},$$

we can use Lemma 2.1 and the fact that $r \geq 1/m(x_0,\mu)$ to obtain

$$\begin{aligned}
I_2 &\leq C\int_B \frac{|a(y)|}{m(y,\mu)^N}\left(\int_{(B^*)^c} \frac{|x-x_0|^{nbq}}{|x-x_0|^{(n+N)q}}\,dx\right)^{1/q} dy \\
&\leq C\int_B \frac{|a(y)|}{m(y,\mu)^N} r^{nb+n/q-n-N}\,dy \\
&\leq C\int_B \bigl|a(y)\bigr|\left\{\frac{1}{m(x_0,\mu)}+|y-x_0|\right\}^N r^{nb+n/q-n-N}\,dy \\
&\leq C\int_B \bigl|a(y)\bigr| r^N r^{nb+n/q-n-N}\,dy \\
&\leq C|B|^{\varepsilon}.
\end{aligned}$$

The estimates for I_1 and I_2 imply that

$$N_{\mathcal{L}}\bigl(\mathcal{L}^{i\gamma}a\bigr) = \bigl\|\mathcal{L}^{i\gamma}a\bigr\|_q^{\varepsilon/b}\bigl\||\cdot-x_0|\mathcal{L}^{i\gamma}a\bigr\|_q^{1-\varepsilon/b} \leq C.$$

Case 2: $r < 1/m(x_0,\mu)$. For this case, the atom a has the canceling property. There exists a positive integer m such that $2^{-m-1}/m(x_0,\mu) \leq r < 2^{-m}/m(x_0,\mu)$. Let $B^\sharp = B(x_0, 2/m(x_0,\mu))$ and $B^* = B(x_0,2r)$. We write

$$\mathcal{L}^{i\gamma}a = \bigl(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\bigr)a + (-\Delta)^{i\gamma}a.$$

We will prove that $(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a$ and $(-\Delta)^{i\gamma}a$ are both moleculars. For $r < 1/m(x_0,\mu)$, any $(1,q)$-atom is a classical atom. By Alverez–Milman [1], $(-\Delta)^{i\gamma}a$ is a $(1,q,\varepsilon)$-molecular. Hence, $(-\Delta)^{i\gamma}a$ is a molecular of $H^1_{\mathcal{L}}(\mathbb{R}^n)$. We write $\|(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a\|_q \leq I_1+I_2+I_3$, where

$$\begin{cases}
I_1 := \|(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a\chi_{B^*}\|_q; \\
I_2 := \|(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a\chi_{B^\sharp\setminus B^*}\|_q; \\
I_3 := \|(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a\chi_{(B^\sharp)^c}\|_q.
\end{cases}$$

We first estimate the term I_1. Because $\delta \in (0,n)$, then $n/q - n + \delta > 0$. Estimate (4.2) implies that

$$\begin{aligned}
I_1 &\leq C\int_B \left(\int_{B^*}\bigl|g(x,y)\bigr|^q\,dx\right)^{1/q}\bigl|a(y)\bigr|\,dy \\
&\leq C\int_B \bigl|a(y)\bigr| m(x_0,\mu)^{\delta}\left(\int_{B^*}|x-y|^{-q(n-\delta)}\,dx\right)^{1/q} dy \\
&\leq C\int_B \bigl|a(y)\bigr| m(x_0,\mu)^{\delta} r^{(n-qn+q\delta)/q}\,dy \\
&\leq Cm(x_0,\mu)^{\delta}\bigl[2^m m(x_0,\mu)\bigr]^{-n/q+n-\delta} \\
&\leq Cm(x_0,\mu)^{n-n/q}.
\end{aligned}$$

Now we deal with I_3. If $x \in (B^\sharp)^c$ and $y \in B$, then $|y-x| \sim |x-x_0|$. By the canceling property of a, we have

$$\begin{aligned}\left|\left[\mathcal{L}^{i\gamma}-(-\Delta)^{i\gamma}\right](a)(x)\right| &\le \left|\int_B \left[K_\gamma^{\mathcal{L}}(x,y)-K_\gamma(x,y)\right]a(y)\,dy\right| \\ &\le \left|\int_B \left[K_\gamma^{\mathcal{L}}(x,x_0)-K_\gamma^{\mathcal{L}}(x,y)\right]a(y)\,dy\right| \\ &\quad + \left|\int_B \left[K_\gamma(x,y)-K_\gamma(x,x_0)\right]a(y)\,dy\right| \\ &\le \int_B \left|a(y)\right|\left[\frac{|y-x_0|^\delta}{|x-x_0|^{n+\delta}}+\frac{|y-x_0|}{|x-x_0|^{n+1}}\right]dy \\ &\le C\left(\frac{r^\delta}{|x-x_0|^{n+\delta}}+\frac{r}{|x-x_0|^{n+1}}\right).\end{aligned}$$

Then, since $rm(x_0,\mu) < 1$, we obtain that

$$\begin{aligned}I_3 &\le C\left[\int_{(B^\sharp)^c}\left(\frac{r^\delta}{|x-x_0|^{n+\delta}}+\frac{r}{|x-x_0|^{n+1}}\right)^q dx\right]^{1/q} \\ &\le C\left\{\left[\int_{(B^\sharp)^c}\frac{r^{q\delta}}{|x-x_0|^{q(n+\delta)}}\,dx\right]^{1/q}+\left[\int_{(B^\sharp)^c}\frac{r^q}{|x-x_0|^{q(n+1)}}\,dx\right]^{1/q}\right\} \\ &\le Cr^\delta m(x_0,\mu)^{\delta+n-n/q}+rm(x_0,\mu)^{\delta+1-n/q} \\ &\le Cm(x_0,\mu)^{n-n/q}.\end{aligned}$$

At last, we estimate I_2. For this case, $x \in B^\sharp \backslash B^*$, then $2r < |x-x_0| < 2/m(x_0,\mu)$ and $|x-y| \sim |x-x_0|$. Applying (4.2) and the canceling property of a again, we get

$$\begin{aligned}\left|\left[\mathcal{L}^{i\gamma}-(-\Delta)^{i\gamma}\right]a(x)\right| &\le C\int_B \left[|y-x_0|m(x_0,\mu)\right]^{\delta'}|x-x_0|^{-n}\left|a(y)\right|dy \\ &\le C\left[m(x_0,\mu)\right]^{\delta'}r^{\delta'}|x-x_0|^{-n}\int_B\left|a(y)\right|dy \\ &\le C\left[m(x_0,\mu)\right]^{\delta'}r^{\delta'}|x-x_0|^{-n} \\ &\le C2^{-m\delta'}|x-x_0|^{-n},\end{aligned}$$

which implies that

$$\begin{aligned}I_2 &\le C\left[\int_{B^\sharp\backslash B^*}\left|\left[\mathcal{L}^{i\gamma}-(-\Delta)^{i\gamma}\right]a(x)\right|^q dx\right]^{1/q} \\ &\le C\left(\int_{B^\sharp\backslash B^*}2^{-m\delta' q}|x-x_0|^{-nq}\,dx\right)^{1/q} \\ &\le Cm(x_0,\mu)^{n-n/q},\end{aligned}$$

where in the last inequality we have used the fact that $q < n/(n-\delta')$.

Finally, it follows from the estimates for I_i, $i=1,2,3$, that

$$\left\|\left[\mathcal{L}^{i\gamma}-(-\Delta)^{i\gamma}\right]a\right\|_q^{1/(n/q-n)} \ge m(x_0,\mu)^{-1}.$$

On the other hand, the L^q-boundedness of $\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}$ gives

$$\big\|\big[\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\big]a\big\|_q \leq \|a\|_q = \left(\int_B |a(y)|^q\,dy\right)^{1/q} \leq r^{n/q-n}.$$

This means that for this case, $(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a$ need not satisfy the canceling condition.

Part III: There exists a constant C such that, for any $(1,\infty)$*-atom, uniformly,*

$$N_{\mathcal{L}}\big(\big[\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\big]a\big) \leq C.$$

We write $b = 1 - 1/q + \varepsilon$, then $\varepsilon - b = 1/q - 1$. We have proved that

$$\big\|\big(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\big)a\big\|_q \leq C\rho(x_0)^{n/q-n} \leq \rho(x_0)^{n(\varepsilon-b)}.$$

Now we split: $\||\cdot - x_0|^{nb}(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a\|_q \leq I_1 + I_2$, where

$$\begin{cases} I_1 := \||\cdot - x_0|^{nb}(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a\|_{L^q(B^\sharp)}; \\ I_2 := \||\cdot - x_0|^{nb}(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma})a\|_{L^q((B^\sharp)^c)}. \end{cases}$$

For I_1, because $B^\sharp = (x_0, 2\rho(x_0))$,

$$I_1 \leq C\rho(x_0)^{nb}\left[\int_{B^\sharp}\big|\big(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\big)a(x)\big|^q\right]^{1/q} \leq C\rho(x_0)^{nb}\rho(x_0)^{n(\varepsilon-b)} \leq C\rho(x_0)^{n\varepsilon}.$$

For I_2, we further split I_2 into $I_{2,1} + I_{2,2}$, where

$$\begin{cases} I_{2,1} := \||\cdot - x_0|^{nb}\mathcal{L}^{i\gamma}a\|_{L^q((B^\sharp)^c)}; \\ I_{2,2} := \||\cdot - x_0|^{nb}(-\Delta)^{i\gamma}a\|_{L^q((B^\sharp)^c)}. \end{cases}$$

Notice that $\varepsilon < \delta/n$ and $nb - (n+\delta) + n/q < 0$. By Theorem 4.3, we have

$$\begin{aligned} I_{2,1} &\leq C\int_B |a(y)|\left[\int_{(B^\sharp)^c}|x - x_0|^{qnb}|\int_B |y - x_0|^{q\delta}|x - x_0|^{-q(n+\delta)}\,dx\right]^{1/q} dy \\ &\leq C\int_B |a(y)||y - x_0|^{\delta}\left[\int_{(B^\sharp)^c}|x - x_0|^{qnb-q(n+\delta)}\,dx\right]^{1/q} dy \\ &\leq C\int_B |a(y)|r^{\delta}\rho(x_0)^{nb-(n+\delta)+n/q}\,dy \\ &\leq C\rho(x_0)^{n\varepsilon}. \end{aligned}$$

For $I_{2,2}$, similarly, we have

$$\begin{aligned} I_{2,2} &\leq C\int_B |a(y)||x - x_0|\left(\int_{|x-x_0|\geq 2\rho(x_0)}|x - x_0|^{q(nb-n-1)}|x - x_0|^{n-1}\,d|x - x_0|\right)^{1/q} dy \\ &\leq C\int_B |a(y)|r\rho(x_0)^{nb-(n+1)+n/q}\,dy \\ &\leq C\rho(x_0)^{n\varepsilon}, \end{aligned}$$

where we have used the fact that $0 < \varepsilon < \min\{\delta/n, 1/n\}$. Finally, we get

$$\big\| |\cdot - x_0|^{nb}\big(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\big)a\big\|_q \le \rho(x_0)^{n\varepsilon},$$

and, hence,

$$\begin{aligned}&\big\|\big(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\big)a\big\|_q^{\varepsilon/b}\big\| |\cdot - x_0|^{nb}\big(\mathcal{L}^{i\gamma} - (-\Delta)^{i\gamma}\big)a\big\|_q^{(1-\varepsilon/b)}\\ &\quad \le C\rho(x_0)^{n(\varepsilon-b)\varepsilon/b}\rho(x_0)^{n\varepsilon(1-\varepsilon/b)} \le C.\end{aligned}$$

Finally, we have proved that, for any $(1,\infty)$-atom, $\mathcal{L}^{i\gamma}a$ is a $(1,q,\varepsilon)$-molecular or the linear combination of finite $(1,q,\varepsilon)$-moleculars. □

4.2 The $H^1_{\mathcal{L}}$-boundedness of Riesz transforms $R_{\mathcal{L}}$

In this section, we prove that Riesz transforms $R_{\mathcal{L}}$ are bounded on $H^1_{\mathcal{L}}(\mathbb{R}^n)$. The Riesz transforms associated with $\mathcal{L}$ are defined as

$$R_{\mathcal{L}} := \nabla(-\Delta + \mu)^{-1/2},$$

where $(-\Delta + \mu)^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(-\Delta + \mu + \lambda)^{-1}\,d\lambda$. Shen proved the following estimate of $R_{\mathcal{L}}$. Assume that μ satisfies (1.2) & (1.3) for some $\delta > 1$. Then $\nabla(-\Delta+\mu)^{-1/2}$ is a Calderón–Zygmund operator. Precisely,

$$\nabla(-\Delta + \mu)^{-1/2}f(x) = \int_{\mathbb{R}^n} R_{\mathcal{L}}(x,y)f(y)\,dy,$$

where

$$R_{\mathcal{L}}(x,y) := \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\nabla_x \Gamma_{\mu+\lambda}(x,y)\,d\lambda.$$

In [12], Shen proved the following results, see [12, (7.20), (7.26), (7.29)], respectively.

Lemma 4.5 *The kernel $R_{\mathcal{L}}(\cdot,\cdot)$ satisfies the following estimates*:

$$\begin{cases}(1) & |R_{\mathcal{L}}(x,y)| \le Ce^{-cd(x,y,\mu)}|x-y|^{-n};\\ (2) & |R_{\mathcal{L}}(x+h,y) - R_{\mathcal{L}}(x,y)| \le C(|h|/|x-y|)^{\delta-1}|x-y|^{-n};\\ (3) & |R_{\mathcal{L}}(x,y+h) - R_{\mathcal{L}}(x,y)| \le C(|h|/|x-y|)^{\delta_1}|x-y|^{-n}, \quad \delta_1 \in (0,1).\end{cases}$$

Theorem 4.6 *Let $\mathcal{L} = -\Delta + \mu$ be a generalized Schrödinger operator, where $\mu \ne 0$ is a nonnegative Radon measure on $\mathbb{R}^n$ satisfying (1.2) & (1.3) for some $\delta > 0$. The Riesz transform $R_{\mathcal{L}}$ is bounded on $H^1_{\mathcal{L}}(\mathbb{R}^n)$.*

Proof Similar to Theorem 4.4, the proof of this theorem is divided into three parts.

Part I: $\||\cdot|^{nb}R_{\mathcal{L}}a\|_q < \infty$, uniformly. For any atom a and $B^* = B(x_0, 2r)$, we write $\||\cdot|^{nb}R_{\mathcal{L}}a\|_q \le I_1 + I_2$, where $I_1 := \||\cdot|^{nb}R_{\mathcal{L}}a\|_{L^q(B^*)}$ and $I_2 := \||\cdot|^{nb}R_{\mathcal{L}}a\|_{L^q((B^*)^c)}$.

By the L^q-boundedness of $R_{\mathcal{L}}$, we have

$$I_1 \le r^{nb}\bigg(\int_{B^*}\big|R_{\mathcal{L}}a(x)\big|^q\,dx\bigg)^{1/q} \le r^{nb}\|a\|_q \le r^{nb}\big|B(x_0,r)\big|^{1/q-1} \le r^{n\varepsilon}.$$

By Lemma 4.5, for any positive $N > 0$,

$$\left|R_{\mathcal{L}}(x,y)\right| \le C|x-y|^{-n-N}\left[m(y,\mu)\right]^{-N}.$$

On the other hand, for $y \in B$ and $x \in (B^*)^c$, $|x-y| \ge |x-x_0|/2$. We can obtain that

$$\begin{aligned}
I_2 &\le C\int_B \left|a(y)\right| \left(\int_{(B^*)^c} |x|^{qnb}\left|R_{\mathcal{L}}(x,y)\right|^q\,dx\right)^{1/q} dy \\
&\le C\int_B \left|a(y)\right| \frac{1}{m(y,\mu)^N}\left(\int_{(B^*)^c} |x|^{qnb}\frac{1}{|x-y|^{(n+N)q}}\,dx\right)^{1/q} dy \\
&\le C\left\{\int_B \left|a(y)\right| \frac{|x_0|^{nb}}{m(y,\mu)^N}\left(\int_{(B^*)^c} \frac{dx}{|x-x_0|^{(n+N)q}}\right)^{1/q} dy\right. \\
&\quad \left. + \int_B \left|a(y)\right| \frac{1}{m(y,\mu)^N}\left(\int_{(B^*)^c} \frac{|x-x_0|^{qnb}}{|x-x_0|^{(n+N)q}}\,dx\right)^{1/q} dy\right\} \\
&\le C\int_B \frac{|a(y)|}{m(y,\mu)^N}\left\{r^{n\epsilon-N} + |x_0|^{-n-N+n/q}\right\} dy.
\end{aligned}$$

Because $y \in B(x_0,r)$,

$$m(y,\mu) \ge \frac{Cm(x_0,\mu)}{\{1+|x-x_0|m(x_0,\mu)\}^{k_0/(k_0+1)}},$$

which implies that $\||\cdot|^{nb}a\|_{L^q((B^*)^c)} < \infty$.

Part II: $N_{\mathcal{L}}(R_{\mathcal{L}}(a)) \le C$. We divide the proof into two cases.

Case1: $r \ge 1/m(x_0,\mu)$. By the boundedness of the Riesz transform $R_{\mathcal{L}}$, we have

$$\begin{aligned}
\|R_{\mathcal{L}}a\|_q &\le C\left\{\|R_{\mathcal{L}}a\|_q + \|\chi_{(B^*)^c}R_{\mathcal{L}}a\|_q\right\} \\
&\le C\left\{\|a\|_q + \|\chi_{(B^*)^c}R_{\mathcal{L}}a\|_q\right\} \\
&\le C\left\{\left|B(x_0,r)\right|^{1/q-1} + \int_B \left|a(y)\right|\left(\int_{(B^\sharp)^c}\left|R_{\mathcal{L}}(x,y)\right|^q\,dx\right)^{1/q} dy\right\}.
\end{aligned}$$

By Lemma 4.5, we can get

$$\begin{aligned}
\int_B \left|a(y)\right|\left(\int_{(B^\sharp)^c}\left|R_{\mathcal{L}}(x,y)\right|^q\,dx\right)^{1/q} dy &\le C\int_B \left|a(y)\right|\left(\int_{2r}^\infty s^{n-qn-1}\,ds\right)^{1/q} dy \\
&\le C\int_B \left|a(y)\right|\,dy \times r^{n/q-n} \\
&\le C\rho(x_0)^{n/q-n},
\end{aligned}$$

which means that $\|R_{\mathcal{L}}a\|_q^{1/(n/q-n)} \ge \rho(x_0)$, i.e., $R_{\mathcal{L}}a$ does not need the canceling condition for this case. Now we split $\||\cdot-x_0|^{nb}R_{\mathcal{L}}a\|_q \le I_1 + I_2$, where

$$\begin{cases}
I_1 := (\int_{B^*} |x-x_0|^{qnb}|R_{\mathcal{L}}a(x)|^q\,dx)^{1/q}; \\
I_2 := (\int_{(B^*)^c} |x-x_0|^{qnb}|R_{\mathcal{L}}a(x)|^q\,dx)^{1/q}.
\end{cases}$$

It is easy to see that

$$I_1 \le r^{nb}\left(\int_{B^*}\left|R_{\mathcal{L}}a(x)\right|^q dx\right)^{1/q} \le r^{nb}\|R_{\mathcal{L}}a\|_q \le r^{nb}\|a\|_q \le r^{n\varepsilon}.$$

For I_2, by Minkowski's inequality,

$$\begin{aligned}
I_2 &\le C\int_B\left|a(y)\right|m(y,\mu)^{-N}\left(\int_{B^*}|x-x_0|^{qnb-q(n+N)}\,dx\right)^{1/q}dy\\
&\le C\int_B\left|a(y)\right|m(y,\mu)^{-N}r^{nb+n/q-n-N}\,dy\\
&\le C\int_B\left|a(y)\right|\left\{m(x_0,\mu)^{-1}+|y-x_0|\right\}^N r^{nb+n/q-n-N}\,dy\\
&\le C\int_B\left|a(y)\right|r^N r^{nb+n/q-n-N}\,dy\\
&\le C|B|^{(\varepsilon-b)\varepsilon/b}|B|^{\varepsilon(1-\varepsilon/b)},
\end{aligned}$$

which gives $N_{\mathcal{L}}(R_{\mathcal{L}}a)=\|R_{\mathcal{L}}a\|_q^{\varepsilon/b}\||\cdot-x_0|^{nb}R_{\mathcal{L}}a\|_q^{1-\varepsilon/b}\le C$.

Case 2: $r\le\rho(x_0)$. Let $B^\sharp=B(x_0,2\rho(x_0))$ and $B^*=B(x_0,2r)$. So $R_{\mathcal{L}}a=R_0a+(R_{\mathcal{L}}-R_0)a$, where $R_0:=\nabla(-\Delta)^{-1/2}$. For any a with the canceling condition, R_0a is a molecular. We only need to deal with $(R_{\mathcal{L}}-R_0)a$. Split $\|(R_{\mathcal{L}}-R_0)a\|_q\le I_1+I_2+I_3$, where

$$\begin{cases}
I_1:=(\int_{B^*}|(R_{\mathcal{L}}-R_0)a(x)|^q\,dx)^{1/q};\\
I_2:=(\int_{B^\sharp\setminus B^*}|(R_{\mathcal{L}}-R_0)a(x)|^q\,dx)^{1/q};\\
I_3:=(\int_{(B^\sharp)^c}|(R_{\mathcal{L}}-R_0)a(x)|^q\,dx)^{1/q}.
\end{cases}$$

We first estimate I_3. For $x\in(B^\sharp)^c$ and $y\in B$, $|x-y|\sim|x-x_0|$. Denote by $R_0(\cdot,\cdot)$ the kernel of $\nabla(-\Delta)^{-1/2}$. We can get

$$\begin{aligned}
&\left|(R_{\mathcal{L}}-R_0)(a)(x)\right|\\
&\quad\le C\left\{\int_B\left|R_{\mathcal{L}}(x,y)-R_{\mathcal{L}}(x,x_0)\right|\left|a(y)\right|dy+\int_B\left|R_0(x,y)-R_0(x,x_0)\right|\left|a(y)\right|dy\right\}\\
&\quad\le C\|a\|_\infty\left(\int_B\frac{|y-x_0|^\delta}{|x-x_0|^{n+\delta}}\,dy+\int_B\frac{|y-x_0|}{|x-x_0|^{n+1}}\,dy\right)\\
&\quad\le C\|a\|_\infty|B|\left(\frac{r^\delta}{|x-x_0|^{n+\delta}}+\frac{r}{|x-x_0|^{n+1}}\right).
\end{aligned}$$

It follows from the above estimate that

$$\begin{aligned}
I_3 &\le C\left\{\left(\int_{(B^\sharp)^c}\frac{r^{q\delta}}{|x-x_0|^{q(n+\delta)}}\,dx\right)^{1/q}+\left(\int_{(B^\sharp)^c}\frac{r^q}{|x-x_0|^{q(n+1)}}\,dx\right)^{1/q}\right\}\\
&\le C\left\{r^\delta m(x_0,\mu)^{n+\delta-n/q}+r\left[m(x_0,\mu)\right]^{n+1-n/q}\right\}\\
&\le C\left[m(x_0,\mu)\right]^{n-n/q}.
\end{aligned}$$

For the estimates of $I_1\&I_2$, we need the following estimate:

$$\left|R_{\mathcal{L}}(y,x)-R_0(y,x)\right| \le C\left\{\frac{1}{r^{n-1}}\int_{B(y,r)}\frac{d\mu(z)}{|z-y|^{n-1}}+\frac{(rm(x,\mu))^{\delta}}{r^n}\right\}. \tag{4.4}$$

For $r=|x-y|/2$,

$$\left|R_{\mathcal{L}}(y,x)-R_0(y,x)\right| \le C\left\{\frac{1}{|x-y|^{n-1}}\int_{B(y,|x-y|/2)}\frac{d\mu(z)}{|z-y|^{n-1}}+\frac{(|x-y|m(x,\mu))^{\delta}}{|x-y|^n}\right\}.$$

We get $I_1 \le \int_B |a(y)|A_1(y)\,dy$, where

$$A_1 := \left\{\int_{B^*}\left|R_{\mathcal{L}}(x,y)-R_0(x,y)\right|^q dx\right\}^{1/q}.$$

Due to (4.4), we further obtain $A_1 \le U_1+U_2$, where

$$\begin{cases} U_1 := \{\int_{B^\sharp}|x-y|^{q(1-n)}(\int_{B(x,|x-y|/2)}|z-x|^{1-n}\,d\mu(z))^q\,dx\}^{1/q}; \\ U_2 := \{\int_{B^\sharp}(|x-y|m(y,\mu))^{q\delta}|x-y|^{-qn}\,dx\}^{1/q}. \end{cases}$$

For U_2, if $y\in B$, then $|y-x_0|<r<2\rho(x_0)$ and $m(y,\mu)\sim m(x_0,\mu)$. On the other hand, because $x\in B^\sharp$, then $|x-y|<3/m(x_0,\mu)$. We can get

$$U_2 \le Cm(x_0,\mu)^{\delta}\left(\int_{B^\sharp}|x-y|^{q\delta-qn}\,dx\right)^{1/q} \le Cm(x_0,\mu)^{n-n/q}.$$

Now we estimate the term U_1. Let $T_j=B(y,2^{j+2}/m(x_0,\mu))$. If $y\in B$ and $x\in B^\sharp$, by the triangle inequality, it is easy to see that $B^\sharp\subset B(y,4/m(x_0,\mu))$. Also, for $x\in T_{j+1}\setminus T_j$, $|x-y|\ge 2^{j+2}/m(x_0,\mu)$. On the other hand, $B(x,|x-y|/2)\subset B(y,3|x-y|/2)$. Then

$$\begin{aligned} U_1 &\le C\sum_{j=-\infty}^{0}\left(\int_{T_{j+1}\setminus T_j}\frac{1}{|x-y|^{q(n-1)}}\left(\int_{B(x,|x-y|/2)}\frac{d\mu(z)}{|z-x|^{n-1}}\right)^q dx\right)^{1/q} \\ &\le C\sum_{j=-\infty}^{0}\left[\frac{m(x_0,\mu)}{2^j}\right]^{n-1}\left(\int_{T_{j+1}\setminus T_j}\left(\int_{B(y,|x-y|/2)}\frac{d\mu(z)}{|z-x|^{n-1}}\right)^q dx\right)^{1/q} \\ &\le C\sum_{j=-\infty}^{0}\left[\frac{m(x_0,\mu)}{2^j}\right]^{n-1}\left(\int_{T_{j+1}\setminus T_j}\left(\int_{B(y,2^{j+2}/m(x_0,\mu))}\frac{d\mu(z)}{|z-x|^{n-1}}\right)^q dx\right)^{1/q} \\ &\le C\sum_{j=-\infty}^{0}\left[\frac{m(x_0,\mu)}{2^j}\right]^{n-1}\frac{\mu(3T_{j+1})}{2^{(j+2)(n-n/q-1)}}\left[m(x_0,\mu)\right]^{n-n/q-1}. \end{aligned}$$

Notice that

$$\mu(T_{j+1})=\mu\bigl(B\bigl(y,2^{j+2}/m(x_0,\mu)\bigr)\bigr)\le\left(2^{j+2}\right)^{n-2+\delta}m(x_0,\mu)^{2-n}.$$

A direct computation gives

$$U_1 \le C\sum_{j=-\infty}^{0}\left[m(x_0,\mu)\right]^{n-n/q}2^{j(n-2+\delta')}2^{-j(2n-1/q-2)}$$

$$
\begin{aligned}
&\le C\big[m(x_0,\mu)\big]^{n-n/q}\sum_{j=-\infty}^{0}2^{j(n-2+\delta'-2n+n/q+2)}\\
&\le C\big[m(x_0,\mu)\big]^{n-n/q},
\end{aligned}
$$

which implies that

$$
I_1 \le C\int_B \big|a(y)\big|A_1(y)\,dy \le C\|a\|_1\big[m(x_0,\mu)\big]^{n-n/q} \le C\big[m(x_0,\mu)\big]^{n-n/q}.
$$

The estimate for I_2 is similar. Then we obtain $\|(R_{\mathcal{L}}-R_0)a\|_q^{1/\{n/q-n\}} \ge C/m(x_0,\mu)$, which means $(R_{\mathcal{L}}-R_0)a$ does not need the canceling condition. What is left to prove is the norm $\||\cdot -x_0|^{nb}(R_{\mathcal{L}}-R_0)a\|_q$. We write $\||\cdot -x_0|^{nb}(R_{\mathcal{L}}-R_0)a\|_q \le E_1+E_2$, where

$$
\begin{cases}
E_1 := (\int_{B^\sharp}|x-x_0|^{qnb}|(R_{\mathcal{L}}-R_0)a(x)|^q\,dx)^{1/q};\\
E_2 := (\int_{(B^\sharp)^c}|x-x_0|^{qnb}|(R_{\mathcal{L}}-R_0)a(x)|^q\,dx)^{1/q}.
\end{cases}
$$

By the L^p-boundedness of $R_{\mathcal{L}}$ and R_0, we get

$$
E_1 \le Cm(x_0,\mu)^{-nb}\big\|(R_{\mathcal{L}}-R_0)a\big\|_q \le Cm(x_0,\mu)^{-nb}\|a\|_q \le Cm(x_0,\mu)^{-n\varepsilon}.
$$

For the term E_2, we have $E_2 \le E_{2,1}+E_{2,2}$, where

$$
\begin{cases}
E_{2,1} := (\int_{(B^\sharp)^c}|x-x_0|^{qnb}|R_{\mathcal{L}}a(x)|^q\,dx)^{1/q};\\
E_{2,2} := (\int_{(B^\sharp)^c}|x-x_0|^{qnb}|R_0a(x)|^q\,dx)^{1/q}.
\end{cases}
$$

A direct computation gives

$$
\begin{aligned}
E_{2,1} &\le C\int_B\big|a(y)\big|\,dy\bigg(\int_{(B^\sharp)^c}|x-x_0|^{qnb}\big|R_{\mathcal{L}}(x,y)-R_{\mathcal{L}}(x,x_0)\big|^q\,dx\bigg)^{1/q}\\
&\le C\int_B\big|a(y)\big||y-x_0|^{\delta}\bigg(\int_{(B^\sharp)^c}\frac{|x-x_0|^{qnb}}{|x-x_0|^{(n+\delta)q}}\,dx\bigg)^{1/q}dy \le Cm(x_0,\mu)^{-n\varepsilon}.
\end{aligned}
$$

For $E_{2,2}$, because R_0 is a Calderón–Zygmund operator, the kernel $K_0(\cdot,\cdot)$ satisfies

$$
\big|R_0(x,y)-R_0(x,x_0)\big| \le C|y-x_0||x-x_0|^{-n-1}.
$$

We can get

$$
\begin{aligned}
E_{2,2} &\le C\int_B\big|a(y)\big|\bigg(\int_{(B^\sharp)^c}|x-x_0|^{q(nb-n)}\,dx\bigg)^{1/q}dy\\
&\le Cr^{n+1-n/q}\bigg(\int_{2/m(x_0,\mu)}^{\infty}s^{-(q-1)n-q+nbq-1}\,ds\bigg)^{1/q} \le Cm(x_0,\mu)^{-n\varepsilon}.
\end{aligned}
$$

Finally, we obtain that

$$
N_L\big((R_{\mathcal{L}}-R_0)a\big) \le Cm(x_0,\mu)^{(n/q-n)(\varepsilon/b)}m(x_0,\mu)^{n\varepsilon(\varepsilon/b-1)} \le C. \qquad \square
$$

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Availability of data and materials
We declare that the materials described in the manuscript, including all relevant raw data, will be freely available to any scientist wishing to use them for non-commercial purposes, without breaching participant confidentiality.

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The authors declare that they have no competing interests.

Authors' contributions
Both authors conceived of the study, participated in its design and coordination, drafted the manuscript, and participated in the sequence alignment. All authors read and approved the final manuscript.

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References

1. Alvarez, J., Milman, M.: H^p continuty properties of Calderón–Zygmund operators. J. Math. Anal. Appl. **118**, 65–79 (1986)
2. Cao, J., Yang, Da.: Hardy spaces $H_L^p(\mathbb{R}^n)$ associated with operators satisfying k-Davies–Gaffney estimates. Sci. China Math. **55**, 1403–1440 (2012)
3. Dong, J., Huang, J., Liu, H.: Boundedness of singular integrals on Hardy type spaces associated with Schrödinger operators. J. Funct. Spaces **2015**, Article ID 409215 (2015)
4. Duong, X., Yan, L.: Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. J. Am. Math. Soc. **18**, 943–973 (2005)
5. Dziubański, J.: Note on H^1 spaces related to degenerate Schrödinger operators. Ill. J. Math. **49**, 1271–1297 (2005)
6. Dziubański, J., Zienkiewicz, J.: Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. Rev. Mat. Iberoam. **15**, 279–296 (1999)
7. Hofmann, S., Lu, G., Mitrea, D., Mitrea, M., Yan, L.: Hardy spaces associated to non-negative self-adjoint operators satisfying Davies–Gaffney estimates. Mem. Am. Math. Soc. **214** no. 1007 (2011)
8. Jiang, R., Yang, Da.: Orlicz–Hardy spaces associated with operators. Sci. China Ser. A **52**, 1042–1080 (2009)
9. Jiang, R., Yang, Da.: Orlicz-Hardy spaces associated with operators satisfying Davies–Gaffney estimates. Commun. Contemp. Math. **13**, 331–373 (2011)
10. Lin, C., Liu, H., Liu, Y.: Hardy spaces associated with Schrödinger operators on the Heisenberg group. arXiv:1106.4960 [math.AP]
11. Liu, Y., Qi, S.: Endpoint estimates of Riesz transforms associated with generalized Schrödinger operators. Can. Math. Bull. **61**, 787–801 (2018)
12. Shen, Z.: On fundamental solutions of generalized Schrödinger operators. J. Funct. Anal. **167**, 521–564 (1999)
13. Song, L., Yan, L.: Riesz transforms associated to Schrödinger operators on weighted Hardy spaces. J. Funct. Anal. **259**, 1466–1490 (2010)
14. Stein, E.M.: Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton Math. Serises, vol. 43. Princeton University Press, Princeton (1993)
15. Wu, L., Yan, L.: Heat kernel, upper bounds and Hardy spaces associated to the generalized Schrödinger operators. J. Funct. Anal. **270**, 3709–3749 (2016)