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Some inequality techniques in handling fixed point problems on unbounded sets via homotopy methods

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Abstract

As is well known, fixed point theorems and problems play important roles in differential equations, mathematical programming, control, and so on. In this paper, by providing some unboundedness conditions and by using some inequality techniques, we can give constructive proofs of the existence of fixed points on unbounded non-convex sets via homotopy methods.

MSC: 34B15; 35A01; 39B05; 65H10; 90C30

Keywords: Inequality techniques; Constructive proofs; Fixed point problems; Homotopy methods

1 Introduction

As is well known, fixed point theorems and problems play important roles in many areas and have attracted extensive attention of more and more researchers [1–11]. The homotopy method now has become an important tool in handling various fixed point theorems and problems, bilevel programming, and so on [12–19], because the general Brouwer fixed point theorem does not require the convexity of the subsets of R^n, which is a necessary condition for the constructive proofs of the Brouwer fixed point theorem given by the classical homotopy methods and is difficult to remove. Recently, to remove the convexity restriction, Yu et al. [20] introduced the normal cone condition and combined the interior point methods with the classical homotopy methods, therefore gave a constructive proof of the general Brouwer fixed point theorem on a class of non-convex sets.

In [21], we applied linear homotopy and Newton homotopy techniques and hence extended the results [20] from the cases only with inequality constraints to more general cases with equality constraints. In [22], we furthermore introduced C^2 mappings $\xi_i(x,y_i) \in R^n$, $i = 1,\dots,m$, $\eta_j(x,z_j) \in R^n$, $j = 1,\dots,m$. With these new mappings, we were able to handle the general Brouwer fixed point theorem in more general non-convex sets. Moreover, we applied the perturbation techniques to the equality and inequality constraints to expand the scope of the choice of initial points. This point improves the computational efficiency of the algorithm greatly. However, to our knowledge, the global convergence results in [21, 22] were obtained under the boundedness assumptions. In this paper, we give two sets of unbounded conditions, with which we extend the results in [21, 22] from the boundedness cases to the unboundedness ones, respectively. Our results may be helpful in dealing with important nonlinear problems because fixed point theorems are widely applied in differential equations, economics, and so on.

Throughout this paper, we need the following notations: $\Omega = \{x \in R^n : g_i(x) \leq 0, i = 1,\dots,m,\ h_j(x) = 0, j = 1,\dots,l\}$, $\Omega^0 = \{x \in R^n : g_i(x) < 0, i = 1,\dots,m, h_j(x) = 0, j = 1,\dots,l\}$ and $\partial\Omega = \Omega \backslash \Omega^0$. The index set is $B(x) = \{i \in \{1,\dots,m\} : g_i(x) = 0\}$.

2 Main results

In [21], we gave a constructive proof of the existence of fixed points on a class of non-convex set Ω. The assumptions are listed as follows:

(A_1) Ω^0 is nonempty and Ω is bounded;

(A_2) For any $x \in \Omega$, if

$$\sum_{i\in I(x,\mu)} y_i \nabla g_i(x) + \nabla h(x)z = 0, \quad y_i \geq 0,$$

then $y_i = 0$, $\forall i \in I(x)$, $z = 0$;

(A_3) For any $x \in \Omega$ and for all $(x,y,z) \in \Omega \times R^m_+ \times R^l$, we have

$$\left\{x + \sum_{i\in B(x)} \nabla g_i(x) y_i + \nabla h(x)z\right\} \cap \Omega = \{x\}.$$

Then the homotopy is constructed as follows:

$$H\bigl(w, w^{(0)}, \mu\bigr) = \begin{pmatrix} (1-\mu)(x - F(x) + \nabla g(x)y) + \nabla h(x)z + \mu(x - x^{(0)}) \\ h(x) \\ Yg(x) - \mu Y^{(0)} g(x^{(0)}) \end{pmatrix} = 0, \tag{1}$$

where $g(x) = (g_1(x),\dots,g_m(x))^T$, $h(x) = (h_1(x),\dots,h_m(x))^T$, $\nabla g(x) = (\nabla g_1(x),\dots,\nabla g_m(x)) \in R^{n\times m}$, $Y = \operatorname{diag}(y) \in R^{m\times m}$, $Y^{(0)} = \operatorname{diag}(y^{(0)}) \in R^{m\times m}$, and $\mu \in (0,1]$.

In this paper, to extend the results in [21] to the unbounded non-convex sets, we use the idea of infinite solutions in [23, 24] and replace the boundedness condition (A_1) with the following unboundedness one:

(A_1') Ω^0 is nonempty; for any given $\eta \in \Omega$ and for any sequences $\{x^{(k)}\} \subset \Omega$,

$$\bigl(x^{(k)} - \eta\bigr)^T \nabla h\bigl(x^{(k)}\bigr) \geq h\bigl(x^{(k)}\bigr)^T - h(\eta)^T.$$

Moreover, there are no sequences $\{x^{(k)}\} \subset \Omega$, if $\|x^{(k)}\| \to \infty$ as $k \to \infty$, such that there exist $y^{(k)} \in R^m_+$ and $z^{(k)} \in R^l$ satisfying

$$\lim_{k\to\infty} \bigl(\eta - x^{(k)}\bigr)^T \bigl(x^{(k)} - F\bigl(x^{(k)}\bigr) + \nabla g\bigl(x^{(k)}\bigr)y^{(k)} + \nabla h\bigl(x^{(k)}\bigr)z^{(k)}\bigr) \geq 0.$$

Using assumptions (A_1'), (A_2), and (A_3), we show that a smooth curve Γ_{w^0} starting from $(w^{(0)}, 1)$ exists by a similar analysis to that in [21].

Lemma 2.1 *Let assumptions (A_1'), (A_2), and (A_3) hold. Then, for almost all $w^{(0)}$, the projection of the smooth curve $\Gamma_{w^{(0)}}$ onto the x-plane is bounded.*

Proof If the conclusion does not hold, there exists a sequence of points $\{(x^{(k)}, y^{(k)}, z^{(k)}, \mu_k)\}_{k=1}^{\infty}$ such that $\|x^{(k)}\| \to \infty$ as $k \to \infty$.

The following conclusion is easily obtained by simple computation:

$$\|x^{(k)} - \eta\|^2 - \|x^{(0)} - \eta\|^2 \le 2(x^{(k)} - \eta)^T (x^{(k)} - x^{(0)}). \tag{2}$$

Then we obtain

$$(1 - \mu_k)(x^{(k)} - F(x^{(k)}) + \nabla g(x^{(k)}) y^{(k)}) + \nabla h(x^{(k)}) z^{(k)} + \mu(x^{(k)} - x^{(0)}) = 0, \tag{3}$$

$$h(x^{(k)}) = 0, \tag{4}$$

$$Y^{(k)} g(x^{(k)}) - \mu_k Y^{(0)} g(x^{(0)}) = 0. \tag{5}$$

Multiplying (3) by $(x^{(k)} - \eta)^T$, we obtain

$$\begin{aligned} &(1 - \mu_k)(x^{(k)} - \eta)^T (x^{(k)} - F(x^{(k)}) + \nabla g(x^{(k)}) y^{(k)}) \\ &\quad + (x^{(k)} - \eta)^T \nabla h(x^{(k)}) z^{(k)} + \mu_k (x^{(k)} - \eta)^T (x^{(k)} - x^{(0)}) = 0, \end{aligned} \tag{6}$$

i.e.,

$$\begin{aligned} \mu_k (x^{(k)} - \eta)^T (x^{(k)} - x^{(0)}) &= -(1 - \mu_k)(x^{(k)} - \eta)^T (x^{(k)} - F(x^{(k)}) + \nabla g(x^{(k)}) y^{(k)}) \\ &\quad - (x^{(k)} - \eta)^T \nabla h(x^{(k)}) z^{(k)}. \end{aligned} \tag{7}$$

So

$$\begin{aligned} &\mu_k (\|x^{(k)} - \eta\|^2 - \|x^{(0)} - \eta\|^2) \\ &\quad \le 2\mu_k (x^{(k)} - \eta)^T (x^{(k)} - x^{(0)}) \\ &\quad = -2(1 - \mu_k)(x^{(k)} - \eta)^T (x^{(k)} - F(x^{(k)}) + \nabla g(x^{(k)}) y^{(k)}) - 2(x^{(k)} - \eta)^T \nabla h(x^{(k)}) z^{(k)} \\ &\quad = -2(1 - \mu_k)(x^{(k)} - \eta)^T (x^{(k)} - F(x^{(k)}) + \nabla g(x^{(k)}) y^{(k)}) \\ &\qquad - 2(1 - \mu_k)(x^{(k)} - \eta)^T \nabla h(x^{(k)}) z^{(k)} + 2\mu_k (\eta - x^{(k)})^T \nabla h(x^{(k)}) z^{(k)} \\ &\quad \le -2(1 - \mu_k)(x^{(k)} - \eta)^T (x^{(k)} - F(x^{(k)}) + \nabla g(x^{(k)}) y^{(k)} + \nabla h(x^{(k)}) z^{(k)}) \\ &\qquad + 2\mu_k (h(\eta)^T - h(x^{(k)})^T) z^{(k)} \\ &\quad = -2(1 - \mu_k)(x^{(k)} - \eta)^T (x^k - F(x^{(k)}) + \nabla g(x^{(k)}) y^k + \nabla h(x^{(k)}) z^{(k)}). \end{aligned} \tag{8}$$

From (8), we obtain

$$\begin{aligned} &(\eta - x^{(k)})^T (x^{(k)} - F(x^{(k)}) + \nabla g(x^{(k)}) y^{(k)} + \nabla h(x^{(k)}) z^{(k)}) \\ &\quad \ge \frac{\mu_k}{2(1 - \mu_k)} (\|x^{(k)} - \eta\|^2 - \|x^{(0)} - \eta\|^2). \end{aligned} \tag{9}$$

When $\|x^{(k)}\| \to \infty$, from (9), we obtain

$$\begin{aligned} &\lim_{k\to\infty} \left(\eta - x^{(k)}\right)^T \left(x^{(k)} - F\left(x^{(k)}\right) + \nabla g\left(x^{(k)}\right) y^{(k)} + \nabla h\left(x^{(k)}\right) z^{(k)}\right) \\ &\quad \geq \lim_{k\to\infty} \frac{\mu_k}{2(1-\mu_k)} \left(\left\|x^{(k)} - \eta\right\|^2 - \left\|x^{(0)} - \eta\right\|^2\right) \geq 0, \end{aligned} \tag{10}$$

which contradicts assumption (A_1'). □

Then by a similar analysis to that in [21], we show that the projections of the smooth curve $\Gamma_{w^{(0)}}$ onto the y and z planes are also bounded, and we can also find a fixed point following the curve $\Gamma_{w^{(0)}}$. As for how to trace $\Gamma_{w^{(0)}}$, one can refer to [25, 26].

To give a constructive proof of the general Brouwer fixed point theorem in more general non-convex sets and to expand the scope of the choice of initial points of the homotopy method, in [22], we introduced continuous mappings $\xi(x,u) = (\xi_1(x,u_1),\ldots,\xi_m(x,u_m)) \in R^{n\times m}$ and $\eta(x,v) = (\eta_1(x,v_1),\ldots,\eta_l(x,v_l)) \in R^{n\times l}$, then applied proper perturbations to the constrained functions $g(x)$ and $h(x)$ by using the following parameters:

$$\gamma_i = \begin{cases} 2g_i(x^{(0)}), & g_i(x^{(0)}) > 0, \\ 1, & g_i(x^{(0)}) = 0, \qquad i = 1,\ldots,m, \\ 0, & g_i(x^{(0)}) < 0, \end{cases}$$

$$\theta_j = \begin{cases} 1, & h_j(x^{(0)}) \neq 0, \\ 0, & h_j(x^{(0)}) = 0, \end{cases} \qquad j = 1,\ldots,l.$$

Then let

$$\begin{aligned} &e_m = (1,\ldots,1)^T \in R^m, \qquad \Upsilon = \operatorname{diag}(\gamma_1,\ldots,\gamma_m) \in R^{m\times m}, \\ &\Theta = \operatorname{diag}(\theta_1,\ldots,\theta_l)^T \in R^{l\times l}, \\ &\Omega(\mu) = \left\{x \in R^n : g(x) - \mu\Upsilon\left(g\left(x^{(0)}\right) + e_m\right) \leq 0, h(x) - \mu\Theta h\left(x^{(0)}\right) = 0\right\}, \\ &\Omega^0(\mu) = \left\{x \in R^n : g(x) - \mu\Upsilon\left(g\left(x^{(0)}\right) + e_m\right) < 0, h(x) - \mu\Theta h\left(x^{(0)}\right) = 0\right\}, \\ &I(x,\mu) = \left\{i \in \{1,\ldots,m\} : g_i(x) - \mu\gamma_i\left(g_i\left(x^{(0)}\right) + 1\right) = 0\right\}. \end{aligned}$$

In [22], we made the following assumptions:

(C_1) $\Omega^0(\mu)$ is nonempty and $\Omega(\mu)$ is bounded;

(C_2) $\xi_i(x,0) = 0$, $i = 1,\ldots,m$, $\eta_j(x,0) = 0$, $j = 1,\ldots,l$; besides, for any $x \in \Omega(\mu)$, if $\|(y,z,u,v)\| \to \infty$, then

$$\left\|\sum_{i\in I(x,\mu)} \left(y_i \nabla g_i(x) + \xi_i(x,u_i)\right) + \nabla h(x) z + \eta(x,v)\right\| \to \infty;$$

(C_3) For any $x \in \Omega(\mu)$, if

$$\sum_{i\in I(x,\mu)} \left(y_i \nabla g_i(x) + \xi_i(x,u_i)\right) + \nabla h(x) z + \eta(x,v) = 0, \quad y_i \geq 0, u_i \geq 0,$$

then $y_i = 0$, $u_i = 0$, $\forall i \in I(x,\mu)$, $z = 0$, $v = 0$;

(C_4) When $\mu = 0, 1$, for any $x \in \Omega(\mu)$, we have

$$\left\{x + \sum_{i \in I(x,\mu)} \xi_i(x, u_i) + \eta(x, v) : u_i \geq 0 \text{ for } i \in I(x, \mu), v \in R^l\right\} \cap \Omega(\mu) = \{x\}.$$

Furthermore, we construct the following homotopy:

$$H\big(w, w^{(0)}, \mu\big) = \begin{pmatrix} (1-\mu)(x - F(x) + (1-\mu)\mu\nabla g(x)y) + \sum_{i=1}^m \xi_i(x, (1-\mu)y_i) \\ + (1-\mu)\mu(\nabla h(x)z + \beta) + \eta(x, z) + \mu(x - x^{(0)}) \\ h(x) - \mu\Theta h(x^{(0)}) \\ Y(g(x) - \mu\Upsilon(g(x^{(0)}) + e_m)) - \mu Y^{(0)}(g(x^{(0)}) - \Upsilon(g(x^{(0)}) + e_m)) \end{pmatrix} = 0, \tag{11}$$

where $\beta \in R^n$ is a given constant vector.

In this paper, to extend the results in [22] to the unbounded sets, we replace the boundedness condition (C_1) with the following unboundedness one:

(C_1') $\Omega^0(\mu)$ is nonempty; for any given $\alpha \in \Omega(\mu)$,

$$\begin{aligned}(\alpha - x)^T \xi_i(x, u_i) \leq \Big(&\big(g_i(\alpha) - \mu\gamma_i\big(g_i\big(x^{(0)}\big) + 1\big)\big)^T \\ &- \big(g_i(x) - \mu\gamma_i\big(g_i\big(x^{(0)}\big) + 1\big)\big)^T\Big)u_i, \quad i = 1, \ldots, m,\end{aligned}$$

and

$$(\alpha - x)^T \eta(x, v) \leq \big(h(x) - \mu\Theta h\big(x^{(0)}\big)\big)^T v - \big(h_j(\alpha) - \mu\Theta h\big(x^{(0)}\big)\big)^T v.$$

In addition, there are no sequences $\{x^{(k)}\} \subset \Omega$, if $\|x^{(k)}\| \to \infty$ as $k \to \infty$, such that there exist $u^{(k)} \in R_+^m$ and $v^{(k)} \in R^l$ satisfying

$$\lim_{k\to\infty} \big(x^{(k)} - \alpha\big)^T \big(x^{(k)} - F\big(x^{(k)}\big) + \nabla g\big(x^{(k)}\big)u^{(k)} + \nabla h\big(x^{(k)}\big)v^{(k)} + \mu\beta\big) \leq 0.$$

In the following, we are devoted to proving the boundedness of x-component of w.

Lemma 2.2 *Let H be defined as in (11), let $g_i(x)$, $i = 1, \ldots, m$, and $h_j(x)$, $j = 1, \ldots, l$ be C^3 functions, let assumptions (C_1'), (C_2)–(C_4) hold, and let $\xi_i(x, u_i)$, $i = 1, \ldots, m$, and $\eta_j(x, v_j)$, $j = 1, \ldots, l$, be C^2 functions. Then, for almost all $w^{(0)}$, the x-component of w is bounded.*

Proof If the conclusion does not hold, then there exists a sequence of points $\{(x^{(k)}, y^{(k)}, z^{(k)}, \mu_k)\}_{k=1}^\infty$ such that $\|x^{(k)}\| \to \infty$ as $k \to \infty$. From the first equation in (11), we obtain

$$\begin{aligned}&(1-\mu_k)\big(x^{(k)} - F\big(x^{(k)}\big) + (1-\mu_k)\mu_k\nabla g\big(x^{(k)}\big)y^{(k)}\big) + \sum_{i=1}^m \xi_i\big(x^{(k)}, (1-\mu_k)y_i^{(k)}\big) \\ &\quad + (1-\mu_k)\mu_k\big(\nabla h\big(x^{(k)}\big)z^{(k)} + \beta\big) + \eta\big(x, z^{(k)}\big) + \mu_k\big(x^{(k)} - x^{(0)}\big) = 0.\end{aligned} \tag{12}$$

Multiplying (12) by $(x^{(k)}-\alpha)^T$, we obtain

$$\begin{aligned}&(1-\mu_k)\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-F\big(x^{(k)}\big)+(1-\mu_k)\mu_k\nabla g\big(x^{(k)}\big)y^{(k)}\big)\\&+\sum_{i=1}^m\big(x^{(k)}-\alpha\big)^T\xi_i\big(x^{(k)},(1-\mu_k)y_i^{(k)}\big)+(1-\mu_k)\mu_k\big(x^{(k)}-\alpha\big)^T\big(\nabla h\big(x^{(k)}\big)z^{(k)}+\beta\big)\\&+\big(x^{(k)}-\alpha\big)^T\eta\big(x,z^{(k)}\big)+\mu_k\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-x^{(0)}\big)=0.\end{aligned}\tag{13}$$

Furthermore, rewrite (13) as

$$\begin{aligned}&\mu_k\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-x^{(0)}\big)\\&=-(1-\mu_k)\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-F\big(x^{(k)}\big)+(1-\mu_k)\mu_k\nabla g\big(x^{(k)}\big)y^{(k)}\big)\\&\quad-\sum_{i=1}^m\big(x^{(k)}-\alpha\big)^T\xi_i\big(x^{(k)},(1-\mu_k)y_i^{(k)}\big)-(1-\mu_k)\mu_k\big(x^{(k)}-\alpha\big)^T\big(\nabla h\big(x^{(k)}\big)z^{(k)}+\beta\big)\\&\quad-\big(x^{(k)}-\alpha\big)^T\eta\big(x,z^{(k)}\big).\end{aligned}\tag{14}$$

By a simple computation, we derive

$$\big\|x^{(k)}-\alpha\big\|^2-\big\|x^{(0)}-\alpha\big\|^2\le 2\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-x^{(0)}\big).\tag{15}$$

Combining (14) and (15), we conclude

$$\begin{aligned}&\mu_k\big(\big\|x^{(k)}-\alpha\big\|^2-\big\|x^{(0)}-\alpha\big\|^2\big)\\&\le 2\mu_k\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-x^{(0)}\big)\\&=-2(1-\mu_k)\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-F\big(x^{(k)}\big)+(1-\mu_k)\mu_k\nabla g\big(x^{(k)}\big)y^{(k)}\big)\\&\quad-\sum_{i=1}^m 2\big(x^{(k)}-\alpha\big)^T\xi_i\big(x^{(k)},(1-\mu_k)y_i^{(k)}\big)\\&\quad-2(1-\mu_k)\mu_k\big(x^{(k)}-\alpha\big)^T\big(\nabla h\big(x^{(k)}\big)z^{(k)}+\beta\big)-2\big(x^{(k)}-\alpha\big)^T\eta\big(x,z^{(k)}\big)\\&=-2(1-\mu_k)\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-F\big(x^{(k)}\big)\\&\quad+(1-\mu_k)\mu_k\nabla g\big(x^{(k)}\big)y^{(k)}+\mu_k\big(\nabla h\big(x^{(k)}\big)z^{(k)}+\beta\big)\big)\\&\quad-\sum_{i=1}^m 2\big(x^{(k)}-\alpha\big)^T\xi_i\big(x^{(k)},(1-\mu_k)y_i^{(k)}\big)-2\big(x^{(k)}-\alpha\big)^T\eta\big(x,z^{(k)}\big).\end{aligned}\tag{16}$$

By assumption (C_1') and (16), we obtain

$$\begin{aligned}&\mu_k\big(\big\|x^{(k)}-\alpha\big\|^2-\big\|x^{(0)}-\alpha\big\|^2\big)\\&\le-2(1-\mu_k)\big(x^{(k)}-\alpha\big)^T\big(x^{(k)}-F\big(x^{(k)}\big)+\nabla g\big(x^{(k)}\big)\big((1-\mu_k)\mu_k y^{(k)}\big)\\&\quad+\nabla h\big(x^{(k)}\big)\big(\mu_k z^{(k)}\big)+\mu_k\beta\big)+\sum_{i=1}^m 2\big(\big(g_i(\alpha)-\mu_k\gamma_i\big(g_i\big(x^{(0)}\big)+1\big)\big)^T\end{aligned}$$

$$
\begin{aligned}
&- \big(g_i\big(x^{(k)}\big) - \mu_k\gamma_i\big(g_i\big(x^{(0)}\big)+1\big)\big)^T\big)(1-\mu_k)y_i^{(k)} \\
&+ \big(h(\alpha) - \mu_k\Theta h\big(x^{(0)}\big)\big)^T z^{(k)} - \big(h\big(x^{(k)}\big) - \mu_k\Theta h\big(x^{(0)}\big)\big)^T z^{(k)}.
\end{aligned} \tag{17}
$$

Because $g_i(\alpha) - \mu_k\gamma_i(g_i(x^{(0)})+1) \le 0$ and $y_i^{(k)} \ge 0$, $i = 1,\dots,m$, then

$$
\begin{aligned}
&\mu_k\big(\|x^{(k)} - \alpha\|^2 - \|x^{(0)} - \alpha\|^2\big) \\
&\quad\le -2(1-\mu_k)\big(x^{(k)} - \alpha\big)^T\big(x^{(k)} - F\big(x^{(k)}\big) + \nabla g\big(x^{(k)}\big)\big((1-\mu_k)\mu_k y^{(k)}\big) \\
&\quad\quad + \nabla h\big(x^{(k)}\big)\big(\mu_k z^{(k)}\big) + \mu_k\beta\big) \\
&\quad\quad - \sum_{i=1}^m 2\big(\big(g_i\big(x^{(k)}\big) - \mu_k\gamma_i\big(g_i\big(x^{(0)}\big)+1\big)\big)^T(1-\mu_k)y_i^{(k)}.
\end{aligned} \tag{18}
$$

It follows from the third equation in (11) that

$$
\big(g_i\big(x^{(k)}\big) - \mu_k\gamma_i\big(g_i\big(x^{(0)}\big)+1\big)\big)y_i^{(k)} = -\mu_k\big(g_i\big(x^{(0)}\big) - \gamma_i\big(g_i\big(x^{(0)}\big)+1\big)y_i^{(0)}. \tag{19}
$$

Then (18) becomes

$$
\begin{aligned}
&\mu_k\big(\|x^{(k)} - \alpha\|^2 - \|x^{(0)} - \alpha\|^2\big) \\
&\quad\le -2(1-\mu_k)\big(x^{(k)} - \alpha\big)^T\big(x^{(k)} - F\big(x^{(k)}\big) + \nabla g\big(x^{(k)}\big)\big((1-\mu_k)\mu_k y^{(k)}\big) \\
&\quad\quad + \nabla h\big(x^{(k)}\big)\big(\mu_k z^{(k)}\big) + \mu_k\beta\big) + \sum_{i=1}^m 2(1-\mu_k)\mu_k\big(\big(g_i\big(x^{(0)}\big) - \mu_k\gamma_i\big(g_i\big(x^{(0)}\big)+1\big)\big)^T y_i^{(0)} \\
&\quad\le -2(1-\mu_k)\big(x^{(k)} - \alpha\big)^T\big(x^{(k)} - F\big(x^{(k)}\big) + \nabla g\big(x^{(k)}\big)\big((1-\mu_k)\mu_k y^{(k)}\big) \\
&\quad\quad + \nabla h\big(x^{(k)}\big)\big(\mu_k z^{(k)}\big) + \mu_k\beta\big).
\end{aligned} \tag{20}
$$

Dividing two sides of (20) by the item $2(1-\mu_k)$, one obtains

$$
\begin{aligned}
&\big(\alpha - x^{(k)}\big)^T\big(x^{(k)} - F\big(x^{(k)}\big) + \nabla g\big(x^{(k)}\big)\big((1-\mu_k)\mu_k y^{(k)}\big) + \nabla h\big(x^{(k)}\big)\big(\mu_k z^{(k)}\big) + \mu_k\beta\big) \\
&\quad\ge \frac{\mu_k}{2(1-\mu_k)}\big(\|x^{(k)} - \alpha\|^2 - \|x^{(0)} - \alpha\|^2\big).
\end{aligned} \tag{21}
$$

From (21), as $\|x^{(k)}\| \to \infty$, one obtains

$$
\begin{aligned}
&\lim_{k\to\infty}\big(\alpha - x^{(k)}\big)^T\big(x^{(k)} - F\big(x^{(k)}\big) + \nabla g\big(x^{(k)}\big)\big((1-\mu_k)\mu_k y^{(k)}\big) + \nabla h\big(x^{(k)}\big)\big(\mu_k z^{(k)}\big) + \mu_k\beta\big) \\
&\quad\ge \lim_{k\to\infty}\frac{\mu_k}{2(1-\mu_k)}\big(\|x^{(k)} - \alpha\|^2 - \|x^{(0)} - \alpha\|^2\big) \ge 0.
\end{aligned}
$$

By assumption (C_1'), this is not possible. □

The following analysis is similar to that in [22], so we omit it in this paper.

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Authors' contributions
The main idea of this paper was proposed by MLS and MJL. MLS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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