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Common fixed point results in function weighted metric spaces

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Abstract

In this paper, we consider a common fixed point result in the context of a very recently defined abstract space: "function weighted metric space". We present also some examples to illustrate the validity of the given results.

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1 Introduction

One of the very natural trends of mathematical research is to refine the framework of the known theorems and results. For instance, Banach observed the first metric fixed point results in the setting of complete normed spaces. Immediate extension of this theorem was given by Caccioppoli who observed the characterization of Banach fixed point theorem in the context of complete metric spaces. After then, for the various abstract spaces, several analogs of the Banach contraction principle have been reported. Among them we can underline some of the interesting abstract structures such as modular metric space, symmetric space, semi-metric space, quasi-metric space, partial metric space, b-metric space, dislocated (metric-like) space, fuzzy metric space, probabilistic metric space, 2-metric space, δ-metric space, G-metric space, S-metric space, function weighted metric space, and so on.

In this paper, we shall restrict ourselves to the recently introduced generalization of a metric space, namely, function weighted metric space [1]. Our aim is to obtain a fixed point result for two mappings. More precisely, we shall consider coincidence points and common fixed points for certain operators in the setting of the function weighted metric space.

For the sake of the self-contained text, we recall the definition of the newly introduced metric space. For this purpose, we first recall two basic notions for functions that we need: A function $f : (0, +\infty) \to \mathbb{R}$ is called logarithmic-like if each sequence $\{t_n\} \subset (0, +\infty)$ satisfies

$$\lim_{n\to+\infty} t_n = 0 \quad \Longleftrightarrow \quad \lim_{n\to+\infty} f(t_n) = -\infty.$$

A function $f:(0,+\infty)\to\mathbb{R}$ is called non-decreasing if

$$0<s<t \quad\Longrightarrow\quad f(s)\le f(t).$$

The letter $\mathfrak{F}$ denotes the set of all functions that are non-decreasing (in symbols, (Δ_1)) and logarithmic-like (in symbols, (Δ_2)).

By using the auxiliary functions of $\mathfrak{F}$, Jleli–Samet [1] introduced a new metric space, more precisely, a function weighted metric space. Indeed, in this new metric space definition, Jleli–Samet [1] proposed a new condition instead of triangle inequality by using a function from the set $\mathfrak{F}$. Henceforth, we presume that X is a nonempty set and avoid to repeat this in all statements. For the sake of the self-contained text, we put the definition here:

Definition 1.1 Let $\delta:X\times X\to[0,+\infty)$ be a given mapping. Suppose that there exist an $f\in\mathfrak{F}$ and a constant $\mathcal{C}\in[0,+\infty)$ such that

- (Δ_1) (Self-distance axiom) $\delta(x,y)=0\Longleftrightarrow x=y$, for $x,y\in X$;
- (Δ_2) (Symmetry axiom) For all $x,y\in X$, we have $\delta(x,y)=\delta(y,x)$;
- (Δ_3) (Generalized function f-weighted triangle inequality axiom) For any pair $(x,y)\in X\times X$ and for any $N\in\mathbb{N}$ with $N\ge 2$, we have

$$\delta(x,y)>0 \quad\Longrightarrow\quad f\bigl(\delta(x,y)\bigr)\le f\left(\sum_{i=1}^{N-1}\delta(u_i,u_{i+1})\right)+\mathcal{C},$$

for every $(u_i)_{i=1}^{N}\subset X$ with $(u_1,u_N)=(x,y)$.

Then, the function δ is called a "function weighted metric" or "$\mathcal{F}$-metric" on X, and the pair (X,δ) is named as a "function weighted metric space" or "$\mathcal{F}$-metric space".

Throughout the text, we prefer to use the name "function weighted metric space" instead of "$\mathcal{F}$-metric space".

As it seen clearly, the only difference between a "standard metric space" and a "function weighted metric space" is the last axiom: In a "function weighted metric space" instead of "the triangle inequality", another axiom has been used, namely "generalized f-weighted triangle inequality axiom." Based on this observation, we also easily conclude that any metric on X is an $\mathcal{F}$-metric on X by letting $f(t)=\ln t$ for the axiom (Δ_3). Indeed, on account of the triangle inequality, for all distinct $x,y\in X$ and for each $N\in\mathbb{N}$ with $N\ge 2$, and for any $(u_i)_{i=1}^{N}\subset X$ with $(u_1,u_N)=(x,y)$, we find

$$d(x,y)>0 \quad\Longrightarrow\quad \ln\bigl(d(x,y)\bigr)\le\ln\left(\sum_{i=1}^{N-1}d(u_i,u_{i+1})\right),$$

since $d(x,y)\le\sum_{i=1}^{N-1}d(u_i,u_{i+1})$, and $f(t)=\ln t$ is non-decreasing. Here, we take $\mathcal{C}=0$.

The main goal of this paper is to obtain some common fixed point result in the context of function weighted metric spaces.

2 Main results

In this section, we establish a common fixed point theorem in the setting of function weighted metric spaces.

Theorem 2.1 *Let $T, g : X \to X$ be self-mappings on a function weighted complete metric space (X, δ) such that $T(X) \subseteq g(X)$. Suppose that there exists $k \in (0,1)$ such that*

$$\delta\big(T(x), T(y)\big) \leq k\delta(gx, gy), \quad (x,y) \in X \times X.$$

Also suppose $g(X)$ is closed. Then, T and g have a unique coincidence point.

Proof Since $T(X) \subseteq g(X)$, we can choose a point $x_1 \in X$ such that $Tx_0 = gx_1$. We shall construct a sequence x_n in X such that

$$y_n = Tx_n = gx_{n+1}, \quad n = 0, 1, 2, \ldots. \tag{2.1}$$

First, observe that T and g possess a unique coincidence point. Indeed, suppose on the contrary that $(u,v) \in X \times X$ are two distinct coincidence points of T and g. Thus, $\delta(u,v) > 0$, $g(u) = T(u)$ and $g(v) = T(v)$. Then from (ii), we have

$$\delta(u,v) = \delta(Tu, Tv) \leq k\delta(gu, gv) = k\delta(u,v) < \delta(u,v),$$

a contradiction.

Suppose $(f, \mathcal{C}) \in \mathfrak{F} \times [0, +\infty)$ so that (Δ_3) is fulfilled. For a given $\epsilon > 0$ and on account of (Δ_2), there exists $\gamma > 0$ such that

$$0 < t < \gamma \quad \Rightarrow \quad f(t) < f(\epsilon) - \mathcal{C}. \tag{2.2}$$

Consider the sequence $\{y_n\} \subset X$ defined in (2.1). Now, without loss of generality, we assume that $\delta(Tx_0, Tx_1) > 0$. Otherwise, x_1 will be a coincidence point of T and g. By the contraction condition,

$$\begin{aligned} \delta(Tx_n, Tx_{n+1}) &\leq k\delta(gx_n, gx_{n+1}) \\ &= k\delta(Tx_{n-1}, Tx_n) \\ &\leq k^2\delta(gx_{n-1}, gx_n), \end{aligned}$$

thereby implying that

$$\delta(Tx_n, Tx_{n+1}) \leq k^n\delta(Tx_0, Tx_1).$$

So we have

$$\sum_{i=n}^{m-1} \delta(Tx_i, Tx_{i+1}) \leq \frac{k^n}{1-k}\delta(Tx_0, Tx_1), \quad m > n.$$

Since

$$\lim_{n\to+\infty} \frac{k^n}{1-k} D(Tx_0, Tx_1) = 0,$$

there exists some $N \in N$ such that

$$0 < \frac{k^n}{1-k}\delta(Tx_0, Tx_1) < \gamma, \quad n \geq N. \tag{2.3}$$

Hence, by (2.2) and (Δ_1), we have

$$\begin{aligned} f\left(\sum_{i=n}^{m-1}\delta(Tx_i, Tx_{i+1})\right) &\leq f\left(\frac{k^n}{1-k}\delta(Tx_0, Tx_1)\right) \\ &< f(\epsilon) - \mathcal{C}, \quad m > n \geq N. \end{aligned} \tag{2.4}$$

Employing (Δ_3) together with (2.4), we find

$$\delta(y_n, y_m) > 0, \quad m > n \geq N \quad \Rightarrow \quad f\big(\delta(y_n, y_m)\big) \leq f\left(\sum_{i=n}^{m-1}\delta(y_i, y_{i+1})\right) + \mathcal{C} < f(\epsilon),$$

which implies by (Δ_1) that

$$\delta(y_n, y_m) < \epsilon, \quad m, n \geq N.$$

This proves that $\{Tx_n\}$ is Cauchy. Since $\{Tx_n\} = \{gx_{n+1}\} \subseteq g(X)$ and $g(X)$ is closed, there exists $z \in X$ such that

$$\lim_{n\to\infty}\delta(gx_n, gz) = 0. \tag{2.5}$$

As a next step, we shall indicate that z is a coincidence point of T and g. On the contrary, assume that $\delta(Tz, gz) > 0$. We have

$$\begin{aligned} f\big(\delta(Tz, gz)\big) &\leq f\big(\delta(Tz, Tx_n) + \delta(Tx_n, gz)\big) + \mathcal{C}, \quad n \in N \\ &\leq f\big(k\delta(gz, gx_n) + \delta(gx_{n+1}, gz)\big) + \mathcal{C}. \end{aligned}$$

As $n \to \infty$ in the inequality above, and due to (2.5), we get

$$\lim_{n\to+\infty} f\big(k\delta(gz, gx_n) + \delta(gx_{n+1}, gz)\big) + \mathcal{C} = -\infty,$$

which is a contradiction. Therefore, we conclude that $\delta(Tz, gz) = 0$, and hence z is a unique coincidence point of T and g. □

Example 2.1 Consider $X = \mathbb{R}_0^+$. Define $\delta : X \times X \to [0, \infty)$ as

$$\delta(x, y) = |x - y|, \quad x, y \in X.$$

So δ is a function weighted metric with $f(t) = \ln(t)$ and $a = 0$. Consider,

$$T(x) = \begin{cases} \frac{x-3}{2} & \text{if } x > -1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} \frac{5-x}{2} & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $x = 4$ is a coincidence point of T and g. We shall show that there exists a $k \in (0,1)$ such that

$$\delta(Tx, Ty) = \left|\frac{x-3}{2} - \frac{y-3}{2}\right| = \frac{1}{2}|x-y| \leq \frac{1}{2}|y-x| = k|gx-gy| = k\delta(gx,gy).$$

Therefore, all the hypothesis of Theorem 2.1 are satisfied. Furthermore, $x = 4$ is a unique coincidence point of T and g.

Next, we present the notion of generalized θ–ψ contractive pair of mappings in the setting of function weighted metric spaces as follows:

Definition 2.1 Let T, g be self-mappings on a function weighted metric space (X, δ). We say that the pair (T,g) is a generalized θ–ψ contractive pair of mappings if there exist $\theta : X \times X \to [0,+\infty)$ and $\psi \in \Psi$ such that

$$\theta(gx,gy)\delta(Tx, Ty) \leq \psi\bigl(M(gx,gy)\bigr), \tag{2.6}$$

for all $x, y \in X$, where

$$M(gx,gy) = \max\left\{\delta(gx,gy), \frac{\delta(gx, Tx) + \delta(gy, Ty)}{2}, \frac{\delta(gx, Ty) + \delta(gy, Tx)}{2}\right\}.$$

Theorem 2.2 *Let (X, δ) be a complete function weighted metric space and $T,g : X \to X$ be such that $T(X) \subseteq g(X)$ where $g(X)$ is closed. Assume that the pair (T,g) is a generalized θ–ψ contractive pair of mappings and the following conditions hold*:

(i) *T is θ-admissible with respect to g;*
(ii) *There exists $x_0 \in X$ such that $\theta(gx_0, Tx_0) \geq 1$;*
(iii) *If $\{gx_n\}$ is a sequence in X such that $\theta(gx_n,gx_{n+1}) \geq 1$ for all n and $gx_n \to gz \in g(X)$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\theta(gx_{n(k)},gz) \geq 1$ for all k.*

Then, T and g have a coincidence point.

Proof In view of condition (ii), let $x_0 \in X$ be such that $\theta(gx_0, Tx_0) \geq 1$. On account of $T(X) \subseteq g(X)$, we choose a point $x_1 \in X$ in a way that $Tx_0 = gx_1$. Iteratively, we build a sequence $\{x_n\}$ in X so that

$$y_n := Tx_n = gx_{n+1}, \quad n = 0, 1, 2, \ldots. \tag{2.7}$$

Since T is θ-admissible with respect to g, we have

$$\theta(gx_0, Tx_0) = \theta(gx_0,gx_1) \geq 1 \quad \Rightarrow \quad \theta(Tx_0, Tx_1) = \theta(gx_1,gx_2) \geq 1.$$

Using mathematical induction, we get

$$\theta(gx_n, gx_{n+1}) \geq 1, \quad \forall n = 0, 1, 2, \ldots. \tag{2.8}$$

If $Tx_{n+1} = Tx_n$ for some n, then by (2.7),

$$Tx_{n+1} = gx_{n+1}, \quad n = 0, 1, 2, \ldots,$$

and then $x = x_{n+1}$ forms a coincidence point of T and g, which completes the proof. Accordingly, we suppose that $\delta(Tx_n, Tx_{n+1}) > 0$ for all n. Applying inequality (2.6) and using (2.8), we obtain

$$\begin{aligned} \delta(Tx_n, Tx_{n+1}) &\leq \theta(gx_n, gx_{n+1})\delta(Tx_n, Tx_{n+1}) \\ &\leq \psi\bigl(M(gx_n, gx_{n+1})\bigr). \end{aligned} \tag{2.9}$$

On the other hand, we have

$$\begin{aligned} M(gx_n, gx_{n+1}) &= \max\biggl\{\delta(gx_n, gx_{n+1}), \frac{\delta(gx_n, Tx_n) + \delta(gx_{n+1}, Tx_{n+1})}{2}, \\ &\qquad \frac{\delta(gx_n, Tx_{n+1}) + \delta(gx_{n+1}, Tx_n)}{2}\biggr\} \\ &\leq \max\bigl\{\delta(Tx_{n-1}, Tx_n), \delta(Tx_n, Tx_{n+1})\bigr\}. \end{aligned}$$

Due to the monotonicity of function ψ and using inequalities (2.9), we have for all $n \geq 1$

$$\delta(Tx_n, Tx_{n+1}) \leq \psi\bigl(\max\bigl\{\delta(Tx_{n-1}, Tx_n), \delta(Tx_n, Tx_{n+1})\bigr\}\bigr). \tag{2.10}$$

If, the inequality $\delta(Tx_{n-1}, Tx_n) \leq \delta(Tx_n, Tx_{n+1})$ is satisfied for some $n \in \mathbb{N}$, from (2.10), we obtain that

$$\delta(Tx_n, Tx_{n+1}) \leq \psi\bigl(\delta(Tx_n, Tx_{n+1})\bigr) < \delta(Tx_n, Tx_{n+1}),$$

a contradiction. Thus, for all $n \geq 1$, we have

$$\delta(Tx_{n-1}, Tx_n) = \max\bigl\{\delta(Tx_{n-1}, Tx_n), \delta(Tx_n, Tx_{n+1})\bigr\}. \tag{2.11}$$

Notice that in view of (2.10) and (2.11), we get for all $n \geq 1$ that

$$\delta(Tx_n, Tx_{n+1}) \leq \psi\bigl(\delta(Tx_{n-1}, Tx_n)\bigr). \tag{2.12}$$

Continuing this process inductively, we obtain

$$\delta(Tx_n, Tx_{n+1}) \leq \psi^n\bigl(\delta(Tx_0, Tx_1)\bigr), \quad \forall n \geq 1. \tag{2.13}$$

So,

$$\sum_{i=n}^{m-1} \delta(Tx_i, Tx_{i+1}) \leq \sum_{p=n}^{m-1} \psi^p\bigl(\delta(Tx_0, Tx_1)\bigr), \quad m > n.$$

Since

$$\lim_{p\to+\infty} \psi^p\big(\delta(Tx_0, Tx_1)\big) = 0,$$

there exists some $N \in N$ such that

$$0 < \psi^p\big(\delta(Tx_0, Tx_1)\big) < \gamma, \quad p \geq N. \tag{2.14}$$

Next, let $(f, \mathcal{C}) \in \mathfrak{F} \times [0, +\infty)$ be such that (Δ_3) is satisfied. Let $\epsilon > 0$ be fixed. By (Δ_2), $\exists \gamma > 0$ such that

$$0 < t < \gamma \quad \Rightarrow \quad f(t) < f(\epsilon) - \mathcal{C}. \tag{2.15}$$

Hence, by (2.15) and (Δ_1), we have

$$\begin{aligned} f\left(\sum_{i=n}^{m-1} \delta(Tx_i, Tx_{i+1})\right) &\leq f\left(\sum_{p=i}^{m-1} \psi^p \delta(Tx_0, Tx_1)\right) \\ &< f(\epsilon) - \mathcal{C}, \quad m > n \geq N. \end{aligned} \tag{2.16}$$

Using (Δ_3) and (2.16), we have

$$\delta(y_n, y_m) > 0, \quad m > n \geq N \quad \Rightarrow \quad f\big(\delta(y_n, y_m)\big) \leq f\left(\sum_{i=n}^{m-1} \delta(y_i, y_{i+1})\right) + \mathcal{C} < f(\epsilon),$$

which implies by (Δ_1) that

$$\delta(y_n, y_m) < \epsilon, \quad m, n \geq N.$$

It proves that $\{y_n\} = \{Tx_n\}$ is Cauchy sequence. Due to the completeness of the considered space, there is $y \in X$ such that

$$\lim_{n\to\infty} \delta(y_n, y) = 0 = \lim_{n\to\infty} \delta(Tx_n, y) = \lim_{n\to\infty} \delta(gx_{n+1}, y).$$

Since $\{Tx_n\} = \{gx_{n+1}\} \subseteq g(X)$ and $g(X)$ is closed, there exists $z \in X$ such that $gz = y$ and

$$\lim_{n\to\infty} \delta(gx_n, gz) = 0. \tag{2.17}$$

Next, we show that z is a coincidence point of T and g. On the contrary, assume that $\delta(Tz, gz) > 0$. We have from (iii) that

$$\begin{aligned} f\big(\delta(Tz, y)\big) &\leq f\big(\delta(Tz, Tx_{n(k)}) + \delta(Tx_{n(k)}, y)\big) + \mathcal{C}, \quad n \in N \\ &\leq f\big(\theta(gx_{n(k)}, gz)\delta(Tz, Tx_{n(k)}) + \delta(Tx_{n(k)}, y)\big) + \mathcal{C} \\ &\leq f\big(\psi\big(M(gz, gx_{n(k)})\big) + \delta(Tx_{n(k)}, y)\big) + \mathcal{C}, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} M(gx_{n(k)}, gz) &= \max\left\{\delta(gx_{n(k)}, gz), \frac{\delta(gx_{n(k)}, Tx_{n(k)}) + \delta(gz, Tz)}{2}, \right. \\ &\quad \left. \frac{\delta(gx_{n(k)}, fz) + \delta(gz, Tx_{n(k)})}{2}\right\} \\ &\le \max\left\{\delta(gx_{n(k)}, gz), \right. \\ &\quad \frac{\delta(gx_{n(k)}, Tx_{n(k)}) + \delta(gz, Tx_{n(k)}) + \delta(Tx_{n(k)}, Tz)}{2}, \\ &\quad \left. \frac{\delta(gx_{n(k)}, fz) + \delta(gz, Tx_{n(k)})}{2}\right\}. \end{aligned}$$

Keeping (2.17) in mind and letting $n \to \infty$ in (2.18), we conclude that the right-hand side tends to ∞. This is a contradiction, and hence z is coincidence point of point T and g. □

We present the following example in support of our theorem:

Example 2.2 Consider $X = \mathbb{R}$. Define $\delta : X \times X \to [0, \infty)$ as

$$\delta(x, y) = \begin{cases} e^{|x-y|} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

So δ is a complete function weighted metric with $f(t) = -\frac{1}{t}$ and $a = 1$. Consider the following self-mappings:

$$T(x) = \begin{cases} \frac{3-x}{2} & \text{if } x > -1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\psi(t) = \sqrt{t}$ for all $t \geq 0$. So all the hypothesis of Theorem 2.2 are satisfied and $x = \frac{3}{2}$ is a unique coincidence point of T and g.

Theorem 2.3 *If we assume that T or g is continuous, in addition to the axioms of in Theorem 2.2, then T and g possess a common point.*

If we take $\theta(gx, gy) = 1$ in Theorem 2.2, then we find the following:

Theorem 2.4 *Let (X, δ) be a complete function weighted metric space and $T, g : X \to X$ be such that $T(X) \subseteq g(X)$ where $g(X)$ is closed. Assume that the pair (T, g) satisfies*

$$\delta(Tx, Ty) \leq \psi\big(M(gx, gy)\big), \tag{2.19}$$

for all $x, y \in X$ *with* $\delta(Tx, Ty) > 0$ *where*

$$M(gx, gy) = \max\left\{\delta(gx, gy), \frac{\delta(gx, Tx) + \delta(gy, Ty)}{2}, \frac{\delta(gx, Ty) + \delta(gy, Tx)}{2}\right\}.$$

Then, T and g possess a coincidence point.

If we take $g(x) = x$ for all $x \in X$ in Theorem 2.4, then we derive the following result.

Theorem 2.5 *Let (X, δ) be a complete function weighted metric space and assume $T : X \to X$ satisfies*

$$\delta(Tx, Ty) \leq \psi\big(M(x, y)\big), \tag{2.20}$$

for all $x, y \in X$ *with* $\delta(Tx, Ty) > 0$ *where*

$$M(x, y) = \max\left\{\delta(x, y), \frac{\delta(x, Tx) + \delta(y, Ty)}{2}, \frac{\delta(x, Ty) + \delta(y, Tx)}{2}\right\}.$$

Then, T possesses a fixed point.

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