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A kind of sharp Wirtinger inequality

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Abstract

In this study, we give a kind of sharp Wirtinger inequality

$$\|f\|_p \leq C_{r,p,q}\left\|f^{(r)}\right\|_q \quad \text{for all } 1 \leq p,q \leq \infty,$$

where f is defined on $[0,1]$ and satisfies $f^{(k_1)}(0) = f^{(k_2)}(0) = \cdots = f^{(k_s)}(0) = f^{(m_{s+1})}(1) = \cdots = f^{(m_r)}(1) = 0$ with $0 \leq k_1 < k_2 < \cdots < k_s \leq r-1$ and $0 \leq m_{s+1} < m_{s+2} < \cdots < m_r \leq r-1$. First, based on the Birkhoff interpolation, we refer the computation of $C_{r,p,q}$ to the norm of an integral-type operator. Second, we refer the values of $C_{r,1,1}$ and $C_{r,\infty,\infty}$ to explicit integral expressions and the value of $C_{r,2,2}$ to the computation of the maximal eigenvalue of a Hilbert–Schmidt operator. Finally, we give three examples to show our method.

MSC: 41A44; 41A80

Keywords: Birkhoff interpolation; L_p-norm; Eigenvalue; Wirtinger inequality

1 Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. For $1 \leq p \leq \infty$, let $L_p[a,b]$ be the space of pth-power Lebesgue-integrable functions $f : [a,b] \to \mathbb{R}$ with the corresponding L_p-norms $\|\cdot\|_p$. Denote by W_p^r, $r \in \mathbb{N}$, the class of all continuous real-valued functions f defined on the interval $[a,b]$ such that $f^{(r-1)}$ (with $f^{(0)} := f$) is absolutely continuous and $\|f^{(r)}\|_p < \infty$.

The relationships among the norms of a function and its derivatives play an important role in the study of harmonic analysis and function approximation theory. There are many well-known inequalities in this area, for example, the Landau–Kolmogorov, Gorny, Wirtinger, Schmidt, Sobolev, Bernstein, and Markov inequalities. Wirtinger-type inequalities is a kind of the most important inequalities in this aspect. The first result appeared in [1, p. 105]: for every locally absolutely continuous and 2π-periodic function f with the first-order derivative $f' \in L_2[0,2\pi]$ and $\int_0^{2\pi} f(x)\,dx = 0$, we have

$$\|f\|_2 \leq \left\|f'\right\|_2,$$

where the equality is valid if and only if $f \in \operatorname{span}\{\cos, \sin\}$.

Since then, many results on Wirtinger-type inequalities appeared. For example, in the case $r = 1$, Schmidt [2] proved the following two results.

Let $0 < p \leq \infty$ and $1 \leq q \leq \infty$. Then for an arbitrary function $f \in W_q^1$ [a,b] satisfying $f(a) = 0$ (or equivalently $f(b) = 0$), we have the sharp inequality

$$\|f\|_p \leq \frac{(1/p + 1/q')^{-1/p-1/q'}(1/p)^{1/p}(1/q')^{1/q'}\Gamma(1 + 1/p + 1/q')}{\Gamma(1 + 1/p)\Gamma(1 + 1/q')} \times (b - a)^{1+1/p-1/q}\|f'\|_q, \tag{1.1}$$

where q' is the conjugate exponent of q, and $1/\infty$ is to be interpreted (in the usual way) as 0. At the same time, for an arbitrary function $f \in W_q^1[a,b]$ satisfying $f(a) = f(b) = 0$, we have the sharp inequality

$$\|f\|_p \leq \frac{1}{2}\frac{(1/p + 1/q')^{-1/p-1/q'}(1/p)^{1/p}(1/q')^{1/q'}\Gamma(1 + 1/p + 1/q')}{\Gamma(1 + 1/p)\Gamma(1 + 1/q')} \times (b - a)^{1+1/p-1/q}\|f'\|_q. \tag{1.2}$$

Further generalizations and applications of (1.1) and (1.2) can be found in [3–12].

In the case $r > 1$ the most important result is that if $f \in W_q^r[a,b]$ with zero a of multiplicity k and zero b of multiplicity $r - k$, $0 \leq k \leq r$, and $1 \leq p,q \leq \infty$, then we have the inequality

$$\|f\|_p \leq C(r,k,p,q)(b - a)^{r+1/p-1/q}\|f^{(r)}\|_q. \tag{1.3}$$

But as far as we know, the best constants $C(r,k,p,q)$ are known only for $q = \infty$ and in some particular cases for $r = 2$. At the same time, some papers closely related to the Wirtinger inequality, such as Shadrin [13] and Waldron [14], considered the problem of estimating the best constant $C(r,j,p,q)$ in the inequality

$$\|(f - H_\Theta f)^{(j)}\|_p \leq C(r,j,p,q)(b - a)^{r-j+1/p-1/q}\|f^{(r)}\|_q \quad \text{for all } f \in W_q^r[a,b], \tag{1.4}$$

where $H_\Theta f$ is the Hermite interpolation to f at some multiset of r points in $[a,b]$ and $0 \leq j < r$. However, the best constant was determined also only for $p = q = \infty$. Xu and Zhang [15] considered the corresponding estimate of (1.4) for the cubic Hermite interpolation (for which the number of points in the multiset is greater than r), but the best constant was determined only in the cases $p = q = \infty$ and $p = q = 1$. Liu, Wu, and Xu [16] obtained the constant in (1.3) for the particular case of $k = r$ and $p = q = 2$. Recently, Xu, Liu, and Xiong [17] obtained the best constant in (1.3) with zeros of multiplicity m at both a and b for $m + 1 \leq r \leq 2m + 2$ and $p = q = 2$.

In this paper, we give an extension of (1.3). Let $1 \leq s \leq r - 1$ be an integer, $0 \leq k_1 < k_2 < \cdots < k_s \leq r - 1$, and $0 \leq m_{s+1} < m_{s+2} < \cdots < m_r \leq r - 1$. Let D_g denote the number of k_i and m_j for which $k_i = g$ or $m_j = g$. We assume that

$$\sum_{g=0}^{m} D_g \geq m + 1 \quad \text{for all } m = 0, 1, \ldots, r - 1. \tag{1.5}$$

For $f \in W_p^r[a,b]$ with $f^{(k_i)}(a) = 0$ for $1 \leq i \leq s$ and $f^{(m_j)}(b) = 0$ for $s + 1 \leq j \leq r$, we will prove that (1.3) also holds and give the corresponding best constant.

The paper is organized as follows. Section 2 contains our main theorem and its proof. In Sect. 3, we give three examples to show our method.

2 Basic concepts and our main theorem

First, we introduce some known facts about Birkhoff interpolation related to our problems (see [18]). Let $x_0, x_1, \dots, x_n$ be points of $[a, b]$, not necessarily distinct. Also, let $k_0, k_1, \dots, k_n$ be integers such that $0 \le k_i \le n - 1$, $i = 0, 1, \dots, n$. The $n + 1$ pairs of numbers $(x_i, k_i)_{i=0}^n$ are supposed to be distinct. Furthermore, let D_g denote the number of pairs of the system $(x_i, k_i)_{i=0}^n$ for which $k_i = g$. We assume that

$$\sum_{g=0}^{m} D_g \ge m + 2 \quad \text{for all } 0 \le m \le n - 1. \tag{2.1}$$

Under assumption (2.1), for every $f \in W_p^n[a, b]$, from [18] we know that there exists a unique polynomial (Birkhoff interpolation) $L_n(f, t)$ of degree $\le n$ that satisfies

$$L_n^{(k_i)}(f, x_i) = f^{(k_i)}(x_i) \quad \text{for all } 0 \le i \le n. \tag{2.2}$$

Now we introduce a remainder theorem about the Birkhoff interpolation (see [18]). Denote $l_i = n - k_i - 1$ for $0 \le i \le n$, and let

$$\Delta_i = (-1)^i \begin{vmatrix} \frac{x_0^{l_0}}{l_0!} & \frac{x_0^{l_0-1}}{(l_0-1)!} & \cdots & \frac{x_0^{l_0-i+1}}{(l_0-i+1)!} & \frac{x_0^{l_0-i}}{(l_0-i)!} & \cdots & \frac{x_0^{l_0-n+1}}{(l_0-n+1)!} \\ \frac{x_1^{l_1}}{l_1!} & \frac{x_1^{l_1-1}}{(l_1-1)!} & \cdots & \frac{x_1^{l_1-i+1}}{(l_1-i+1)!} & \frac{x_1^{l_1-i}}{(l_1-i)!} & \cdots & \frac{x_1^{l_1-n+1}}{(l_1-n+1)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{x_{i-1}^{l_{i-1}}}{l_{i-1}!} & \frac{x_{i-1}^{l_{i-1}-1}}{(l_{i-1}-1)!} & \cdots & \frac{x_{i-1}^{l_{i-1}-i+1}}{(l_{i-1}-i+1)!} & \frac{x_{i-1}^{l_{i-1}-i}}{(l_{i-1}-i)!} & \cdots & \frac{x_{i-1}^{l_{i-1}-n+1}}{(l_{i-1}-n+1)!} \\ \frac{x_{i+1}^{l_{i+1}}}{l_{i+1}!} & \frac{x_{i+1}^{l_{i+1}-1}}{(l_{i+1}-1)!} & \cdots & \frac{x_{i+1}^{l_{i+1}-i+1}}{(l_{i+1}-i+1)!} & \frac{x_{i+1}^{l_{i+1}-i}}{(l_{i+1}-i)!} & \cdots & \frac{x_{i+1}^{l_{i+1}-n+1}}{(l_{i+1}-n+1)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{x_n^{l_n}}{l_n!} & \frac{x_n^{l_n-1}}{(l_n-1)!} & \cdots & \frac{x_n^{l_n-i+1}}{(l_n-i+1)!} & \frac{x_n^{l_n-i}}{(l_n-i)!} & \cdots & \frac{x_n^{l_n-n+1}}{(l_n-n+1)!} \end{vmatrix}, \tag{2.3}$$

where $\frac{1}{s!} = 0$ for $s < 0$. Besides, we define the discontinuous function $\chi_t(s)$ as follows:

$$\chi_t(s) = \begin{cases} 1, & t \le s, \\ 0, & t > s. \end{cases}$$

Then from (8) in [18] we obtain

$$\sum_{i=0}^{n} \Delta_i \left(f^{(k_i)}(x_i) - \int_a^b f^{(n)}(t) \cdot \chi_t(x_i) \cdot \frac{(x_i - t)^{l_i}}{l_i!} \, dt \right) = 0. \tag{2.4}$$

In particular, if $f^{(k_i)}(x_i) = 0$ for $1 \le i \le n$ and $\Delta_0 \neq 0$, $x_0 = x$, $k_0 = 0$, then (2.4) turns into

$$f(x) = \int_a^b f^{(n)}(t) B(x, t) \, dt, \tag{2.5}$$

where

$$B(x,t) = \sum_{i=0}^{n} \chi_t(x_i) \cdot \frac{(x_i - t)^{l_i}}{l_i!} \frac{\Delta_i}{\Delta_0}.$$

Combining (2.3) with the last equality, we obtain

$$B(x,t) = \Delta_0^{-1} \begin{vmatrix} \chi_t(x) \cdot \frac{(x-t)^{n-1}}{(n-1)!} & \frac{x^{n-1}}{(n-1)!} & \cdots & \frac{x^{n-i}}{(n-i)!} & \cdots & 1 \\ \chi_t(x_1) \cdot \frac{(x_1-t)^{l_1}}{l_1!} & \frac{x_1^{l_1}}{l_1!} & \cdots & \frac{x_1^{l_1-i}}{(l_1-i)!} & \cdots & \frac{x_1^{l_1-n+1}}{(l_1-n+1)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_t(x_i) \cdot \frac{(x_i-t)^{l_i}}{l_i!} & \frac{x_i^{l_i}}{l_i!} & \cdots & \frac{x_i^{l_i-i}}{(l_i-i)!} & \cdots & \frac{x_i^{l_i-n+1}}{(l_i-n+1)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_t(x_n) \cdot \frac{(x_n-t)^{l_n}}{l_n!} & \frac{x_n^{l_n!}}{l_n!} & \cdots & \frac{x_n^{l_n-i}}{(l_n-i)!} & \cdots & \frac{x_n^{l_n-n+1}}{(l_n-n+1)!} \end{vmatrix}. \tag{2.6}$$

In particular, for r pairs of numbers $(0,k_1),(0,k_2),\ldots,(0,k_s),(1,m_{s+1}),\ldots,(1,m_r)$, we obtain the problem presented in the introduction (here $a = 0$ and $b = 1$). For simplicity, we represent these r pairs of numbers as $\{k_1,\ldots,k_s\} \cup \{m_{s+1},\ldots,m_r\}$.

Now we introduce some information about the norms of the integral operators. Let $K(x,t)$ be a piecewise continuous function on $[0,1] \times [0,1]$. We define

$$S(f,x) = \int_0^1 K(x,t)f(t)\,dt. \tag{2.7}$$

It is known that S is a linear continuous operator from $L_q[0,1]$ to $L_p[0,1]$ for all $1 \le p,q \le \infty$. Let $\|S\|_{p,q}$ be the operator norm of S from $L_q[0,1]$ to $L_p[0,1]$. It is known that

$$\|S\|_{1,1} = \sup_{f\in L_1[0,1], f\neq 0} \frac{\|Sf\|_1}{\|f\|_1} = \sup_{0\le t\le 1} \int_0^1 \bigl|K(x,t)\bigr|\,dx, \tag{2.8}$$

$$\|S\|_{\infty,\infty} = \sup_{f\in L_\infty[0,1], f\neq 0} \frac{\|Sf\|_\infty}{\|f\|_\infty} = \sup_{0\le x\le 1} \int_0^1 \bigl|K(x,t)\bigr|\,dt. \tag{2.9}$$

Besides, let S^* be the dual operator of S, and let

$$W(f,x) := S^*S(f,x) = \int_0^1 K^*(x,t)f(t)\,dt, \tag{2.10}$$

where

$$K^*(x,t) = \int_0^1 K(\tau,x)K(\tau,t)\,d\tau.$$

Then W is a Hilbert–Schmidt operator. Let $\{(\lambda_j,e_j)\}_{j\in\mathbb{N}}$ be the sequence of eigenpairs of W with nonincreasing eigenvalues, that is, $\lambda_1 \ge \lambda_2 \ge \cdots$ and $W(e_j) = \lambda_j e_j$. Then (see [16])

$$\|S\|_{2,2} = \sup_{f\in L_2[0,1], f\neq 0} \frac{\|Sf\|_2}{\|f\|_2} = \sqrt{\lambda_1}. \tag{2.11}$$

In this paper, we obtain the following results.

Theorem 2.1 *Let* $\{k_1,\dots,k_s\} \cup \{m_{s+1},\dots,m_r\}$ *satisfy* (1.5). *Then for an arbitrary* $f \in W_q^r[0,1]$ *with* $f^{(k_i)}(0) = 0$ *for* $1 \le i \le s$ *and* $f^{(m_j)}(1) = 0$ *for* $s+1 \le j \le r$, *we have the sharp inequality*

$$\|f\|_p \le C(p,q)\big\|f^{(r)}\big\|_q \quad \textit{for all } 1 \le p,q \le \infty, \tag{2.12}$$

where $C(p,q)$ *is the norm of the operator* (*with* $B(x,t)$ *given by* (2.6))

$$T(g,x) = \int_0^1 B(x,t)g(t)\,dt \quad \textit{for } x \in [0,1] \tag{2.13}$$

from $L_q[0,1]$ *to* $L_p[0,1]$, *that is*, $C(p,q) = \|T\|_{p,q}$ *depends on* $\{k_1,\dots,k_s\} \cup \{m_{s+1},\dots,m_r\}$. *Furthermore, the following relations hold*:

$$C(1,1) = \sup_{0\le t\le 1}\int_0^1 \big|B(x,t)\big|\,dx, \tag{2.14}$$

$$C(\infty,\infty) = \sup_{0\le x\le 1}\int_0^1 \big|B(x,t)\big|\,dt, \tag{2.15}$$

$$C(2,2) = \sqrt{\lambda_1}, \tag{2.16}$$

where λ_1 *is the maximal eigenvalue of the operator*

$$W(g,x) = \int_0^1 K^*(x,t)g(t)\,dt \quad \textit{for all } x \in [0,1] \tag{2.17}$$

with kernel

$$K^*(x,t) = \int_0^1 B(\tau,x)B(\tau,t)\,d\tau. \tag{2.18}$$

Proof If $f^{(k_i)}(0) = 0$ for $1 \le i \le s$ and $f^{(m_j)}(1) = 0$ for $s+1 \le j \le r$, then it follows from (2.5) with $n = r$ that

$$f(x) = \int_0^1 f^{(r)}(t)B(x,t)\,dt = T\big(f^{(r)},x\big). \tag{2.19}$$

From (2.19) and (2.13) we conclude that

$$\|f\|_p = \big\|Tf^{(r)}\big\|_p \le C(p,q)\big\|f^{(r)}\big\|_q. \tag{2.20}$$

On the other hand, for any $g \in L_q[0,1]$, let

$$\bar{f}(x) = \frac{1}{(r-1)!}\int_0^x (x-t)^{r-1}g(t)\,dt,$$

and let

$$f(x) = \bar{f}(x) - L_r(\bar{f},x),$$

where L_r is the Birkhoff interpolation based on $\{k_1,\dots,k_s\}\cup\{m_{s+1},\dots,m_r\}$. It is known that $L_r(\bar{f},x)$ is an algebraic polynomial of degree at most $n=r-1$. Then we easily check that $f^{(r)}=g$ and $f^{(k_i)}(0)=0$ for $1\le i\le s$ and $f^{(m_j)}(1)=0$ for $s+1\le j\le r$. Hence (2.19) becomes

$$f(x)=T\bigl(f^{(r)},x\bigr)=T(g,x). \tag{2.21}$$

From (2.21) we obtain

$$\sup_{f\in W_q^r[0,1],f^{(k_i)}(0)=f^{(m_j)}(1)=0,1\le i\le s,s+1\le j\le r}\frac{\|f\|_p}{\|f^{(r)}\|_q}\ge\sup_{g\in L_q[0,1],g\ne 0}\frac{\|Tg\|_p}{\|g\|_q}=C(p,q). \tag{2.22}$$

From (2.20) and (2.22) we obtain (2.12). Besides, from (2.13) and (2.8), (2.9), (2.11) we obtain (2.14)–(2.16), respectively. This completes the proof of Theorem 2.1. □

For $f\in W_q^r[a,b]$ with $f^{(k_i)}(a)=0$ for $1\le i\le s$ and $f^{(m_j)}(b)=0$ for $s+1\le j\le r$, letting $g(t)=f(a+(b-a)t)$, we obtain the following result.

Corollary 2.2 *Let* $\{k_1,\dots,k_s\}\cup\{m_{s+1},\dots,m_r\}$ *satisfy* (1.5). *Then for an arbitrary* $f\in W_q^r[a,b]$ *with* $f^{(k_i)}(a)=0$ *for* $1\le i\le s$ *and* $f^{(m_j)}(b)=0$ *for* $s+1\le j\le r$, *we have the sharp inequality*

$$\|f\|_p\le C(p,q)(b-a)^{r+1/p-1/q}\bigl\|f^{(r)}\bigr\|_q\quad\textit{for all } 1\le p,q\le\infty. \tag{2.23}$$

Remark 2.3 It is obvious that if $k_i=i-1$ for $1\le i\le s$ and $m_j=j-s-1$ for $s+1\le j\le r$, then Corollary 2.2 is the case (1.3) for $k=s$. In this case, we give the best constant $C(r,k,p,q)$ in (1.3) by the corresponding $C(p,q)$.

3 Some examples

In this section, we give three examples showing how to compute the values of $C(1,1)$, $C(\infty,\infty)$, and $C(2,2)$. It is obvious that these three examples are not in the case of (1.3).

Example 1 For $\{0\}\cup\{1\}$, we have $C(1,1)=\frac{1}{2}$, $C(\infty,\infty)=\frac{1}{2}$, and $C(2,2)=\frac{4}{\pi^2}$.

Proof Since $x_1=0$, $x_2=1$, $k_1=0$, and $k_2=1$, it follows from (2.6) that (2.13) holds with

$$B(x,t)=\begin{vmatrix}0&1\\1&0\end{vmatrix}^{-1}\begin{vmatrix}\chi_t(x)(x-t)&x&1\\0&0&1\\1&1&0\end{vmatrix}=-\min\{x,t\}. \tag{3.1}$$

We first consider $C(1,1)$. From (2.14) and (3.1) by a direct computation we obtain

$$C(1,1)=\|T\|_{1,1}=\sup_{0\le t\le 1}\int_0^1\bigl|B(x,t)\bigr|\,dx=\sup_{0\le t\le 1}\left(-\frac{t^2}{2}+t\right)=\frac{1}{2}. \tag{3.2}$$

Now we consider $C(\infty,\infty)$. From (2.15) and (3.1) it follows that

$$C(\infty,\infty)=\|T\|_{\infty,\infty}=\sup_{0\le x\le 1}\int_0^1\bigl|B(x,t)\bigr|\,dt=\sup_{0\le x\le 1}\left(-\frac{x^2}{2}+x\right)=\frac{1}{2}. \tag{3.3}$$

Finally, we consider $C(2,2)$. Since $\min\{x,t\}$ is a reproducing kernel, it follows from the computation of [19, p. 55] that

$$C(2,2) = \|T\|_{2,2} = \frac{4}{\pi^2}.$$

This completes the proof of Example 1. □

Example 2 For $\{0,2\} \cup \{1\}$, we have $C(1,1) = \frac{1}{3}$, $C(\infty,\infty) = \frac{1}{3}$, and $C(2,2) = \frac{8}{\pi^3}$.

Proof Since $x_1 = x_2 = 0$, $x_3 = 1$, $k_1 = 0$, $k_2 = 2$, and $k_3 = 1$, it follows from (2.6) that (2.13) holds with

$$B(x,t) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}^{-1} \begin{vmatrix} \chi_t(x) \cdot \frac{(x-t)^2}{2!} & \frac{x^2}{2!} & x & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1-t & 1 & 1 & 0 \end{vmatrix} = \begin{cases} \frac{(x-t)^2}{2!} - x(1-t), & t \le x, \\ -x(1-t), & x < t. \end{cases} \tag{3.4}$$

We first consider $C(1,1)$. From (2.14) and (3.4) by a direct computation it follows that

$$C(1,1) = \sup_{0\le t\le 1} \int_0^1 |B(x,t)|\,dx = \sup_{0\le t\le 1} \left(\frac{t^3}{6} - \frac{t^2}{2} + \frac{1}{3}\right) = \frac{1}{3}. \tag{3.5}$$

Now we consider $C(\infty,\infty)$. From (2.15) and (3.4) by a direct computation it follows that

$$C(\infty,\infty) = \sup_{0\le x\le 1} \int_0^1 |B(x,t)|\,dt = \sup_{0\le x\le 1} \left(-\frac{x^3}{6} + \frac{x}{2}\right) = \frac{1}{3}. \tag{3.6}$$

Finally, we consider $C(2,2)$. We will use (2.16). From (3.4) and (2.18) by a direct computation we obtain

$$\begin{aligned} K^*(x,t) &= \int_0^1 B(\tau,t) \cdot B(\tau,x)\,d\tau \\ &= \frac{2}{15} - \frac{x^2}{6} - \frac{t^2}{6} + \frac{x^4}{24} + \frac{t^4}{24} + \frac{x^2t^2}{4} \\ &\quad - \frac{x^2t^2 \cdot \max\{x,t\}}{12} - \frac{xt \cdot \min\{x^3,t^3\}}{24} - \frac{\max\{x^5,t^5\}}{120}. \end{aligned} \tag{3.7}$$

Let g be an eigenvector corresponding to a positive eigenvalue λ of the operator W given by (2.17), that is, $\lambda g(x) = W(g,x)$. Then from (3.7) and the relations $\max\{a,b\} = \frac{a+b+|a-b|}{2}$ and $\min\{a,b\} = \frac{a+b-|a-b|}{2}$ by a direct computation it follows that

$$\begin{aligned} \lambda g(x) &= \int_0^x (x-t)\left(-\frac{x^2t^2}{40} + \frac{x^3t}{60} + \frac{xt^3}{60} - \frac{x^4}{240} - \frac{t^4}{240}\right)g(t)\,dt \\ &\quad + \int_x^1 (t-x)\left(-\frac{x^2t^2}{40} + \frac{x^3t}{60} + \frac{xt^3}{60} - \frac{x^4}{240} - \frac{t^4}{240}\right)g(t)\,dt \\ &\quad - x^5 \int_0^1 \frac{1}{240} g(t)\,dt + x^4 \int_0^1 \left(-\frac{t}{48} + \frac{1}{24}\right)g(t)\,dt - x^3 \int_0^1 \frac{t^2}{24} g(t)\,dt \end{aligned}$$

$$+x^2\int_0^1\left(-\frac{t^3}{24}+\frac{t^2}{4}-\frac{1}{6}\right)g(t)\,dt-x\int_0^1\frac{t^4}{48}g(t)\,dt$$
$$+\int_0^1\left(-\frac{t^5}{240}+\frac{t^4}{24}-\frac{t^2}{6}+\frac{2}{15}\right)g(t)\,dt. \tag{3.8}$$

Taking $x=1$ in (3.8), we obtain

$$g(1)=0. \tag{3.9}$$

Differentiating both sides of (3.8), we obtain

$$\begin{aligned}\lambda g'(x)={}&\int_0^x\left[\left(\frac{x^3t+xt^3}{60}-\frac{x^2t^2}{40}-\frac{x^4+t^4}{240}\right)+(x-t)\left(\frac{t^3}{60}-\frac{x^3}{60}-\frac{xt^2}{20}+\frac{x^2t}{20}\right)\right]g(t)\,dt\\&+\int_x^1\left[-\left(\frac{x^3t+xt^3}{60}-\frac{x^2t^2}{40}-\frac{x^4+t^4}{240}\right)\right.\\&\left.+(t-x)\left(-\frac{xt^2}{20}+\frac{x^2t}{20}+\frac{t^3}{60}-\frac{x^3}{60}\right)\right]g(t)\,dt\\&-x^4\int_0^1\frac{1}{48}g(t)\,dt+x^3\int_0^1\left(-\frac{t}{12}+\frac{1}{6}\right)g(t)\,dt-x^2\int_0^1\frac{t^2}{8}g(t)\,dt\\&+x\int_0^1\left(-\frac{t^3}{12}+\frac{t^2}{2}-\frac{1}{3}\right)g(t)\,dt-\int_0^1\frac{t^4}{48}g(t)\,dt.\end{aligned} \tag{3.10}$$

Let $x=0$ in (3.10). Then we obtain

$$g'(0)=0. \tag{3.11}$$

Differentiating both sides of (3.10), we obtain

$$\begin{aligned}\lambda g''(x)={}&\int_0^x\left[\left(-\frac{xt^2}{10}+\frac{x^2t}{10}+\frac{t^3}{30}-\frac{x^3}{30}\right)+(x-t)\left(-\frac{t^2}{20}+\frac{xt}{10}-\frac{x^2}{20}\right)\right]g(t)\,dt\\&+\int_x^1\left[-\left(-\frac{xt^2}{10}+\frac{x^2t}{10}+\frac{t^3}{30}-\frac{x^3}{30}\right)+(t-x)\left(-\frac{t^2}{20}+\frac{xt}{10}-\frac{x^2}{20}\right)\right]g(t)\,dt\\&-x^3\int_0^1\frac{1}{12}g(t)\,dt+x^2\int_0^1\left(-\frac{t}{4}+\frac{1}{2}\right)g(t)\,dt\\&-x\int_0^1\frac{t^2}{4}g(t)\,dt+\int_0^1\left(-\frac{t^3}{12}+\frac{t^2}{2}-\frac{1}{3}\right)g(t)\,dt.\end{aligned} \tag{3.12}$$

Let $x=1$ in (3.12). Then we obtain

$$g''(1)=0. \tag{3.13}$$

Differentiating both sides of (3.12), we obtain

$$\begin{aligned}\lambda g^{(3)}(x) &= \int_0^x \left[\left(\frac{t}{10}-\frac{x}{10}\right)(x-t)+\left(-\frac{3t^2}{20}+\frac{3xt}{10}-\frac{3x^2}{20}\right)\right]g(t)\,dt \\ &\quad+\int_x^1 \left[\left(\frac{t}{10}-\frac{x}{10}\right)(t-x)-\left(-\frac{3t^2}{20}+\frac{3xt}{10}-\frac{3x^2}{20}\right)\right]g(t)\,dt \\ &\quad-x^2\int_0^1 \frac{1}{4}g(t)\,dt+x\int_0^1\left(-\frac{t}{2}+1\right)g(t)\,dt-\int_0^1\frac{t^2}{4}g(t)\,dt. \end{aligned} \tag{3.14}$$

Let $x = 0$ in (3.14). Then we obtain

$$g^{(3)}(0) = 0. \tag{3.15}$$

Differentiating both sides of (3.14), we obtain

$$\lambda g^{(4)}(x) = \int_0^x \frac{t-x}{2}g(t)\,dt+\int_x^1\frac{x-t}{2}g(t)\,dt-\frac{x}{2}\int_0^1 g(t)\,dt+\int_0^1\left(-\frac{t}{2}+1\right)g(t)\,dt. \tag{3.16}$$

Let $x = 1$ in (3.16). Then we obtain

$$g^{(4)}(1) = 0. \tag{3.17}$$

Differentiating both sides of (3.16), we obtain

$$\lambda g^{(5)}(x) = -\int_0^x \frac{1}{2}g(t)\,dt+\int_x^1\frac{1}{2}g(t)\,dt-\int_0^1\frac{1}{2}g(t)\,dt. \tag{3.18}$$

Let $x = 0$ in (3.18). Then we obtain

$$g^{(5)}(0) = 0. \tag{3.19}$$

Differentiating both sides of (3.18), we obtain

$$\lambda g^{(6)}(x) = -g(x). \tag{3.20}$$

Let $\mu = \frac{1}{\sqrt[6]{\lambda}}$. Then the general solution of equation (3.20) is

$$\begin{aligned} g(x) &= C_1\cos\mu x + C_2\sin\mu x \\ &\quad+e^{\frac{\sqrt{3}}{2}\mu x}\left(C_3\cos\frac{\mu x}{2}+C_4\sin\frac{\mu x}{2}\right)+e^{-\frac{\sqrt{3}}{2}\mu x}\left(C_5\cos\frac{\mu x}{2}+C_6\sin\frac{\mu x}{2}\right). \end{aligned} \tag{3.21}$$

From (3.21) it follows that

$$\begin{aligned} g'(x) &= \mu(C_2\cos\mu x - C_1\sin\mu x) \\ &\quad+\mu e^{\frac{\sqrt{3}}{2}\mu x}\left(\left(\frac{\sqrt{3}C_3}{2}+\frac{C_4}{2}\right)\cos\frac{\mu x}{2}+\left(\frac{\sqrt{3}C_4}{2}-\frac{C_3}{2}\right)\sin\frac{\mu x}{2}\right) \\ &\quad+\mu e^{-\frac{\sqrt{3}}{2}\mu x}\left(\left(-\frac{\sqrt{3}C_5}{2}+\frac{C_6}{2}\right)\cos\frac{\mu x}{2}+\left(-\frac{\sqrt{3}C_6}{2}-\frac{C_5}{2}\right)\sin\frac{\mu x}{2}\right), \end{aligned} \tag{3.22}$$

$$
\begin{aligned}
g''(x) &= -\mu^2(C_1\cos\mu x + C_2\sin\mu x) \\
&\quad + \mu^2 e^{\frac{\sqrt{3}}{2}\mu x}\left(\left(\frac{\sqrt{3}C_4}{2} + \frac{C_3}{2}\right)\cos\frac{\mu x}{2} + \left(\frac{C_4}{2} - \frac{\sqrt{3}C_3}{2}\right)\sin\frac{\mu x}{2}\right) \\
&\quad + \mu^2 e^{-\frac{\sqrt{3}}{2}\mu x}\left(\left(\frac{C_5}{2} - \frac{\sqrt{3}C_6}{2}\right)\cos\frac{\mu x}{2} + \left(\frac{C_6}{2} + \frac{\sqrt{3}C_5}{2}\right)\sin\frac{\mu x}{2}\right), \qquad (3.23)
\end{aligned}
$$

$$
\begin{aligned}
g^{(3)}(x) &= -\mu^3(C_2\cos\mu x - C_1\sin\mu x) \\
&\quad + \mu^3 e^{\frac{\sqrt{3}}{2}\mu x}\left(C_4\cos\frac{\mu x}{2} - C_3\sin\frac{\mu x}{2}\right) \\
&\quad + \mu^3 e^{-\frac{\sqrt{3}}{2}\mu x}\left(C_6\cos\frac{\mu x}{2} - C_5\sin\frac{\mu x}{2}\right), \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
g^{(4)}(x) &= \mu^4(C_1\cos\mu x + C_2\sin\mu x) \\
&\quad + \mu^4 e^{\frac{\sqrt{3}}{2}\mu x}\left(\left(\frac{\sqrt{3}C_4}{2} - \frac{C_3}{2}\right)\cos\frac{\mu x}{2} + \left(-\frac{C_4}{2} - \frac{\sqrt{3}C_3}{2}\right)\sin\frac{\mu x}{2}\right) \\
&\quad + \mu^4 e^{-\frac{\sqrt{3}}{2}\mu x}\left(\left(-\frac{C_5}{2} - \frac{\sqrt{3}C_6}{2}\right)\cos\frac{\mu x}{2} + \left(-\frac{C_6}{2} + \frac{\sqrt{3}C_5}{2}\right)\sin\frac{\mu x}{2}\right), \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
g^{(5)}(x) &= \mu^5(C_2\cos\mu x - C_1\sin\mu x) \\
&\quad + \mu^5 e^{\frac{\sqrt{3}}{2}\mu x}\left(\left(\frac{C_4}{2} - \frac{\sqrt{3}C_3}{2}\right)\cos\frac{\mu x}{2} + \left(-\frac{\sqrt{3}C_4}{2} - \frac{C_3}{2}\right)\sin\frac{\mu x}{2}\right) \\
&\quad + \mu^5 e^{-\frac{\sqrt{3}}{2}\mu x}\left(\left(\frac{\sqrt{3}C_5}{2} + \frac{C_6}{2}\right)\cos\frac{\mu x}{2} + \left(\frac{\sqrt{3}C_6}{2} - \frac{C_5}{2}\right)\sin\frac{\mu x}{2}\right). \qquad (3.26)
\end{aligned}
$$

Substituting (3.9), (3.11), (3.13), (3.15), (3.17), and (3.19) into (3.21)–(3.26), respectively, by a simplification we obtain the following linear equations in the six unknown numbers C_j, $1 \le j \le 6$:

$$
\begin{aligned}
&C_1\cos\mu + C_2\sin\mu \\
&\quad = -e^{\frac{\sqrt{3}}{2}\mu}\left(C_3\cos\frac{\mu}{2} + C_4\sin\frac{\mu}{2}\right) - e^{-\frac{\sqrt{3}}{2}\mu}\left(C_5\cos\frac{\mu}{2} + C_6\sin\frac{\mu}{2}\right), \qquad (3.27)
\end{aligned}
$$

$$
2C_2 + \sqrt{3}C_3 + C_4 - \sqrt{3}C_5 + C_6 = 0, \qquad (3.28)
$$

$$
\begin{aligned}
&C_1\cos\mu + C_2\sin\mu \\
&\quad = e^{\frac{\sqrt{3}}{2}\mu}\left(\left(\frac{\sqrt{3}C_4}{2} + \frac{C_3}{2}\right)\cos\frac{\mu}{2} + \left(\frac{C_4}{2} - \frac{\sqrt{3}C_3}{2}\right)\sin\frac{\mu}{2}\right) \\
&\qquad + e^{-\frac{\sqrt{3}}{2}\mu}\left(\left(\frac{C_5}{2} - \frac{\sqrt{3}C_6}{2}\right)\cos\frac{\mu}{2} + \left(\frac{C_6}{2} + \frac{\sqrt{3}C_5}{2}\right)\sin\frac{\mu}{2}\right), \qquad (3.29)
\end{aligned}
$$

$$
C_2 - C_4 - C_6 = 0. \qquad (3.30)
$$

$$
\begin{aligned}
&C_1\cos\mu + C_2\sin\mu \\
&\quad = -e^{\frac{\sqrt{3}}{2}\mu}\left(\left(\frac{\sqrt{3}C_4}{2} - \frac{C_3}{2}\right)\cos\frac{\mu}{2} + \left(-\frac{C_4}{2} - \frac{\sqrt{3}C_3}{2}\right)\sin\frac{\mu}{2}\right) \\
&\qquad - e^{-\frac{\sqrt{3}}{2}\mu}\left(\left(-\frac{C_5}{2} - \frac{\sqrt{3}C_6}{2}\right)\cos\frac{\mu}{2} + \left(\frac{\sqrt{3}C_5}{2} - \frac{C_6}{2}\right)\sin\frac{\mu}{2}\right), \qquad (3.31)
\end{aligned}
$$

$$-2C_2 + \sqrt{3}C_3 - C_4 - \sqrt{3}C_5 - C_6 = 0. \tag{3.32}$$

Subtracting two sides of equation (3.31) from two sides of equation (3.29), by a simple simplification we obtain

$$C_3 e^{\frac{\sqrt{3}}{2}\mu} \sin\frac{\mu}{2} - C_4 e^{\frac{\sqrt{3}}{2}\mu} \cos\frac{\mu}{2} - C_5 e^{-\frac{\sqrt{3}}{2}\mu} \sin\frac{\mu}{2} + C_6 e^{-\frac{\sqrt{3}}{2}\mu} \cos\frac{\mu}{2} = 0. \tag{3.33}$$

Subtracting two sides of equation (3.27) from two sides of equation (3.29) and adding $\frac{\sqrt{3}}{2}$ times two sides of equation (3.33), we obtain

$$C_3 e^{\frac{\sqrt{3}}{2}\mu} \cos\frac{\mu}{2} + C_4 e^{\frac{\sqrt{3}}{2}\mu} \sin\frac{\mu}{2} + C_5 e^{-\frac{\sqrt{3}}{2}\mu} \cos\frac{\mu}{2} + C_6 e^{-\frac{\sqrt{3}}{2}\mu} \sin\frac{\mu}{2} = 0. \tag{3.34}$$

Equations (3.27) and (3.34) imply that

$$C_1 \cos\mu + C_2 \sin\mu = 0. \tag{3.35}$$

Thus by (3.28), (3.30), (3.32), (3.33), (3.34), and (3.35), if λ is a positive eigenvalue of the operator W and g is a nonzero eigenfunction corresponding to λ, then $\mu = \lambda^{-\frac{1}{6}}$ is a positive zero of the function F defined by

$$\begin{aligned} F(t) &= \begin{vmatrix} 0 & 2 & \sqrt{3} & 1 & -\sqrt{3} & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & -2 & \sqrt{3} & -1 & -\sqrt{3} & -1 \\ 0 & 0 & e^{\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} & -e^{\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} & -e^{-\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} & e^{-\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} \\ 0 & 0 & e^{\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} & e^{\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} & e^{-\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} & e^{-\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} \\ \cos t & \sin t & 0 & 0 & 0 & 0 \end{vmatrix} \\ &= -6\sqrt{3}\cos t\left(\left(e^{\frac{\sqrt{3}}{2}t} - e^{-\frac{\sqrt{3}}{2}t}\right)^2 \sin^2\frac{t}{2} + \left(e^{\frac{\sqrt{3}}{2}t} + e^{-\frac{\sqrt{3}}{2}t}\right)^2 \cos^2\frac{t}{2}\right). \end{aligned} \tag{3.36}$$

Due to $(e^{\frac{\sqrt{3}}{2}t} - e^{-\frac{\sqrt{3}}{2}t})^2 \sin^2\frac{t}{2} + (e^{\frac{\sqrt{3}}{2}t} + e^{-\frac{\sqrt{3}}{2}t})^2 \cos^2\frac{t}{2} > 0$, by (3.36) we see that the set of all positive eigenvalues of W is a subset of the set

$$\left\{\left(\left(k - \frac{1}{2}\right)\pi\right)^{-6} : k \in \mathbb{N}\right\}, \tag{3.37}$$

which is the set of all positive roots of the equation $\cos t = 0$. Furthermore, it is easy to verify that $g_k(x) = \cos(k - \frac{1}{2})\pi x$ is an eigenvector corresponding to the eigenvalue $((k - \frac{1}{2})\pi)^{-6}$ for all $k \in \mathbb{N}$. Therefore the set of all positive eigenvalues of W is given by (3.37), and hence the maximal eigenvalue of W is $\lambda_1 = (\frac{\pi}{2})^{-6}$. Combining this fact with (2.16), we obtain that $C(2,2) = \frac{8}{\pi^3}$. □

Example 3 For $\{0,1\} \cup \{1\}$, we have $C(1,1) = \frac{\sqrt{3}}{27}$, $C(\infty,\infty) = \frac{1}{12}$, and

$$C(2,2) = \mu_1^{-3}, \tag{3.38}$$

where μ_1 is the minimal positive root of the equation

$$(3\sin t - \sqrt{3}\cos t)e^{-\sqrt{3}t} + 2\sqrt{3}\cos 2t - (3\sin t + \sqrt{3}\cos t)e^{\sqrt{3}t} = 0.$$

Proof Since $x_1 = x_2 = 0$, $x_3 = 1$, $k_1 = 0$, and $k_2 = k_3 = 1$, it follows from (2.6) that (2.13) holds with

$$B(x,t) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}^{-1} \begin{vmatrix} \chi_t(x)\cdot\frac{(x-t)^2}{2!} & \frac{x^2}{2!} & x & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1-t & 1 & 1 & 0 \end{vmatrix} = \begin{cases} \frac{(x-t)^2}{2!} - \frac{x^2(1-t)}{2}, & t \le x, \\ -\frac{x^2(1-t)}{2}, & x < t. \end{cases} \tag{3.39}$$

We first consider $C(1,1)$. From (2.14) and (3.39) by a direct computation it follows that

$$C(1,1) = \sup_{0\le t\le 1}\int_0^1 \left|B(x,t)\right| dx = \sup_{0\le t\le 1}\left(\frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{3}\right) = \frac{\sqrt{3}}{27}. \tag{3.40}$$

Now we consider $C(\infty,\infty)$. From (2.15) and (3.39) by a direct computation it follows that

$$C(\infty,\infty) = \sup_{0\le x\le 1}\int_0^1 \left|B(x,t)\right| dt = \sup_{0\le x\le 1}\left(\frac{x^2}{4} - \frac{x^3}{6}\right) = \frac{1}{12}. \tag{3.41}$$

Finally, we consider $C(2,2)$. We will use (2.16). From (3.39) and (2.18) by a direct computation we obtain

$$\begin{aligned} K^*(x,t) &= \int_0^1 B(\tau,t)\cdot B(\tau,x)\,d\tau \\ &= \frac{2xt}{15} - \frac{xt^2}{6} - \frac{x^2t}{6} + \frac{x^2t^2}{4} - \frac{x^5t}{120} - \frac{xt^5}{120} \\ &\quad - \frac{x^2t^2\max\{x,t\}}{12} + \frac{xt\max\{x^3,t^3\}}{24} + \frac{\min\{x^5,t^5\}}{120}. \end{aligned} \tag{3.42}$$

Let g be an eigenvector corresponding to a positive eigenvalue λ of the operator W given by (2.17). Then from (3.42) and the relations $\max\{a,b\} = \frac{a+b+|a-b|}{2}$ and $\min\{a,b\} = \frac{a+b-|a-b|}{2}$ by a direct computation it follows that

$$\begin{aligned} \lambda g(x) &= \int_0^x (x-t)\left(-\frac{x^2t^2}{40} + \frac{x^3t}{60} + \frac{xt^3}{60} - \frac{x^4}{240} - \frac{t^4}{240}\right)g(t)\,dt \\ &\quad + \int_x^1 (t-x)\left(-\frac{x^2t^2}{40} + \frac{x^3t}{60} + \frac{xt^3}{60} - \frac{x^4}{240} - \frac{t^4}{240}\right)g(t)\,dt \\ &\quad - x^5\int_0^1\left(\frac{t}{120} - \frac{1}{240}\right)g(t)\,dt + x^4\int_0^1 \frac{t}{48}g(t)\,dt - x^3\int_0^1 \frac{t^2}{24}g(t)\,dt \\ &\quad + x^2\int_0^1\left(-\frac{t^3}{24} + \frac{t^2}{4} - \frac{t}{6}\right)g(t)\,dt + x\int_0^1\left(-\frac{t^5}{120} + \frac{t^4}{48} - \frac{t^2}{6} + \frac{2t}{15}\right)g(t)\,dt \\ &\quad + \int_0^1 \frac{t^5}{240}g(t)\,dt. \end{aligned} \tag{3.43}$$

Taking $x = 0, 1$ in (3.43), we obtain

$$g(0) = g(1) = 0. \tag{3.44}$$

Differentiating twice both sides of (3.43), we obtain

$$\begin{aligned}
\lambda g''(x) &= \int_0^x \left[\left(-\frac{xt^2}{10} + \frac{x^2t}{10} + \frac{t^3}{30} - \frac{x^3}{30}\right) + (x-t)\left(-\frac{t^2}{20} + \frac{xt}{10} - \frac{x^2}{20}\right)\right] g(t)\,dt \\
&\quad + \int_x^1 \left[-\left(-\frac{xt^2}{10} + \frac{x^2t}{10} + \frac{t^3}{30} - \frac{x^3}{30}\right) + (t-x)\left(-\frac{t^2}{20} + \frac{xt}{10} - \frac{x^2}{20}\right)\right] g(t)\,dt \\
&\quad + x^3 \int_0^1 \left(-\frac{t}{6} + \frac{1}{12}\right) g(t)\,dt + x^2 \int_0^1 \frac{t}{4} g(t)\,dt - x \int_0^1 \frac{t^2}{4} g(t)\,dt \\
&\quad + \int_0^1 \left(-\frac{t^3}{12} + \frac{t^2}{2} - \frac{t}{3}\right) g(t)\,dt.
\end{aligned} \tag{3.45}$$

Let $x = 1$ in (3.45). Then we obtain

$$g''(1) = 0. \tag{3.46}$$

Differentiating both sides of (3.45), we obtain

$$\begin{aligned}
\lambda g^{(3)}(x) &= \int_0^x \left[\left(\frac{t}{10} - \frac{x}{10}\right)(x-t) + \left(-\frac{3t^2}{20} + \frac{3xt}{10} - \frac{3x^2}{20}\right)\right] g(t)\,dt \\
&\quad + \int_x^1 \left[\left(\frac{t}{10} - \frac{x}{10}\right)(t-x) - \left(-\frac{3t^2}{20} + \frac{3xt}{10} - \frac{3x^2}{20}\right)\right] g(t)\,dt \\
&\quad + x^2 \int_0^1 \left(-\frac{t}{2} + \frac{1}{4}\right) g(t)\,dt + x \int_0^1 \frac{t}{2} g(t)\,dt - \int_0^1 \frac{t^2}{4} g(t)\,dt.
\end{aligned} \tag{3.47}$$

Let $x = 0$ in (3.47). Then we obtain

$$g^{(3)}(0) = 0. \tag{3.48}$$

Differentiating both sides of (3.47), we obtain

$$\lambda g^{(4)}(x) = \int_0^x \frac{t-x}{2} g(t)\,dt + \int_x^1 \frac{x-t}{2} g(t)\,dt + x \int_0^1 \frac{1-2t}{2} g(t)\,dt + \int_0^1 \frac{t}{2} g(t)\,dt. \tag{3.49}$$

Let $x = 0, 1$ in (3.49). Then we obtain

$$g^{(4)}(0) = g^{(4)}(1) = 0. \tag{3.50}$$

Differentiating twice both sides of (3.49), we obtain (3.21). By (3.21) we obtain (3.22)–(3.26). Substituting (3.44), (3.46), (3.48), and (3.50) into (3.21), (3.23), (3.24), and (3.25), respectively, by a simple simplification we obtain the following linear equations in the six unknown numbers C_j, $1 \le j \le 6$:

$$C_1 = -C_3 - C_5, \tag{3.51}$$

$$\begin{aligned} &C_1\cos\mu + C_2\sin\mu \\ &\quad = -e^{\frac{\sqrt{3}}{2}\mu}\left(C_3\cos\frac{\mu}{2} + C_4\sin\frac{\mu}{2}\right) - e^{-\frac{\sqrt{3}}{2}\mu}\left(C_5\cos\frac{\mu}{2} + C_6\sin\frac{\mu}{2}\right), \end{aligned} \tag{3.52}$$

$$\begin{aligned} &C_1\cos\mu + C_2\sin\mu \\ &\quad = e^{\frac{\sqrt{3}}{2}\mu}\left(\left(\frac{\sqrt{3}C_4}{2} + \frac{C_3}{2}\right)\cos\frac{\mu}{2} + \left(\frac{C_4}{2} - \frac{\sqrt{3}C_3}{2}\right)\sin\frac{\mu}{2}\right) \\ &\qquad + e^{-\frac{\sqrt{3}}{2}\mu}\left(\left(\frac{C_5}{2} - \frac{\sqrt{3}C_6}{2}\right)\cos\frac{\mu}{2} + \left(\frac{C_6}{2} + \frac{\sqrt{3}C_5}{2}\right)\sin\frac{\mu}{2}\right), \end{aligned} \tag{3.53}$$

$$C_2 = C_4 + C_6, \tag{3.54}$$

$$C_1 - \frac{C_3}{2} + \frac{\sqrt{3}C_4}{2} - \frac{C_5}{2} - \frac{\sqrt{3}C_6}{2} = 0, \tag{3.55}$$

$$\begin{aligned} &C_1\cos\mu + C_2\sin\mu \\ &\quad = -e^{\frac{\sqrt{3}}{2}\mu}\left(\left(\frac{\sqrt{3}C_4}{2} - \frac{C_3}{2}\right)\cos\frac{\mu}{2} + \left(-\frac{C_4}{2} - \frac{\sqrt{3}C_3}{2}\right)\sin\frac{\mu}{2}\right) \\ &\qquad - e^{-\frac{\sqrt{3}}{2}\mu}\left(\left(-\frac{C_5}{2} - \frac{\sqrt{3}C_6}{2}\right)\cos\frac{\mu}{2} + \left(\frac{\sqrt{3}C_5}{2} - \frac{C_6}{2}\right)\sin\frac{\mu}{2}\right). \end{aligned} \tag{3.56}$$

Substituting (3.51) into (3.55), by a simple simplification we obtain

$$3C_3 - \sqrt{3}C_4 + 3C_5 + \sqrt{3}C_6 = 0. \tag{3.57}$$

Subtracting both sides of equation (3.56) from both sides of equation (3.53), by a simple simplification we obtain

$$C_3e^{\frac{\sqrt{3}}{2}\mu}\sin\frac{\mu}{2} - C_4e^{\frac{\sqrt{3}}{2}\mu}\cos\frac{\mu}{2} - C_5e^{-\frac{\sqrt{3}}{2}\mu}\sin\frac{\mu}{2} + C_6e^{-\frac{\sqrt{3}}{2}\mu}\cos\frac{\mu}{2} = 0. \tag{3.58}$$

Subtracting both sides of equation (3.52) from both sides of equation (3.53) and adding $\frac{\sqrt{3}}{2}$ times both sides of equation (3.58), we obtain

$$C_3e^{\frac{\sqrt{3}}{2}\mu}\cos\frac{\mu}{2} + C_4e^{\frac{\sqrt{3}}{2}\mu}\sin\frac{\mu}{2} + C_5e^{-\frac{\sqrt{3}}{2}\mu}\cos\frac{\mu}{2} + C_6e^{-\frac{\sqrt{3}}{2}\mu}\sin\frac{\mu}{2} = 0. \tag{3.59}$$

Equations (3.51), (3.52), and (3.59) imply

$$C_3\cos\mu - C_4\sin\mu + C_5\cos\mu - C_6\sin\mu = 0. \tag{3.60}$$

Thus by (3.51), (3.54), (3.57), (3.60), (3.58), and (3.59), if λ is a positive eigenvalue of the operator W and if g is a nonzero eigenfunction corresponding to λ, then $\mu = \lambda^{-1/6}$ is a positive zero of the function F_1 defined by

$$\begin{aligned} F_1(t) &= \begin{vmatrix} 3 & -\sqrt{3} & 3 & \sqrt{3} \\ \cos t & -\sin t & \cos t & -\sin t \\ e^{\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} & -e^{\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} & -e^{-\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} & e^{-\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} \\ e^{\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} & e^{\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} & e^{-\frac{\sqrt{3}}{2}t}\cos\frac{t}{2} & e^{-\frac{\sqrt{3}}{2}t}\sin\frac{t}{2} \end{vmatrix} \\ &= (3\sin t - \sqrt{3}\cos t)e^{-\sqrt{3}t} + 2\sqrt{3}\cos 2t - (3\sin t + \sqrt{3}\cos t)e^{\sqrt{3}t}. \end{aligned}$$

On the other hand, if $F_1(\mu) = 0$, then we can find $\overline{C}_3, \overline{C}_4, \overline{C}_5, \overline{C}_6$ such that (3.57)–(3.60) hold, and let $\overline{C}_1 = -\overline{C}_3 - \overline{C}_5$, $\overline{C}_2 = \overline{C}_4 + \overline{C}_6$. Then, instead of $C_1, C_2, C_3, C_4, C_5, C_6$ in (3.21), take $\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4, \overline{C}_5, \overline{C}_6$ such that the corresponding $g(x)$ satisfies (3.44), (3.46), (3.48), and (3.50). It is easy to verify that this $g(x)$ is an eigenfunction of W corresponding to the eigenvalue $\lambda = \mu^{-6}$. Therefore the set of all positive eigenvalues of W is

$$\left\{\lambda_k = \mu_k^{-6} : k \in \mathbb{N}\right\},$$

where $\{\mu_k\}$ is the sequence of all positive zeros of the function F_1, and $\mu_1 \le \mu_2 \le \cdots$. This shows that $\lambda_1 = \mu_1^{-6}$, and hence (2.16) implies (3.38). □

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