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The eigenvalue product bounds of the Lyapunov matrix differential equation and the stability of a class of time-varying nonlinear system

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Abstract

The Lyapunov matrix differential equation plays an important role in many scientific and engineering fields. In this paper, we first give a class relation between the eigenvalue of functional matrix derivative and the derivative of functional matrix eigenvalue. Using this relation, we convert the Lyapunov matrix differential equation into an eigenvalue differential equation. Further, by the Schur theorem, combining Hölder's integral inequality with arithmetic–geometric average inequality, we provide several lower and upper bounds on the eigenvalue product for the solution of the Lyapunov matrix differential equation. As an application in control and optimization, we show that our bounds can be used to discuss the stability of a class of time-varying nonlinear systems. Finally, we illustrate the superiority and effectiveness of the derived bounds by a numerical example.

Keywords: Lyapunov matrix differential equation; Majorization inequality; Eigenvalue; Bounds

1 Introduction

Consider the following Lyapunov matrix differential equation [1]:

$$\dot{P}(t) = A^H(t)P(t) + P(t)A(t) + Q(t), \qquad P(t_0) = P_0 = P_0^H \geq 0, \tag{1}$$

where $A(t) \in \mathbb{C}^{n\times n}$, $Q(t) = Q^H(t) \in \mathbb{C}^{n\times n}$, and $Q(t) \geq 0$ are continuous functions of t, and $P(t) \in \mathbb{C}^{n\times n}$ is the Hermitian positive semidefinite solution of (1).

It is well known that the linear matrix differential equation (1) has many express styles, and it usually can be found in time-varying nonlinear and linear systems. For one thing, consider the time-varying nonlinear system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \Delta f(x(t), \Phi(t)), \\ \dot{\Phi}(t) = -RB^T(t)P(t)x(t)r(t)^T, \end{cases} \tag{2}$$

where $x(t), r(t) \in \mathbb{C}^n$, and $P(t)$ is the positive definite solution of the Lyapunov matrix differential equation (1). Here $\Delta f(x(t), \Phi(t))$ is a nondeterministic term. In Sect. 5, we will show that the eigenvalue bounds of the Lyapunov matrix differential equation (1) can be used to discuss the stability of the time-varying nonlinear system (2).

For another, sometimes we need to consider the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{3}$$

where $A, B : [t_0, \infty) \to \mathbb{C}^{n\times n}$ are given time-varying matrices. Based on (3), we will also consider the optimal control problem

$$J(u) = \int_{t_0}^{t_f} \left[x^H(t)Q(t)x(t) + u^H(t)R(t)u(t)\right] dt + x^H(t)Sx(t), \tag{4}$$

where $Q(t) \in \mathbb{C}^{n\times n}$ is Hermitian positive semidefinite, and $R(t), S \in \mathbb{C}^{n\times n}$ are Hermitian positive definite.

Kucera [2] showed that the optimal value of (3) can be obtained by taking $u(t)$ of thye form

$$u(t) = -B^H(t)P(t)x(t), \quad t \in [t_0, t_f],$$

where $P(t)$ is the unique Hermitian positive definite solution of the following Riccati matrix differential equation:

$$\dot{P}(t) = A^H(t)P(t) + P(t)A(t) + Q(t) + P(t)B(t)R^{-1}(t)B^H(t)P(t) \tag{5}$$

with terminal condition $P(t_0) = S$.

Observe that when $B(t) = 0$ and $P(t_0) = P_0$, (4) reduces to (1).

In addition, (1) is also expressed in the following form [3]:

$$\dot{P}(t) = A^H P(t) + P(t)A + Q, \tag{6}$$

where $A, Q \in \mathbb{C}^{n\times n}$, $Q \geq 0$. Obviously, (1) is more general than (6).

Considering their applications, the authors of [1, 3–12] have paid their attention to the bounds of the solution for the Lyapunov and Riccati matrix differential equations. These include eigenvalue bounds (Mori et al. [3], Hmamed [4], Dai and Bai [5], Zhang and Liu [6], Liu et al. [7]), trace bounds (Liu and He [8], Zhu and Pagilla [9]), norm bounds (Patel and Toda [10]), and matrix bounds (Jodar and Ponsoda [1], Lee [11], Chen et al. [12]).

The paper is organized as follows. In Sect. 2, we show some notation and lemmas. In Sect. 3, we derive lower bounds on the eigenvalue product for (1). In Sect. 4, we present upper bounds on the eigenvalue product for (1). In Sect. 5, we show that our bounds can be used to discuss the stability of a class of time-varying nonlinear systems. In Sect. 6, we give a corresponding numerical example to show the superiority and effectiveness of the derived bounds.

2 Notation and lemmas

Throughout this paper, let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex numbers. For $a \in \mathbb{C}$, $\mathrm{Re}(a)$ is the real part of a. Let $x = (x_1, x_2, \ldots, x_n)$ be a real n-element array reordered in nonincreasing order, that is, $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. For

$X \in \mathbb{C}^{n\times n}$, we denote by X^H, X^{-1}, $\operatorname{tr}(X)$, and $\det(X)$ the conjugate transpose, the inverse, the trace, and the determinant of X, respectively. Further, suppose the diagonal elements and eigenvalues of X are $d(X) = (d_1(X), d_2(X), \ldots, d_n(X))$ and $\lambda(X) = (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))$, respectively. Assume that $\operatorname{Re}(d_{[1]}(X)) \geq \operatorname{Re}(d_{[2]}(X)] \geq \cdots \geq \operatorname{Re}(d_{[n]}(X))$, $\operatorname{Re}(\lambda_{[1]}(X)) \geq \operatorname{Re}(\lambda_{[2]}(X)) \geq \cdots \geq \operatorname{Re}(\lambda_{[n]}(X))$. The inequality $X > (\geq)\ 0$ means that X is a Hermitian positive (nonnegative) matrix. The identity matrix of appropriate dimensions is denoted I.

Let k be an integer with $1 \leq k \leq n$. We denote by $Q_{k,n}$ the set of all increasing sequences, that is,

$$Q_{k,n} = \bigl\{\omega = \{\omega_1, \ldots, \omega_k\} : 1 \leq \omega_1 < \cdots < \omega_k \leq n\bigr\}.$$

Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, and let k and r be integers satisfying $1 \leq k \leq m$ and $1 \leq r \leq n$, respectively. For $\alpha \in Q_{k,m}$ and $\beta \in Q_{r,n}$, we denote by $A[\alpha, \beta]$ the $k \times r$ matrix whose (i,j)th entry is a_{α_i, β_j}. If α is equal to β, then we simplify the notation to $A[\alpha]$.

Definition 2.1 ([13, p. 502]) Let $A \in \mathbb{C}^{n\times n}$ and $1 \leq r \leq n$. The rth compound matrix $C_r(A)$ of A is the matrix with the (j_α, j_β)th entry $\det A[\alpha, \beta]$, where $\alpha \in Q_{r,m}$ and $\beta \in Q_{r,n}$ are ordered lexicographically; $C_r(A)$ is a C_n^rth matrix ($C_n^r = \frac{n!}{r!(n-r)!}$).

Definition 2.2 ([13, p. 10]) Let x, y be two real n-element arrays. If they satisfy

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n,$$

then we say that x is weakly majorized by y and denote this by $x \prec_w y$.

Definition 2.3 ([13, p. 10]) If they satisfy

$$\sum_{i=1}^{k} x_{[n-i+1]} \geq \sum_{i=1}^{k} y_{[n-i+1]}, \quad k = 1, 2, \ldots, n,$$

then we say that x is weakly submajorized by y and denote this by $x \prec^w y$.

Definition 2.4 ([13, p. 11]) If $x \prec_w y$ and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

then we say that x is majorized by y and denote this by $x \prec y$.

Lemma 2.1 ([14, p. 69]) *If $X = X^H \in \mathbb{C}^{n\times n}$, then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$X = U^H \operatorname{diag}\bigl[\lambda_1(X), \ldots, \lambda_n(X)\bigr] U.$$

Lemma 2.2 ([15, p. 490]) *If $X(t), Y(t) \in \mathbb{C}^{n\times n}$, then*

$$\frac{d[X(t)Y(t)]}{dt} = \frac{d[X(t)]}{dt} Y(t) + \frac{d[Y(t)]}{dt} X(t).$$

Lemma 2.3 ([16]) *If* $(x_1, x_2, \ldots, x_k) > 0$, $(y_1, y_2, \ldots, y_k) > 0$, *then for any* $k = 1, \ldots, n$,

$$\left[\prod_{j=1}^{k}(x_j + y_j)\right]^{\frac{1}{k}} \geq \left(\prod_{j=1}^{k} x_j\right)^{\frac{1}{k}} + \left(\prod_{j=1}^{k} y_j\right)^{\frac{1}{k}}.$$

Lemma 2.4 ([17]) *If* $f_1, f_2, \ldots, f_n$ *are real positive continuous functions on* $[a, b]$ *and* $r_1, r_2, \ldots, r_n$ *are real positive numbers with* $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, *then we get the following Hölder integral inequality*:

$$\int_a^b \prod_{i=1}^{n} f_i(x)\,dx \leq \prod_{i=1}^{n}\left(\int_a^b f_i^{r_i}(x)\,dx\right)^{\frac{1}{r_i}},$$

where the equality holds if and only if $f_1^{r_1}, f_2^{r_2}, \ldots, f_n^{r_n}$ *are effectively proportional.*

Lemma 2.5 ([13, p. 300, B.1]) *Let* $X = X^H \in \mathbb{C}^{n\times n}$. *Then*

$$d(X) \prec \lambda(X),$$

that is,

$$d(X) \prec_w \lambda(X), \quad \textit{and} \quad d(X) \prec^w \lambda(X).$$

Lemma 2.6 ([13, p. 92, C.1]) *If* $x \prec y$, *then*

$$\left(e^{x_1}, e^{x_2}, \ldots, e^{x_n}\right) \prec \left(e^{y_1}, e^{y_2}, \ldots, e^{y_n}\right).$$

Lemma 2.7 ([13, p. 166, A.1.d]) *If* $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]} > 0$, $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]} > 0$, *and* $x \prec y$, *then*

$$\prod_{i=1}^{k} x_{[i]} \geq \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n.$$

Lemma 2.8 ([13, p. 208, A.4]) *If* $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]} > 0$ *and* $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]} > 0$, *then for any* $k = 1, \ldots, n$,

$$\prod_{i=1}^{k}(x_{[i]} + y_{[i]}) \leq \prod_{i=1}^{k}(x_i + y_i) \leq \prod_{i=1}^{k}(x_{[i]} + y_{[n-i+1]}).$$

Lemma 2.9 ([13, p. 655]) *If* $a_i > 0$ $(i = 1, \ldots, n)$, *then for* $k = 1, \ldots, n$,

$$\prod_{i=1}^{k} a_i \leq \left(\frac{1}{k}\sum_{i=1}^{k} a_i\right)^{k}.$$

Lemma 2.10 ([13, p. 136, H.3.b]) *If* $x_{[1]} \geq \cdots \geq x_{[n]}$, $y_{[1]} \geq \cdots \geq y_{[n]}$, *and* $x \prec_w y$, *then for any real array* $u_{[1]} \geq \cdots \geq u_{[n]} \geq 0$,

$$\sum_{i=1}^{n} x_{[i]} u_{[i]} \leq \sum_{i=1}^{n} y_{[i]} u_{[i]}.$$

Lemma 2.11 ([13, p. 503, F.2.c]) *If $X \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1,\ldots,\lambda_n$, then the eigenvalues of $C_r(X)$ are $\lambda_{i_1}\cdots\lambda_{i_r}$, $1 \le i_1 < \cdots < i_r \le n$.*

3 Lower bounds on eigenvalue product

In this section, we offer some new lower bounds on the eigenvalue product of the solution of the Lyapunov matrix differential equation (1).

Theorem 3.1 *Let $P(t)$ be a Hermitian positive semidefinite solution of (1), then for any $k = 1,\ldots,n$ and $t \ge t_0$,*

$$\prod_{i=1}^{k}\lambda_{[n-i+1]}\big[P(t)\big] \ge \Bigg[\bigg(e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[A(\xi)+A^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[n-i+1]}\big[P(t_0)\big]\bigg)^{\frac{1}{k}} + \int_{t_0}^{t}\bigg(e^{\int_{\tau}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[A(\xi)+A^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[n-i+1]}\big[Q(\tau)\big]\bigg)^{\frac{1}{k}}\,d\tau\Bigg]^{k}. \tag{7}$$

Proof By Lemma 2.1 there exists a unitary matrix $U(t) \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} P(t) &= U^H(t)\,\mathrm{diag}\big(\lambda_{[1]}\big[P(t)\big],\ldots,\lambda_{[n]}\big[P(t)\big]\big)U(t) \\ &\overset{\text{def.}}{=} U^H(t)D_{P(t)}U(t). \end{aligned} \tag{8}$$

Using Lemma 2.2 and substituting (8) into (1) yield

$$\begin{aligned} &\dot{U}^H(t)D_{P(t)}U(t) + U^H(t)\dot{D}_{P(t)}U(t) + U^H(t)D_{P(t)}\dot{U}(t) \\ &\quad = A^H(t)U^H(t)D_{P(t)}U(t) + U^H(t)D_{P(t)}U(t)A(t) + Q(t). \end{aligned} \tag{9}$$

Multiplying $U(t)$ and $U^H(t)$ on the left and right of (9), respectively, we have

$$\begin{aligned} &U(t)\dot{U}^H(t)D_{P(t)} + \dot{D}_{P(t)} + D_{P(t)}\dot{U}(t)U^H(t) \\ &\quad = U(t)A^H(t)U^H(t)D_{P(t)} + D_{P(t)}U(t)A(t)U^H(t) + U(t)Q(t)U^H(t) \\ &\quad = \widetilde{A}^H(t)D_{P(t)} + D_{P(t)}\widetilde{A}(t) + \widetilde{Q}(t), \end{aligned} \tag{10}$$

where

$$\widetilde{A}(t) = U(t)A(t)U^H(t), \qquad \widetilde{Q}(t) = U(t)Q(t)U^H(t). \tag{11}$$

Since $U(t)U^H(t) = I$, from (10) it it easy to see that

$$\begin{aligned} &\dot{U}(t)U^H(t)D_{P(t)} + U(t)\dot{U}^H(t)D_{P(t)} + U(t)U^H(t)\dot{D}_{P(t)} + D_{P(t)}\dot{U}(t)U^H(t) - \dot{U}(t)U^H(t)D_{P(t)} \\ &\quad = \widetilde{A}^H(t)D_{P(t)} + D_{P(t)}\widetilde{A}(t) + \widetilde{Q}(t). \end{aligned} \tag{12}$$

Since $D_{P(t)} = U(t)U^H(t)D_{P(t)}$, we have

$$\dot{D}_{P(t)} = \dot{U}(t)U^H(t)D_{P(t)} + U(t)\dot{U}^H(t)D_{P(t)} + U(t)U^H(t)\dot{D}_{P(t)}. \tag{13}$$

Substituting (13) into (12) implies

$$\dot{D}_{P(t)} + D_{P(t)}\dot{U}(t)U^H(t) - \dot{U}(t)U^H(t)D_{P(t)} = \widetilde{A}^H(t)D_{P(t)} + D_{P(t)}\widetilde{A}(t) + \widetilde{Q}(t). \tag{14}$$

Obviously,

$$\bigl[\widetilde{A}^H(t)D_{P(t)} + D_{P(t)}\widetilde{A}(t) + \widetilde{Q}(t)\bigr]^H = \widetilde{A}^H(t)D_{P(t)} + D_{P(t)}\widetilde{A}(t) + \widetilde{Q}(t), \qquad [\dot{D}_{P(t)}]^H = \dot{D}_{P(t)},$$

and thus by (14) we obtain

$$\bigl[D_{P(t)}\dot{U}(t)U^H(t) - \dot{U}(t)U^H(t)D_{P(t)}\bigr]^H = D_{P(t)}\dot{U}(t)U^H(t) - \dot{U}(t)U^H(t)D_{P(t)}.$$

By computation we easily see that the diagonal elements of $D_{P(t)}\dot{U}(t)U^H(t) - \dot{U}(t)U^H(t) \times D_{P(t)}$ are all 0. Thus, by (14),

$$\frac{d\{\lambda_{[i]}[P(t)]\}}{dt} = 2\,\mathrm{Re}\bigl[\widetilde{a}_{ii}(t)\bigr]\lambda_{[i]}\bigl[P(t)\bigr] + \widetilde{q}_{ii}(t), \tag{15}$$

where $\widetilde{a}_{ii}(t)$, $\widetilde{q}_{ii}(t)$ $(i = 1, 2, \dots, n)$ are the diagonal elements of $\widetilde{A}(t)$ and $\widetilde{Q}(t)$, respectively.

According to (15), for all $t \ge t_0$, it is evident that

$$e^{-2\int_{t_0}^t \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\left(\frac{d(\lambda_{[i]}[P(t)])}{dt} - 2\,\mathrm{Re}\bigl[\widetilde{a}_{ii}(t)\bigr]\lambda_{[i]}\bigl[P(t)\bigr]\right) = e^{-2\int_{t_0}^t \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\widetilde{q}_{ii}(t),$$

which is equivalent to

$$\frac{d(e^{-2\int_{t_0}^t \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\lambda_{[i]}[P(t)])}{dt} = e^{-2\int_{t_0}^t \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\widetilde{q}_{ii}(t).$$

By solving this differential equation we get

$$e^{-2\int_{t_0}^t \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\lambda_{[i]}\bigl[P(t)\bigr] = \lambda_{[i]}\bigl[P(t_0)\bigr] + \int_{t_0}^t e^{-2\int_{t_0}^\tau \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\widetilde{q}_{ii}(\tau)\,d\tau.$$

Hence

$$\lambda_{[i]}\bigl[P(t)\bigr] = e^{2\int_{t_0}^t \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\lambda_{[i]}\bigl[P(t_0)\bigr] + \int_{t_0}^t e^{2\int_{\tau}^t \mathrm{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\widetilde{q}_{ii}(\tau)\,d\tau,$$

which implies that

$$\begin{aligned}
&\prod_{i=1}^k \lambda_{[n-i+1]}\bigl[P(t)\bigr] \\
&\quad = \prod_{i=1}^k \Bigl(e^{2\int_{t_0}^t \mathrm{Re}\,\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi}\lambda_{[n-i+1]}\bigl[P(t_0)\bigr] \\
&\qquad + \int_{t_0}^t e^{2\int_{\tau}^t \mathrm{Re}\,\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi}\widetilde{q}_{n-i+1,n-i+1}(\tau)\,d\tau\Bigr)
\end{aligned}$$

$$= \left[\prod_{i=1}^{k} \left(e^{2\int_{t_0}^{t} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \lambda_{[n-i+1]}\bigl[P(t_0)\bigr] + \int_{t_0}^{t} e^{2\int_{\tau}^{t} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \widetilde{q}_{n-i+1,n-i+1}(\tau)\,d\tau \right)^{\frac{1}{k}} \right]^{k}. \tag{16}$$

Applying Lemma 2.3 to (16) yields

$$\prod_{i=1}^{k} \lambda_{[n-i+1]}\bigl[P(t)\bigr] \geq \left[\prod_{i=1}^{k} \bigl(e^{2\int_{t_0}^{t} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \lambda_{[n-i+1]}\bigl[P(t_0)\bigr]\bigr)^{\frac{1}{k}} + \prod_{i=1}^{k} \left(\int_{t_0}^{t} e^{2\int_{\tau}^{t} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \widetilde{q}_{n-i+1,n-i+1}(\tau)\,d\tau \right)^{\frac{1}{k}} \right]^{k}. \tag{17}$$

By Lemma 2.4 (let $r_i = k$) and (17) we have

$$\begin{aligned}
&\prod_{i=1}^{k} \lambda_{[n-i+1]}\bigl[P(t)\bigr] \\
&\geq \left[\prod_{i=1}^{k} \bigl(e^{2\int_{t_0}^{t} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \lambda_{[n-i+1]}\bigl[P(t_0)\bigr]\bigr)^{\frac{1}{k}} + \int_{t_0}^{t} \prod_{i=1}^{k} e^{\frac{2}{k}\int_{\tau}^{t} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \bigl(\widetilde{q}_{n-i+1,n-i+1}(\tau)\bigr)^{\frac{1}{k}}\,d\tau \right]^{k} \\
&= \left[\left(e^{2\int_{t_0}^{t} \sum_{i=1}^{k} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \prod_{i=1}^{k} \lambda_{[n-i+1]}\bigl[P(t_0)\bigr] \right)^{\frac{1}{k}} + \int_{t_0}^{t} e^{\frac{2}{k}\int_{\tau}^{t} \sum_{i=1}^{k} \operatorname{Re}\widetilde{a}_{n-i+1,n-i+1}(\xi)\,d\xi} \left(\prod_{i=1}^{k} \widetilde{q}_{n-i+1,n-i+1}(\tau) \right)^{\frac{1}{k}} d\tau \right]^{k} \\
&\geq \left[\left(e^{2\int_{t_0}^{t} \sum_{i=1}^{k} \operatorname{Re}\widetilde{a}_{[n-i+1,n-i+1]}(\xi)\,d\xi} \prod_{i=1}^{k} \lambda_{[n-i+1]}\bigl[P(t_0)\bigr] \right)^{\frac{1}{k}} + \int_{t_0}^{t} e^{\frac{2}{k}\int_{\tau}^{t} \sum_{i=1}^{k} \operatorname{Re}\widetilde{a}_{[n-i+1,n-i+1]}(\xi)\,d\xi} \left(\prod_{i=1}^{k} \widetilde{q}_{[n-i+1,n-i+1]}(\tau) \right)^{\frac{1}{k}} d\tau \right]^{k}.
\end{aligned} \tag{18}$$

From Lemma 2.5 we have

$$\begin{aligned}
&\bigl(2\operatorname{Re}\widetilde{a}_{[nn]}(\xi), \ldots, 2\operatorname{Re}\widetilde{a}_{[n-k+1,n-k+1]}(\xi)\bigr) \\
&\quad \prec^{w} \bigl(\lambda_{[n]}\bigl[\widetilde{A}(\xi) + \widetilde{A}^{H}(\xi)\bigr], \ldots, \lambda_{[n-k+1]}\bigl[\widetilde{A}(\xi) + \widetilde{A}^{H}(\xi)\bigr]\bigr).
\end{aligned}$$

Then by Lemma 2.6

$$\begin{aligned}
&\bigl(e^{2\int_{t_0}^{t} \operatorname{Re}\widetilde{a}_{[nn]}(\xi)\,d\xi}, \ldots, e^{2\int_{t_0}^{t} \operatorname{Re}\widetilde{a}_{[n-k+1,n-k+1]}(\xi)\,d\xi}\bigr) \\
&\quad \prec^{w} \bigl(e^{\int_{t_0}^{t} \lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^{H}(\xi)]\,d\xi}, \ldots, e^{\int_{t_0}^{t} \lambda_{[n-k+1]}[\widetilde{A}(\xi)+\widetilde{A}^{H}(\xi)]\,d\xi}\bigr),
\end{aligned}$$

that is,

$$e^{2\int_{t_0}^{t}\sum_{i=1}^{k}\operatorname{Re}\widetilde{a}_{[n-i+1,n-i+1]}(\xi)\,d\xi} \geq e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}. \tag{19}$$

By Lemma 2.5 we have

$$\bigl(\widetilde{q}_{[nn]}(\tau),\ldots,\widetilde{q}_{[n-k+1,n-k+1]}(\tau)\bigr) \prec^{w} \bigl(\lambda_{[n]}\bigl[\widetilde{Q}(\tau)\bigr],\ldots,\lambda_{[n-k+1]}\bigl[\widetilde{Q}(\tau)\bigr]\bigr).$$

Using Lemma 2.7, we obtain

$$\prod_{i=1}^{k}\widetilde{q}_{[n-i+1,n-i+1]}(\tau) \geq \prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[\widetilde{Q}(\tau)\bigr]. \tag{20}$$

By (11), substituting (19) and (20) into (18), we have

$$\begin{aligned}
&\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[P(t)\bigr] \\
&\quad\geq \Biggl[\Biggl(e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[P(t_0)\bigr]\Biggr)^{\frac{1}{k}} \\
&\qquad + \int_{t_0}^{t} e^{\frac{1}{k}\int_{\tau}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\Biggl(\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[\widetilde{Q}(\tau)\bigr]\Biggr)^{\frac{1}{k}}\,d\tau\Biggr]^{k} \\
&\quad= \Biggl[\Biggl(e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[A(\xi)+A^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[P(t_0)\bigr]\Biggr)^{\frac{1}{k}} \\
&\qquad + \int_{t_0}^{t}\Biggl(e^{\int_{\tau}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[A(\xi)+A^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[Q(\tau)\bigr]\Biggr)^{\frac{1}{k}}\,d\tau\Biggr]^{k},
\end{aligned}$$

which completes the proof. □

By Theorem 3.1 we easily get the following result.

Theorem 3.2 *Let $P(t)$ be a Hermitian positive semidefinite solution of* (1). *Then for any $k = 1,\ldots,n$ and $t \geq t_0$,*

$$\begin{aligned}
\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[P(t)\bigr] \geq{}& e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[A(\xi)+A^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[P(t_0)\bigr] \\
&+ \Biggl[\int_{t_0}^{t}\Biggl(e^{\int_{\tau}^{t}\sum_{i=1}^{k}\lambda_{[n-i+1]}[A(\xi)+A^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[n-i+1]}\bigl[Q(\tau)\bigr]\Biggr)^{\frac{1}{k}}\,d\tau\Biggr]^{k}.
\end{aligned} \tag{21}$$

By Theorems 3.1 and 3.2 we immediately get the following corollaries.

Corollary 3.1 *Let $P(t)$ be a Hermitian positive semidefinite solution of* (1). *Then for any $k = 1,\dots,n$ and $t \geq t_0$,*

$$\det[P(t)] \geq \Bigl[\bigl(e^{\int_{t_0}^{t} \operatorname{tr}[A(\xi)+A^H(\xi)]\,d\xi} \det[P(t_0)]\bigr)^{\frac{1}{n}} + \int_{t_0}^{t} \bigl(e^{\int_{\tau}^{t} \operatorname{tr}[A(\xi)+A^H(\xi)]\,d\xi} \det[Q(\tau)]\bigr)^{\frac{1}{n}}\,d\tau\Bigr]^n. \tag{22}$$

Corollary 3.2 *Let $P(t)$ be a Hermitian positive semidefinite solution of* (1). *Then for any $k = 1,\dots,n$ and $t \geq t_0$,*

$$\det[P(t)] \geq e^{\int_{t_0}^{t} \operatorname{tr}[A(\xi)+A^H(\xi)]\,d\xi} \det[P(t_0)] + \Bigl(\int_{t_0}^{t} e^{\frac{1}{n}\int_{\tau}^{t} \operatorname{tr}[A(\xi)+A^H(\xi)]\,d\xi} \bigl(\det[Q(\tau)]\bigr)^{\frac{1}{n}}\,d\tau\Bigr)^n. \tag{23}$$

4 Upper bounds on eigenvalue product

In this section, we offer some new upper bounds on the eigenvalue product of the solution for the Lyapunov matrix differential equation (1).

Theorem 4.1 *Let $P(t)$ be a Hermitian positive definite solution of* (1). *Then for any $k = 1,\dots,n$ and $t \geq t_0$,*

$$\begin{aligned}\prod_{i=1}^{k} \lambda_{[i]}[P(t)] &\leq e^{\int_{t_0}^{t} \sum_{i=1}^{k} \lambda_{[i]}[A(\xi)+A^H(\xi)]\,d\xi} \prod_{i=1}^{k} \lambda_{[i]}[P(t_0)] \\ &\quad \cdot \Bigl(1 + \frac{\sum_{i=1}^{k} \lambda_{[k-i+1]}^{-1}[P(t_0)] \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} \lambda_{[n]}[A(\xi)+A^H(\xi)]\,d\xi} \lambda_{[i]}[Q(\tau)]\,d\tau}{k}\Bigr)^k \\ &= l_k(t).\end{aligned} \tag{24}$$

Proof By (16) we get

$$\begin{aligned}&\prod_{i=1}^{k} \lambda_{[i]}[P(t)] \\ &\quad = \prod_{i=1}^{k} \Bigl[e^{2\int_{t_0}^{t} \operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi} \lambda_{[i]}[P(t_0)] + \int_{t_0}^{t} e^{2\int_{\tau}^{t} \operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi} \widetilde{q}_{ii}(\tau)\,d\tau\Bigr] \\ &\quad = \prod_{i=1}^{k} \bigl(e^{2\int_{t_0}^{t} \operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi} \lambda_{[i]}[P(t_0)]\bigr) \\ &\qquad \cdot \prod_{i=1}^{k} \Bigl(1 + \lambda_{[i]}^{-1}[P(t_0)] \int_{t_0}^{t} e^{-2\int_{t_0}^{\tau} \operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi} \widetilde{q}_{ii}(\tau)\,d\tau\Bigr) \\ &\quad = e^{2\int_{t_0}^{t} \sum_{i=1}^{k} \operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi} \prod_{i=1}^{k} \lambda_{[i]}[P(t_0)] \\ &\qquad \cdot \prod_{i=1}^{k} \Bigl(1 + \lambda_{[i]}^{-1}[P(t_0)] \int_{t_0}^{t} e^{-2\int_{t_0}^{\tau} \operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi} \widetilde{q}_{ii}(\tau)\,d\tau\Bigr).\end{aligned} \tag{25}$$

By Lemma 2.5 we get

$$\lambda_{[n]}\big[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)\big] \le 2\operatorname{Re}\big(\widetilde{a}_{[nn]}(\xi)\big) \le 2\operatorname{Re}\big[\widetilde{a}_{ii}(\xi)\big].$$

Then

$$e^{-\int_{t_0}^{\tau}\lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi} \ge e^{-\int_{t_0}^{\tau}2\operatorname{Re}(\widetilde{a}_{[nn]}(\xi))\,d\xi} \ge e^{-\int_{t_0}^{\tau}2\operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}. \tag{26}$$

Substituting (26) into (25) gives

$$\begin{aligned}
&\prod_{i=1}^{k}\lambda_{[i]}\big[P(t)\big] \\
&\quad\le e^{2\int_{t_0}^{t}\sum_{i=1}^{k}\operatorname{Re}[\widetilde{a}_{ii}(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[i]}\big[P(t_0)\big] \\
&\qquad\cdot\prod_{i=1}^{k}\left(1+\lambda_{[i]}^{-1}\big[P(t_0)\big]\int_{t_0}^{t}e^{-\int_{t_0}^{\tau}\lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\widetilde{q}_{ii}(\tau)\,d\tau\right) \\
&\quad\le e^{2\int_{t_0}^{t}\sum_{i=1}^{k}\operatorname{Re}[\widetilde{a}_{[ii]}(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[i]}\big[P(t_0)\big] \\
&\qquad\cdot\prod_{i=1}^{k}\left(1+\lambda_{[i]}^{-1}\big[P(t_0)\big]\int_{t_0}^{t}e^{-\int_{t_0}^{\tau}\lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\widetilde{q}_{[ii]}(\tau)\,d\tau\right).
\end{aligned} \tag{27}$$

From Lemma 2.5 we have

$$\begin{aligned}
&\big(2\operatorname{Re}\widetilde{a}_{[11]}(\xi),\dots,2\operatorname{Re}\widetilde{a}_{[kk]}(\xi)\big) \\
&\quad\prec_w \big(\lambda_{[1]}\big[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)\big],\dots,\lambda_{[k]}\big[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)\big]\big).
\end{aligned}$$

Then by Lemma 2.6 we have

$$\begin{aligned}
&\big(e^{2\int_{t_0}^{t}\operatorname{Re}\widetilde{a}_{[11]}(\xi)\,d\xi},\dots,e^{2\int_{t_0}^{t}\operatorname{Re}\widetilde{a}_{[kk]}(\xi)\,d\xi}\big) \\
&\quad\prec_w \big(e^{\int_{t_0}^{t}\lambda_{[1]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi},\dots,e^{\int_{t_0}^{t}\lambda_{[k]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\big),
\end{aligned}$$

that is,

$$e^{2\int_{t_0}^{t}\sum_{i=1}^{k}\operatorname{Re}\widetilde{a}_{[ii]}(\xi)\,d\xi} \le e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[i]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}. \tag{28}$$

By Lemma 2.5 we have

$$\big(\widetilde{q}_{[11]}(\tau),\dots,\widetilde{q}_{[kk]}(\tau)\big) \prec_w \big(\lambda_{[1]}\big[\widetilde{Q}(\tau)\big],\dots,\lambda_{[k]}\big[\widetilde{Q}(\tau)\big]\big).$$

Using Lemma 2.7, we obtain

$$\prod_{i=1}^{k}\widetilde{q}_{[ii]}(\tau) \le \prod_{i=1}^{k}\lambda_{[i]}\big[\widetilde{Q}(\tau)\big]. \tag{29}$$

Substituting (28) and (29) into (27), by Lemma 2.8 we get

$$\prod_{i=1}^{k}\lambda_{[i]}\big[P(t)\big]\le e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[i]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[i]}\big[P(t_0)\big] \cdot\prod_{i=1}^{k}\Big(1+\lambda_{[k-i+1]}^{-1}\big[P(t_0)\big]\int_{t_0}^{t}e^{-\int_{t_0}^{\tau}\lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\widetilde{q}_{[ii]}(\tau)\,d\tau\Big). \tag{30}$$

Applying Lemma 2.9 to (30), we get

$$\prod_{i=1}^{k}\lambda_{[i]}\big[P(t)\big]\le e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[i]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[i]}\big[P(t_0)\big] \cdot\Big(1+\frac{\sum_{i=1}^{k}\lambda_{[k-i+1]}^{-1}[P(t_0)]\int_{t_0}^{t}e^{-\int_{t_0}^{\tau}\lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\widetilde{q}_{[ii]}(\tau)\,d\tau}{k}\Big)^k. \tag{31}$$

By (31) let

$$u_{[i]}=\lambda_{[k-i+1]}^{-1}\big[P(t_0)\big]\cdot e^{-\int_{t_0}^{\tau}\lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi},\qquad x_{[i]}=\widetilde{q}_{[ii]}(\tau),\qquad y_{[i]}=\lambda_{[i]}\big[\widetilde{Q}(\tau)\big]$$

in Lemma 2.10. Then

$$\prod_{i=1}^{k}\lambda_{[i]}\big[P(t)\big]\le e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[i]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[i]}\big[P(t_0)\big] \cdot\Big(1+\frac{\sum_{i=1}^{k}\lambda_{[k-i+1]}^{-1}[P(t_0)]\int_{t_0}^{t}e^{-\int_{t_0}^{\tau}\lambda_{[n]}[\widetilde{A}(\xi)+\widetilde{A}^H(\xi)]\,d\xi}\lambda_{[i]}[\widetilde{Q}(\tau)]\,d\tau}{k}\Big)^k. \tag{32}$$

In terms of (11) and (32), we get

$$\prod_{i=1}^{k}\lambda_{[i]}\big[P(t)\big]\le e^{\int_{t_0}^{t}\sum_{i=1}^{k}\lambda_{[i]}[A(\xi)+A^H(\xi)]\,d\xi}\prod_{i=1}^{k}\lambda_{[i]}\big[P(t_0)\big] \cdot\Big(1+\frac{\sum_{i=1}^{k}\lambda_{[k-i+1]}^{-1}[P(t_0)]\int_{t_0}^{t}e^{-\int_{t_0}^{\tau}\lambda_{[n]}[A(\xi)+A^H(\xi)]\,d\xi}\lambda_{[i]}[Q(\tau)]\,d\tau}{k}\Big)^k,$$

which completes the proof. □

By Theorem 4.1 we immediately have the following corollary.

Corollary 4.1 *Let $P(t)$ be a Hermitian positive definite solution of* (1). *Then for any $k=1,\dots,n$ and $t\ge t_0$,*

$$\det\big[P(t)\big]\le e^{\int_{t_0}^{t}\operatorname{tr}[A(\xi)+A^H(\xi)]\,d\xi}\cdot\det\big[P(t_0)\big] \cdot\Big(1+\frac{\sum_{i=1}^{n}\lambda_{[n-i+1]}^{-1}[P(t_0)]\int_{t_0}^{t}e^{-\int_{t_0}^{\tau}\lambda_{[n]}(A(\xi)+A^H(\xi))\,d\xi}\lambda_{[i]}[Q(\tau)]\,d\tau}{n}\Big)^n. \tag{33}$$

5 The stability analysis of a class of time-varying nonlinear systems

In this section, we give an application of our eigenvalue bounds in a class of time-varying nonlinear systems. The time-varying nonlinear systems have wide applications in control and optimization, such as quantum mechanics, solid mechanics, parameter identification, electrical systems, and automatic control systems and can be used to discuss the stability of the time-varying nonlinear system (2). Next, we show that our eigenvalue bounds can be used to discuss the stability of a class of time-varying nonlinear systems.

For system (2), take

$$\Delta f\bigl(x(t),\Phi(t)\bigr)=\bigl\|x(t)\bigr\|^2P(t)EC_k\bigl(P(t)\bigr)E^Tx(t)+B(t)\Phi(t)r(t),$$

where $C_k(P(t))$ denotes the kth compound matrix of $P(t)$, $P(t)$ is the positive definite solution of the Lyapunov matrix differential equation (1)

$$\dot{P}(t)=A^H(t)P(t)+P(t)A(t)+Q(t),$$

and $E=(I_n\ 0)$ is an $n\times C_n^k$th matrix. For E, we have the following three special conditions:

(i) When $k=1$, $C_1(P(t))=P(t)$, and E is the identity matrix.
(ii) When $k=n-1$, E is the identity matrix.
(iii) When $k=n$, $C_n(P(t))=\det(P(t))$, and $E=1$.

In the following theorem, we give a condition under which the time-varying nonlinear system (2) is uniformly asymptotic stable.

Theorem 5.1 *For the time-varying nonlinear system* (2), *if*

$$\frac{-1}{l_1^2(t)}+2\bigl\|x(t)\bigr\|^2l_k(t)<0, \tag{34}$$

then the large range for system (2) *is uniformly asymptotically stable, where* $l_k(t)$ ($k=1,\ldots,n$) *are defined by Theorem* 4.1.

Proof Choose the Lyapunov function

$$V\bigl[x(t),t\bigr]=x^T(t)P^{-1}(t)x(t)+\operatorname{tr}\bigl(\Phi^T(t)R^{-1}\Phi(t)\bigr).$$

Then

$$\begin{aligned}\dot{V}\bigl[x(t),t\bigr]={}&\dot{x}^T(t)P^{-1}(t)x(t)+x^T(t)\dot{P}^{-1}(t)x(t)+x^T(t)P^{-1}(t)\dot{x}(t)\\&+\operatorname{tr}\bigl(\dot{\Phi}^T(t)R^{-1}\Phi(t)\bigr)+\operatorname{tr}\bigl(\Phi^T(t)R^{-1}\dot{\Phi}(t)\bigr).\end{aligned} \tag{35}$$

As $P(t)$ is positive definite, we have $P(t)P^{-1}(t)=I$. Hence

$$\dot{P}(t)P^{-1}(t)+P(t)\dot{P}^{-1}(t)=0,$$

which means that

$$\dot{P}^{-1}(t)=-P^{-1}(t)\dot{P}(t)P^{-1}(t). \tag{36}$$

Substituting (1), (2), and (36) into (35) yields

$$\begin{aligned}
\dot{V}[x(t),t] &= \dot{x}^T(t)P^{-1}(t)x(t) + x^T(t)\dot{P}^{-1}(t)x(t) + x^T(t)P^{-1}(t)\dot{x}(t) \\
&\quad + \mathrm{tr}\big(\dot{\Phi}^T(t)R^{-1}\Phi(t)\big) + \mathrm{tr}\big(\Phi^T(t)R^{-1}\dot{\Phi}(t)\big) \\
&= \big[A(t)x(t) + \Delta f\big(x(t),\Phi(t)\big)\big]^T P^{-1}(t)x(t) - x^T(t)\big[P^{-1}(t)\dot{P}(t)P^{-1}(t)\big]x(t) \\
&\quad + x^T(t)P^{-1}(t)\big[A(t)x(t) + \Delta f\big(x(t),\Phi(t)\big)\big] \\
&\quad - \mathrm{tr}\big\{\big[RB^T(t)P(t)x(t)r(t)^T\big]^T R^{-1}\Phi(t)\big\} \\
&\quad - \mathrm{tr}\big\{\Phi^T(t)R^{-1}\big[RB^T(t)P(t)x(t)r(t)^T\big]\big\} \\
&= \big[A(t)x(t) + \|x(t)\|^2 P(t)EC_k\big(P(t)\big)E^T x(t) + B(t)\Phi(t)r(t)\big]^T P^{-1}(t)x(t) \\
&\quad - x^T(t)\big[P^{-1}(t)\big[A^H(t)P(t) + P(t)A(t) + Q(t)\big]P^{-1}(t)\big]x(t) \\
&\quad + x^T(t)P^{-1}(t)\big[A(t)x(t) + \|x(t)\|^2 P(t)EC_k\big(P(t)\big)E^T x(t) + B(t)\Phi(t)r(t)\big] \\
&\quad - \mathrm{tr}\big\{\big[RB^T(t)P(t)x(t)r(t)^T\big]^T R^{-1}\Phi(t)\big\} \\
&\quad - \mathrm{tr}\big\{\Phi^T(t)R^{-1}\big[RB^T(t)P(t)x(t)r(t)^T\big]\big\} \\
&= -x^T(t)P^{-1}(t)Q(t)P^{-1}(t)x(t) + 2x^T(t)EC_k\big(P(t)\big)E^T x(t)\cdot\|x(t)\|^2.
\end{aligned} \tag{37}$$

Applying Lyapunov stability theory, a sufficient condition of uniform asymptotic stability in large range for system (2) is that, for any $x(t) \neq 0$,

$$\dot{V}[x(t),t] < 0.$$

Using Theorem 4.1, we get

$$\lambda_{[1]}\big[P(t)\big] \leq l_1(t) \tag{38}$$

and

$$\prod_{i=1}^{k}\lambda_{[i]}\big[P(t)\big] \leq l_k(t). \tag{39}$$

In terms of (37), (38), and (39), since $Q(t) = I$, using Lemma 2.11, we obtain

$$\begin{aligned}
\dot{V}[x(t),t] &= -x^T(t)P^{-1}(t)QP^{-1}(t)x(t) + 2x^T(t)EC_k\big(P(t)\big)E^T x(t)\cdot\|x(t)\|^2 \\
&\leq -\lambda_n\big[\big(P(t)\big)^{-1}I\big(P(t)\big)^{-1}\big]\cdot\|x(t)\|^2 + 2\lambda_1\big[C_k\big(P(t)\big)\big]\cdot\|x(t)\|^4 \\
&\leq \frac{-1}{\lambda_1^2(P(t))} + 2\|x(t)\|^2\prod_{i=1}^{k}\lambda_i\big(P(t)\big) \\
&\leq \frac{-1}{l_1^2(t)} + 2\|x(t)\|^2 l_k(t).
\end{aligned} \tag{40}$$

If condition (34) is satisfied, then substitution of (34) into (40) yields

$$\dot{V}[x(t),t] < 0.$$

Hence, using Lyapunov stability theory, the large range for system (2) is uniformly asymptotically stable. □

Remark 5.1 For Theorem 5.1, we have the following three special conditions:

(i) When $k = 1$, $C_1(P(t)) = P(t)$, and E is the identity matrix. For system (2), take

$$\Delta f\big(x(t), \Phi(t)\big) = \|x(t)\|^2 P^2(t)x(t) + B(t)\Phi(t)r(t).$$

If

$$\frac{-1}{l_1^2(t)} + 2\|x(t)\|^2 l_1(t) < 0, \tag{41}$$

then system (2) is uniformly asymptotically stable, where $l_1(t)$ is defined in Theorem 4.1.

(ii) When $k = n-1$, E is the identity matrix. For system (2), take

$$\Delta f\big(x(t), \Phi(t)\big) = \|x(t)\|^2 P(t)C_{n-1}\big(P(t)\big)x(t) + B(t)\Phi(t)r(t).$$

If

$$\frac{-1}{l_1^2(t)} + 2\|x(t)\|^2 l_{n-1}(t) < 0, \tag{42}$$

then system (2) is uniformly asymptotically stable, where $l_1(t)$ and $l_{n-1}(t)$ are defined in Theorem 4.1.

(iii) When $k = n$, $C_n(P(t)) = \det(P(t))$ and $E = 1$. For system (2), take

$$\Delta f\big(x(t), \Phi(t)\big) = \|x(t)\|^2 \det\big(P(t)\big)P(t)x(t) + B(t)\Phi(t)r(t).$$

If

$$\frac{-1}{l_1^2(t)} + 2\|x(t)\|^2 l_n(t) < 0, \tag{43}$$

then system (2) is uniformly asymptotically stable, where $l_1(t)$ and $l_n(t)$ are defined in Theorem 4.1.

6 A numerical example

In this section, we demonstrate the effectiveness of our results and compare our eigenvalue bounds with those of the previous results by a real application example.

Example 6.1 Consider the time-varying nonlinear system (2) with

$$A(t) = \begin{pmatrix} 0 & \frac{2}{3t} & 0 \\ -\frac{1}{3t} & \frac{1}{4t} & 0 \\ 0 & 0 & \frac{3}{8t} \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x(t), r(t) \in \mathbb{C}^3$ and

$$Q(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Choosing $t_0 = \frac{1}{16}$, we have

$$P(t_0) = \begin{pmatrix} \frac{1}{16} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

The solution of (1) can be expressed as

$$P(t) = e^{A^T(t-t_0)} P(t_0) e^{A(t-t_0)} + \int_{t_0}^{t} e^{A^T(t-\tau)} Q(\tau) e^{A(t-\tau)}\, d\tau. \tag{44}$$

Using Mathematical tool and (44), we get that

$$P(t) = \begin{pmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{pmatrix},$$

where

$$P_{11} = \frac{-4879 + 4096 e^{\frac{t^2}{4}} + 64 e^{\frac{t^2}{4}} t + e^{\frac{t^2}{4}} (783 + 55t) \operatorname{Cos}(\frac{\sqrt{119} t^2}{12}) - 3\sqrt{119} e^{\frac{t^2}{4}} (-23 + t) \operatorname{Sin}(\frac{\sqrt{119} t^2}{12})}{1904 t},$$

$$P_{22} = \frac{16(-119 + 4 e^{\frac{t^2}{4}} (32 + t)) + e^{\frac{t^2}{4}} (-144 + 55t) \operatorname{Cos}(\frac{\sqrt{119} t^2}{12}) + 3\sqrt{119} e^{\frac{t^2}{4}} (16 + t) \operatorname{Sin}(\frac{\sqrt{119} t^2}{12})}{952 t},$$

$$P_{33} = \frac{-16 + e^{\frac{3t^2}{4}} (16 + 3t)}{12 t}.$$

Obviously, when $t > 0$, $P(t)$ is positive definite.

(i) First, we show that the eigenvalue bounds for $P(t)$ have wide applications in the time-varying nonlinear systems. Using Theorem 4.1, we get

$$\lambda_{[1]}\big[P(t)\big] \le \frac{1 + 4t}{4} t^{\frac{3}{4}} = l_1(t),$$

$$\prod_{i=1}^{2} \lambda_{[i]}\big[P(t)\big] \le 2^{\frac{2}{3}} t^{\frac{17}{12}} \cdot \left(\frac{9}{7} \cdot 2^{-\frac{8}{3}} t^{\frac{7}{6}} + \frac{19}{28}\right)^2 = l_2(t),$$

$$\prod_{i=1}^{3} \lambda_{[i]}\big[P(t)\big] \le \frac{1}{16} t^{\frac{5}{4}} \left(2^{\frac{11}{3}} t^{\frac{7}{6}} + \frac{1}{2}\right)^3 = l_3(t).$$

(1) When $k = 1$, take

$$\Delta f\big(x(t), \Phi(t)\big) = \big\|x(t)\big\|^2 P^2(t) x(t) + B(t)\Phi(t) r(t).$$

If

$$\frac{-1}{l_1^2(t)} + 2\|x(t)\|^2 l_1(t) = -\frac{16}{(1+4t)^2 t^{\frac{3}{2}}} + \|x(t)\|^2 \frac{1+4t}{2} t^{\frac{3}{4}} < 0,$$

then using Theorem 5.1, we get that system (2) is uniformly asymptotically stable.

(2) When $k = 2$, take

$$\Delta f\big(x(t), \Phi(t)\big) = \|x(t)\|^2 P(t) C_2\big(P(t)\big) x(t) + B(t)\Phi(t) r(t).$$

If

$$\frac{-1}{l_1^2(t)} + 2\|x(t)\|^2 l_2(t) = -\frac{16}{(1+4t)^2 t^{\frac{3}{2}}} + 2\|x(t)\|^2 2^{\frac{2}{3}} t^{\frac{17}{12}} \cdot \left(\frac{9}{7} \cdot 2^{-\frac{8}{3}} t^{\frac{7}{6}} + \frac{19}{28}\right)^2 < 0,$$

then using Theorem 5.1, we get that system (2) is uniformly asymptotically stable.

(3) When $k = 3$, take

$$\Delta f\big(x(t), \Phi(t)\big) = \|x(t)\|^2 \det\big(P(t)\big) P(t) x(t) + B(t)\Phi(t) r(t).$$

If

$$\frac{-1}{l_1^2(t)} + 2\|x(t)\|^2 l_3(t) = -\frac{16}{(1+4t)^2 t^{\frac{3}{2}}} + \|x(t)\|^2 \frac{1}{8} t^{\frac{5}{4}} \left(2^{\frac{11}{3}} t^{\frac{7}{6}} + \frac{1}{2}\right)^3 < 0,$$

then using Theorem 5.1, we get that system (2) is uniformly asymptotically stable.

(ii) Next, we give our upper and lower eigenvalue bounds.

If $k = 2$, then by Theorems 3.2 and 4.1 we obtain

$$\prod_{i=1}^{2} \lambda_{[n-i+1]}\big[P(t)\big] \geq \frac{1}{32} t^{\frac{1}{2}} + \left(\frac{4}{3} t - \frac{1}{6} t^{\frac{1}{4}}\right)^2,$$

$$\prod_{i=1}^{2} \lambda_{[i]}\big[P(t)\big] \leq 2^{\frac{2}{3}} t^{\frac{17}{12}} \cdot \left(\frac{9}{7} \cdot 2^{-\frac{8}{3}} t^{\frac{7}{6}} + \frac{19}{28}\right)^2.$$

If $k = 3$, then by Theorems 3.2 and 4.1 we have

$$\det\big[P(t)\big] \geq 32 t^{\frac{5}{4}} + \left(\frac{12}{7} t - \frac{3}{7} \cdot 2^{-\frac{1}{3}} t^{\frac{5}{12}}\right)^3 \tag{45}$$

and

$$\det\big[P(t)\big] \leq \frac{1}{16} t^{\frac{5}{4}} \left(2^{\frac{11}{3}} t^{\frac{7}{6}} + \frac{1}{2}\right)^3. \tag{46}$$

In addition, by solving equation (1) we obtain

$$\det\big[P(t)\big] = 8t^3. \tag{47}$$

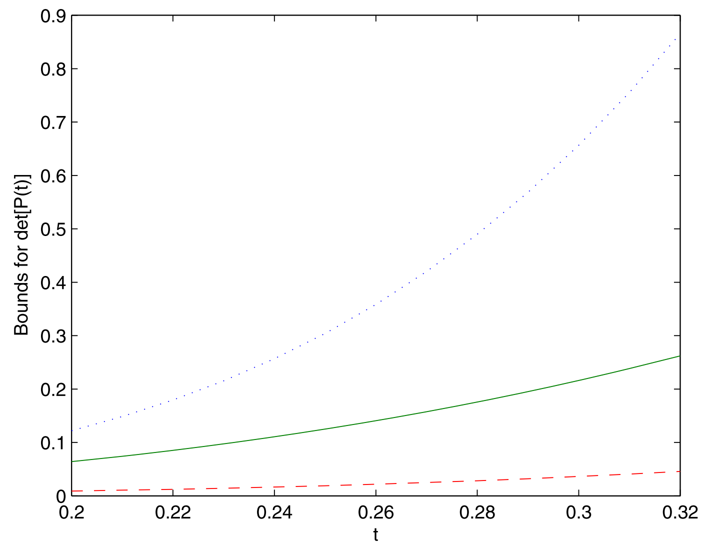

Figure 1 Eigenvalue bounds in Example 6.1

We depict the plot (see Fig. 1) based on all these results. In the plot, the dashed line presents the result of (45). The dot line denotes the result of (46). The real line shows the result of (47). From the plot we can get the bounds of the solution of (1).

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The authors have no any competing interests in the subject of this research.

Authors' contributions
All the authors participated in every phase of the research conducted for this paper. All authors read and approved the final manuscript.

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