Cheng *Journal of Inequalities and Applications* (2019) 2019:169
https://doi.org/10.1186/s13660-019-2118-3

Journal of Inequalities and Applications
a SpringerOpen Journal

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Parallel hybrid viscosity method for fixed point problems, variational inequality problems and split generalized equilibrium problems

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Abstract

In this paper, we first propose a new parallel hybrid viscosity iterative method for finding a common element of three solution sets: (i) finite split generalized equilibrium problems; (ii) finite variational inequality problems; and (iii) fixed point problem of a finite collection of demicontractive operators. And we prove that the sequence generated by the iterative scheme strongly converges to a common solution of the above-mentioned problems. Also, we present numerical examples to demonstrate the effectiveness of our algorithm. Our results presented in this paper improve and extend many recent results in the literature.

Keywords: Split generalized equilibrium problems; Variational inequality problems; Fixed point problems; Parallel hybrid viscosity method; Lipschitzian mappings

1 Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two infinite dimensional real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C and Q be a nonempty closed convex subset of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $T: C \to C$ be a mapping. The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$.

In what follows, we recall some definitions of classes of operators often used in fixed point theory.

Definition 1.1 Let $T: C \to C$ be a mapping. Then

(i) T is *ρ-Lipschitzian* with $\rho > 0$ if

$$\|Tx - Ty\| \le \rho\|x - y\|, \quad \forall x, y \in C;$$

If $\rho \in (0, 1)$, then T is *ρ-contractive* and if $\rho = 1$, then T is *nonexpansive*.

(ii) T is *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C;$$

(iii) T is *κ-strictly pseudo-contractive* with $\kappa \in [0,1)$ if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa \big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

Definition 1.2 Let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. Then

(i) T is *directed* if

$$\|Tx - z\|^2 \le \|x - z\|^2 - \|x - Tx\|^2, \quad \forall z \in F(T), x \in C;$$

(ii) T is *quasi-nonexpansive* if

$$\|Tx - z\| \le \|x - z\|, \quad \forall z \in F(T), x \in C;$$

(iii) T is *β-demicontractive* with $\beta < 1$ if

$$\|Tx - z\|^2 \le \|x - z\|^2 + \beta\|x - Tx\|^2, \quad \forall z \in F(T), x \in C.$$

Note that the class of demicontractive operators contains important classes of operators: directed operator (firmly nonexpansive operator with nonempty fixed points set) for $\beta = -1$, quasi-nonexpansive operator (nonexpansive operator with nonempty fixed points set) for $\beta = 0$, and strictly pseudo-contractive operator with nonempty fixed points set for $\beta \in (0,1)$; the class of quasi-nonexpansive operators also contains nonspreading mappings with nonempty fixed point set and N-generalized hybrid mappings with nonempty fixed point set.

It is well known that every nonexpansive operator $T : \mathcal{H}_1 \to \mathcal{H}_1$ satisfies the following inequality;

$$\big\langle (I - T)x - (I - T)y, Ty - Tx\big\rangle \le \frac{1}{2}\big\|(I - T)y - (I - T)x\big\|^2,$$

for all $x, y \in \mathcal{H}_1$. Therefore, for all $x \in \mathcal{H}_1$, $y \in F(T)$,

$$\big\langle (I - T)x, y - Tx\big\rangle \le \frac{1}{2}\big\|(T - I)x\big\|^2. \tag{1.1}$$

We also know that $F(T)$ of nonexpansive mapping T is closed and convex.

The fixed point problem (FPP) for the mapping T is to find $x \in C$ such that

$$Tx = x.$$

Many iterative algorithms has been introduced for finding fixed points of nonexpansive mappings, quasi-nonexpansive mappings, firmly nonexpansive mappings, demicontractive mappings (see [1–6]), including the since recently popular viscosity iterative algorithms, which formally consist of the sequence $\{x_n\}$ given by the iteration

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \forall n \ge 0, \tag{1.2}$$

where f is a contraction, $\{\alpha_n\} \subset (0,1)$ is a slowly vanishing sequence, i.e., $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_n \alpha_n = \infty$. The above method was first considered with regard to the special case when $f = \mu$ (μ being any given element), in 1967 by Halpern [7] (for $\mu = 0$). There is an extensive literature regarding the convergence analysis of (1.2), with several types of operator T, in the setting of Hilbert spaces and Banach spaces. This procedure can be regarded as a regularization process for fixed point iterations which is supposed to induce the convergence in norm of the iterates. Another advantage of this method is that it allows one to select a particular fixed point of T which satisfies some variational inequality.

Given a nonlinear mapping $B : C \to \mathcal{H}_1$. Recall that B is said to be monotone if

$$\langle x - y, Bx - By\rangle \geq 0, \quad \forall x, y \in C;$$

B is said to be α-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Bx - By\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C;$$

B is said to be α-inverse strongly monotone (for short, α-ism) if there exists $\alpha > 0$ such that

$$\langle x - y, Bx - By\rangle \geq \alpha \|Bx - By\|^2, \quad \forall x, y \in C.$$

We can easily see that

(i) if B is nonexpansive, then $I - B$ is monotone;

(ii) if B is an α-inverse-strongly monotone mapping, then it must be a $\frac{1}{\alpha}$-Lipschitz operator. Moreover, $I - rB$ is nonexpansive when $0 < r \leq 2\alpha$.

The variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The solution set of (1.3) is denoted by $\mathrm{VI}(C, B)$. In fact,

$$\begin{aligned}
& x^* \in \mathrm{VI}(C, B) \\
& \qquad \Updownarrow \\
& \langle Bx^*, y - x^*\rangle \geq 0, \quad \forall y \in C \\
& \qquad \Updownarrow \\
& \langle -\lambda Bx^*, y - x^*\rangle \leq 0, \quad \forall \lambda > 0, \forall y \in C \\
& \qquad \Updownarrow \\
& \langle (I - \lambda Bx^*)x^* - x^*, y - x^*\rangle \leq 0, \quad \forall \lambda > 0, \forall y \in C \\
& \qquad \Updownarrow \\
& \langle (I - \lambda Bx^*)x^* - x^*, x^* - y\rangle \geq 0, \quad \forall \lambda > 0, \forall y \in C \\
& \qquad \Updownarrow \\
& x^* = P_C(I - \lambda B)x^*, \quad \forall \lambda > 0.
\end{aligned}$$

It is well known that if B is strongly monotone and Lipschitz continuous mapping on C, then (1.3) has a unique solution. There are several different approaches towards solving this problem in finite dimensional and infinite dimensional spaces see [8–14] and the research in this direction is intensively continued.

The equilibrium problem for a bifunction $f : C \times C \to R$ is to find a point $x \in C$ such that

$$f(x,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

We denote EP(f) by the solution set of (1.4). It is easy to see that $\mathrm{EP}(f) = \mathrm{VI}(C,B)$ when $f(x,y) = \langle Bx, y - x\rangle$ for all $x, y \in C$. Let $h : C \times C \to R$ be a nonlinear bifunction, then the generalized equilibrium problem (GEP) is to find $x^* \in C$ such that

$$f\bigl(x^*,x\bigr) + h\bigl(x^*,x\bigr) \geq 0, \quad \forall x \in C. \tag{1.5}$$

We denote the solution set of generalized equilibrium problem (1.5) by GEP(f,h). Note that this problem reduces to the equilibrium problem when the bifunction h is a zero mapping; this problem reduces to the mixed equilibrium problem when the bifunction $h(x^*,x) = \varphi(x) - \varphi(x^*)$, where $\varphi : C \to R \cup \{+\infty\}$ is for proper lower semicontinuous and convex functions.

The split generalized equilibrium problem (SGEP) introduced by Kazmi and Rizvi [15] in 2013 is the following problem: find $x^* \in C$

$$f\bigl(x^*,x\bigr) + h\bigl(x^*,x\bigr) \geq 0, \quad \forall x \in C,$$

such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F\bigl(y^*,y\bigr) + H\bigl(y^*,y\bigr) \geq 0, \quad \forall y \in Q,$$

where $f, h : C \times C \to R$ and $F, H : Q \times Q \to R$ are four nonlinear bifunctions and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. The solution set of the split generalized equilibrium problem SGEP is denoted by

$$\Omega = \bigl\{x^* \in \mathrm{GEP}(f,h) : Ax^* \in \mathrm{GEP}(F,H)\bigr\}.$$

If $H = 0$ and $F = 0$, then the split generalized equilibrium problem reduces to the generalized equilibrium problem considered by Cianciaruso et al. [16]; If $h = 0$ and $H = 0$, then the split generalized equilibrium problem reduces to the split equilibrium problem introduced in 2011 by Moudafi [17]; if $h = \varphi(\cdot,\cdot)$ and $H = \phi(\cdot,\cdot)$, where $\varphi : C \to R \cup \{+\infty\}$ and $\phi : Q \to R \cup \{+\infty\}$ are proper lower semicontinuous and convex functions, then the split generalized equilibrium problem reduces to the split mixed equilibrium problem (SEP).

In this paper, we are interested in finding the common solution for a finite family of the split generalized equilibrium problems, that is, find a $x^* \in C$,

$$f_i\bigl(x^*,x\bigr) + h_i\bigl(x^*,x\bigr) \geq 0, \quad \forall x \in C,$$

such that

$$y^* = A_i x^* \in Q \quad \text{solves} \quad F_i\bigl(y^*,y\bigr) + H_i\bigl(y^*,y\bigr) \geq 0, \quad \forall y \in Q,$$

where $f_i, h_i : C \times C \to R$ and $F_i, H_i : Q \times Q \to R$ are nonlinear bifunctions, $A_i : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, for $1 \leq i \leq N_1$.

In 2017, Majee and Nahak [18] introduced a hybrid viscosity iterative method to approximate a common solution of a split equilibrium problem and a fixed point problem of a finite collection of nonexpansive mappings; Onjai-uea and Phuengrattana [19] studied iterative algorithms for solving split mixed equilibrium problems and fixed point problems of hybrid multivalued mappings in real Hilbert spaces; Sitthithakerngkiet et al. [20] proposed an iterative method for finding a common solution of a single split generalized equilibrium problem, variational inequality problem and fixed point problem of nonexpansive mapping in Hilbert spaces. For recent developments in the analysis technique and algorithm design, see [21–25] and the references therein.

Motivated by the above related results in this field, in this paper, we first propose a new parallel hybrid viscosity method for finding a common element of the set of solutions of a finite family of split generalized equilibrium problems, variational inequality problems and the set of common fixed points of a finite family of demicontractive operators in Hilbert spaces.

The rest of the paper is organized as follows: Sect. 2 describes several definitions and lemmas which will be used in proving our main results; Sect. 3 presents a new parallel hybrid viscosity method for finding a common element of the set of solutions of a finite family of split generalized equilibrium problems, variational inequality problems and the set of common fixed points of a finite family of demicontractive operators in Hilbert spaces, and establish the corresponding strong convergence theorem under suitable conditions. Section 4 gives numerical examples to demonstrate the convergence of our algorithm.

2 Preliminaries

Throughout the paper, let the symbol $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence, respectively. In addition, $F(T)$ and $\omega_w(x_n)$ denote the fixed point set of T and the weak ω-limit set of the sequence $\{x_n\}$, respectively, that is, $F(T) = \{x : Tx = x\}$ and $\omega_w(x_n) = \{u : \exists x_{n_j} \rightharpoonup u\}$. In order to prove our main results, we recall some basic definitions and lemmas, which will be needed in the sequel.

Definition 2.1 ([26]) Assume that $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator, then $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in $\mathcal{H}$, the following implication holds:

$$x_n \rightharpoonup x \quad \text{and} \quad (I - T)x_n \to 0 \quad \Rightarrow \quad x \in F(T).$$

Lemma 2.2 ([27]) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $U : C \to C$ be a β-strict pseudo-contractive. Then $I - U$ is demiclosed at* 0.

Lemma 2.3 ([28]) *Suppose that $U : \mathcal{H} \to \mathcal{H}$ is a β-demicontractive mapping. Then the fixed point set $F(U)$ of U is closed and convex.*

Recall that P_C is the metric projection from $\mathcal{H}$ into C, then, for each point $x \in \mathcal{H}$, the unique point $P_C x \in C$ satisfies the property:

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Lemma 2.4 ([29]) *For a given* $x \in \mathcal{H}$:

(i) $z = P_C x$ *if and only if* $\langle x - z, z - y \rangle \geq 0$, $\forall y \in C$;

(ii) $z = P_C x$ *if and only if* $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall x, y \in C$;

(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in \mathcal{H}$.

It is obvious that P_C is nonexpansive and monotone.

Lemma 2.5 ([30]) *Let* $f : C \times C \to R$ *be a bifunction satisfying the following assumptions*:

(i) $f(x, x) \geq 0$, $\forall x \in C$;

(ii) f *is monotone, that is*, $f(x, y) + f(y, x) \leq 0$, $\forall x, y \in C$;

(iii) f *is upper hemicontinuous, that is, for each* $\forall x, y, z \in C$,

$$\limsup_{t \to 0} f\bigl(tz + (1 - t)x\bigr), y) \leq f(x, y);$$

(iv) *For each* $x \in C$ *fixed, the function* $y \mapsto f(x, y)$ *is convex and lower semicontinuous.*

Suppose that $h : C \times C \to R$ *is a bifunction satisfying the following assumptions*:

(i) $h(x, x) \geq 0$, $\forall x \in C$;

(ii) *for each* $y \in C$ *fixed, the function* $x \mapsto h(x, y)$ *is upper semicontinuous*;

(iii) *for each* $x \in C$ *fixed, the function* $y \mapsto h(x, y)$ *is convex and lower semicontinuous.*

Then, for fixed $r > 0$ *and* $z \in C$, *there exist a nonempty compact convex subset* K *of* $\mathcal{H}_1$ *and* $x \in C \cap K$ *such that*

$$f(y, x) + h(y, x) + \frac{1}{r}\langle y - x, x - z \rangle < 0, \quad \forall y \in C \setminus K.$$

Lemma 2.6 *Assume that* $f, h : C \times C \to R$ *satisfying Lemma* 2.5. *Let* $r > 0$ *and* $x \in \mathcal{H}_1$, *Then there exists* $z \in C$ *such that*

$$f(z, y) + h(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 ([31]) *Assume that* $f, h : C \times C \to R$ *satisfying Lemma* 2.5 *and* h *is monotone. For* $r > 0$ *and* $x \in \mathcal{H}_1$, *define the mapping* $T_r^{f,h} : H_1 \to C$ *as follows*:

$$T_r^{f,h}(x) := \left\{ z \in C : f(z, y) + h(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.$$

Then the following statements hold:

(i) $T_r^{f,h}$ *is single-valued*;

(ii) $T_r^{f,h}$ *is firmly nonexpansive, that is*,

$$\bigl\| T_r^{f,h}(x) - T_r^{f,h}(y) \bigr\|^2 \leq \bigl\langle T_r^{f,h}(x) - T_r^{f,h}(y), x - y \bigr\rangle, \quad \forall x, y \in \mathcal{H}_1;$$

(iii) $F(T_r^{f,h}) = \mathrm{GEP}(f, h)$;

(iv) $\mathrm{GEP}(f, h)$ *is compact and convex.*

Let $F, H : Q \times Q \to R$ satisfying Lemma 2.5. From the previous lemma, we can define a mapping $T_s^{F,H} : H_2 \to Q$ as follows:

$$T_s^{F,H}(w) := \left\{ d \in Q : F(d, e) + H(d, e) + \frac{1}{s}\langle e - d, d - w \rangle \geq 0, \forall e \in Q \right\},$$

where $s > 0$ and $w \in \mathcal{H}_2$, Then $T_s^{F,H} : \mathcal{H}_2 \to Q$ also satisfies the same properties in Lemma 2.7. Further, it is easy to prove that Ω is a closed and convex set. We see that Lemma 3.5 in [16] is a special case of Lemmas 2.6 and 2.7; for more details see [32].

Lemma 2.8 ([33]) *Let* $\{B_i\}_{i=1}^N$ *be a finite family of inverse strongly monotone mappings from* C *to* $\mathcal{H}$ *with the constants* $\{\beta_i\}_{i=1}^N$ *and assume that* $\bigcap_{i=1}^N \mathrm{VI}(C,B_i) \neq \emptyset$. *Let* $B = \sum_{i=1}^N \alpha_i B_i$, $\{\alpha_i\}_{i=1}^N \subset (0,1)$ *and* $\sum_{i=1}^N \alpha_i = 1$. *Then* $B : C \to \mathcal{H}$ *is a* β*-inverse strongly monotone mapping with* $\beta = \min\{\beta_1, \dots, \beta_N\}$ *and* $\mathrm{VI}(C,B) = \bigcap_{i=1}^N \mathrm{VI}(C,B_i)$.

A linear bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called *strongly positive* if and only if there exists $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in \mathcal{H}$. and we call such an A a strongly positive operator with coefficient $\overline{\gamma}$.

Lemma 2.9 ([34]) *Let* $\mathcal{H}$ *be a Hilbert space and let* A *be a strongly positive bounded linear operator on* $\mathcal{H}$ *with coefficient* $\overline{\gamma} > 0$. *If* $0 < \delta \leq \|A\|^{-1}$, *then* $\|I - \delta A\| \leq 1 - \delta\overline{\gamma}$.

Lemma 2.10 ([18]) *Let* $\mathcal{H}$ *be a Hilbert space. Let* $f : C \to C$ *be a* ρ*-Lipschitzian mapping and* $A : \mathcal{H} \to \mathcal{H}$ *be a strongly positive bounded linear operator with coefficient* $\delta > 0$. *If* $\mu\delta > \eta\rho$, *then*

$$\bigl\langle(\mu A - \eta f)x - (\mu A - \eta f)y, x - y\bigr\rangle \geq (\mu\delta - \eta\rho)\|x - y\|^2$$

for all $x, y \in \mathcal{H}$. *That is,* $\mu A - \eta f$ *is strongly monotone with coefficient* $\mu\delta - \eta\rho$.

Lemma 2.11 ([35]) *The following inequality holds in a Hilbert space* $\mathcal{H}$:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in \mathcal{H}.$$

Lemma 2.12 ([36]) *For each* $x_1, \dots, x_m \in \mathcal{H}$ *and* $\alpha_1, \dots, \alpha_m \in [0,1]$ *with* $\sum_{i=1}^n \alpha_i = 1$, *we have the equality*

$$\|\alpha_1 x_1 + \cdots + \alpha_m x_m\|^2 = \sum_{i=1}^m \alpha_i\|x_i\|^2 - \sum_{1 \leq i < j \leq m} \alpha_i\alpha_j\|x_i - x_j\|^2.$$

Lemma 2.13 ([37]) *Let* $\{a_n\}$ *be a sequence of non-negative real numbers, such that there exists a subsequence* $\{a_{n_j}\}$ *of* $\{a_n\}$, *such that* $a_{n_j} < a_{n_j+1}$ *for all* $j \in N$. *Then there exists a nondecreasing sequence* $\{m_k\}$ *of* N, *such that* $\lim_{k\to\infty} m_k = \infty$, *and the following properties are satisfied by all (sufficiently large) numbers* $k \in N$:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1, 2, \dots, k\}$, *such that* $a_n \leq a_{n+1}$.

Lemma 2.14 ([38]) *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy the conditions*:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *or equivalently*, $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$;
(ii) $\limsup_{n\to\infty} \beta_n \le 0$;
(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

3 Main results

Theorem 3.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert space. Let C and Q be nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $A_i : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and $A_i^* : \mathcal{H}_2 \to \mathcal{H}_1$ be the adjoint of A_i. Assume that $f_i, h_i : C \times C \to R$ and $F_i, H_i : Q \times Q \to R$ are bifunctions satisfying Lemma 2.5; h_i, H_i are monotone and F_i is upper semicontinuous for $1 \le i \le N_1$. Let $S_j : C \to C$ be a κ_j-demicontractive mappings such that $S_j - I$ is demiclosed at 0 for all $1 \le j \le N_2$. $B_l : C \to \mathcal{H}_1$ is a σ_l-inverse strongly monotone operator for all $1 \le l \le N_3$. Suppose that $\Gamma = \bigcap_{i=1}^{N_1} \Omega_i \cap (\bigcap_{j=1}^{N_2} F(S_j)) \cap (\bigcap_{l=1}^{N_3} \mathrm{VI}(C, B_l)) \neq \emptyset$. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a Lipschitzian mapping with coefficient $\rho \ge 0$. Let $L : \mathcal{H}_1 \to \mathcal{H}_1$ be a strongly positive bounded linear operator with coefficient $\delta > 0$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in \mathcal{H}_1$ by the following algorithm*:

$$\begin{cases} \omega_{n,i} = T_{r_{n,i}}^{f_i,h_i}(x_n + \xi_{n,i} A_i^*(T_{r_{n,i}}^{F_i,H_i} - I)A_i x_n), \\ \omega_n = \omega_{n,i_n}, \quad i_n = \arg\max_{1\le i\le N_1}\{\|\omega_{n,i} - x_n\|\}, \\ z_{n,j} = \beta_{n,j}\omega_n + (1-\beta_{n,j})S_j\omega_n, \\ y_n = P_C(I - \lambda_n(\sum_{l=1}^{N_3} \mu_l B_l))(\sum_{j=1}^{N_2} \nu_{n,j} z_{n,j}), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \eta L) y_n. \end{cases} \tag{3.1}$$

Also, the following conditions are satisfied:
(i) $\eta\delta > \gamma\rho$;
(ii) $\mu_l \in (0,1)$, $\sum_{l=1}^{N_3} \mu_l = 1$;
(iii) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iv) $r_{n,i} \subset (0,\infty)$, $\liminf_{n\to\infty} r_{n,i} > 0$;
(v) $\{\beta_{n,j}\} \subset (0,1)$, $\{\nu_{n,j}\} \subset (0,1)$, $\sum_{j=1}^{N_2} \nu_{n,j} = 1$, *for* $\forall n \ge 1$. $\liminf_{n\to\infty} \nu_{n,j}(1-\beta_{n,j})(\beta_{n,j} - \kappa_j) > 0$ *for* $\forall j \in \{1,\dots,N_2\}$;
(vi) $0 < \liminf_{n\to\infty} \xi_{n,i} \le \limsup_{n\to\infty} \xi_{n,i} < \frac{2}{\|A\|^2}$ *for* $1 \le i \le N_1$;
(vii) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\sigma$, $\sigma = \max_{1\le l\le N_3}\{\sigma_l\}$.
Then $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$, which is the unique solution of the following variational inequality:

$$\langle (\eta L - \gamma f)x^*, x^* - x \rangle \le 0, \quad \forall x \in \Gamma. \tag{3.2}$$

Equivalently, we have $x^* = P_\Gamma(I - \eta L + \gamma f)x^*$.

Proof First, we show the uniqueness of the solution of the variational inequality(3.2). We show it by contradiction. Suppose $\hat{x} \in \Gamma$ and $\tilde{x} \in \Gamma$ be two solution of (3.2) with $\hat{x} \neq \tilde{x}$. Then we have

$$\langle (\eta L - \gamma f)\hat{x}, \hat{x} - \tilde{x} \rangle \le 0$$

and

$$\langle (\eta L - \gamma f)\tilde{x}, \tilde{x} - \hat{x} \rangle \le 0.$$

We can obtain

$$\langle (\eta L - \gamma f)\hat{x} - (\eta L - \gamma f)\tilde{x}, \hat{x} - \tilde{x} \rangle \le 0.$$

From $\eta\delta > \gamma\rho$ and Lemma 2.9, we can get

$$\langle (\eta L - \gamma f)\hat{x} - (\eta L - \gamma f)\tilde{x}, \hat{x} - \tilde{x} \rangle \ge (\eta\delta - \gamma\rho)\|\hat{x} - \tilde{x}\|^2 \ge 0.$$

This leads to a contradiction. Hence, the variational inequality problem (3.2) has a unique solution and we denote it by $x^* \in \Gamma$.

We have

$$\langle (\eta L - \gamma f)x^*, x^* - x \rangle \le 0 \quad \Leftrightarrow \quad \langle \hat{x} - (I - \eta L + \gamma f)x^*, x^* - x \rangle \le 0, \quad \forall x \in \Gamma.$$

It is easy to verify Γ is closed and convex. From Lemma 2.4, we obtain $x^* = P_C(I - \eta L + \gamma f)x^*$.

Next, we show that the sequence $\{x_n\}$ is bounded. Let $p \in \Gamma$, that is, $p \in \Omega_i$, for $\forall i \in \{1, 2, \dots, N_1\}$, and we have $p = T_{r_{n,i}}^{f_i,h_i} p$ and $A_i p = T_{r_{n,i}}^{F_i,H_i} A_i p$. Observe that

$$\begin{aligned}
\|\omega_{n,i} - p\|^2 &= \big\| T_{r_{n,i}}^{f_i,h_i}\big(x_n + \xi_{n,i} A_i^*\big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n\big) - p \big\|^2 \\
&\le \big\| x_n + \xi_{n,i} A_i^*\big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n - p \big\|^2 \\
&= \|x_n - p\|^2 + 2\xi_{n,i}\big\langle A_i^*\big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n, x_n - p \big\rangle \\
&\quad + \xi_{n,i}^2 \big\| A_i^*\big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\|^2 \\
&\le \|x_n - p\|^2 + 2\xi_{n,i}\big\langle \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n, A_i x_n - A_i p \big\rangle \\
&\quad + \xi_{n,i}^2 \|A_i\|^2 \big\| \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\|^2.
\end{aligned}$$

Denoting $\Lambda = 2\xi_{n,i}\langle (T_{r_{n,i}}^{F_i,H_i} - I)A_i x_n, A_i x_n - A_i p \rangle$ and using (1.1), we get

$$\begin{aligned}
\Lambda &= 2\xi_{n,i}\big\langle \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n, A_i x_n - A_i p \big\rangle \\
&= 2\xi_{n,i}\big\langle \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n, A_i x_n - A_i p \\
&\quad + \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n - \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\rangle \\
&= 2\xi_{n,i}\big\{\big\langle \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n, T_{r_{n,i}}^{F_i,H_i} A_i x_n - A_i p \big\rangle - \big\| \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\|^2 \big\} \\
&\le 2\xi_{n,i}\left\{ \frac{1}{2}\big\| \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\|^2 - \big\| \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\|^2 \right\} \\
&\le -\xi_{n,i}\big\| \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\|^2.
\end{aligned}$$

Therefore, for $\forall i \in \{1, 2, \dots, N_1\}$, we have

$$\|\omega_{n,i} - p\|^2 \le \|x_n - p\|^2 + \big(\xi_{n,i}^2 \|A_i\|^2 - \xi_{n,i}\big)\big\| \big(T_{r_{n,i}}^{F_i,H_i} - I\big)A_i x_n \big\|^2. \tag{3.3}$$

From the condition (v), we obtain

$$\|\omega_n - p\| = \|\omega_{n,i_n} - p\| \le \|x_n - p\|. \tag{3.4}$$

From Lemma 2.12 and the definition of S_j, we have

$$\begin{aligned}
\|z_{n,j} - p\|^2 &= \bigl\|\beta_{n,j}\omega_n + (1-\beta_{n,j})S_j\omega_n - p\bigr\|^2 \\
&= \beta_{n,j}\|\omega_n - p\|^2 + (1-\beta_{n,j})\|S_j\omega_n - p\|^2 \\
&\quad - \beta_{n,j}(1-\beta_{n,j})\|\omega_n - S_j\omega_n\|^2 \\
&\le \beta_{n,j}\|\omega_n - p\|^2 + (1-\beta_{n,j})\bigl[\|\omega_n - p\|^2 + \kappa_j\|\omega_n - S_j\omega_n\|^2\bigr] \\
&\quad - \beta_{n,j}(1-\beta_{n,j})\|\omega_n - S_j\omega_n\|^2 \\
&= \|\omega_n - p\|^2 + (1-\beta_{n,j})(\kappa_j - \beta_{n,j})\|\omega_n - S_j\omega_n\|^2.
\end{aligned}$$

It follows from the condition (iv) that

$$\|z_{n,j} - p\| \le \|\omega_n - p\| \le \|x_n - p\|, \quad \forall j \in \{1, 2, \dots, N_2\}. \tag{3.5}$$

Let $B = \sum_{l=1}^{N_3} \mu_l B_l$ and $\sigma = \min\{\sigma_1, \dots, \sigma_{N_3}\}$, by Lemma 2.8, we know that B is σ-ism, and from the condition $0 < \lambda_n < 2\sigma$, we see that $I - \lambda_n B$ is nonexpansive, and $P_C(I - \lambda_n B)$ is also nonexpansive. We have $p \in \Gamma$, that is, $p \in \bigcap_{l=1}^{N_3} \mathrm{VI}(C, B_l) = \mathrm{VI}(C, B)$. Then from Lemma 2.12, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|P_C(I - \lambda_n B)\left(\sum_{j=1}^{N_2} \nu_{n,j} z_{n,j}\right) - p\right\|^2 \\
&= \left\|P_C(I - \lambda_n B)\left(\sum_{j=1}^{N_2} \nu_{n,j} z_{n,j}\right) - P_C(I - \lambda_n B)p\right\|^2 \\
&\le \left\|\sum_{j=1}^{N_2} \nu_{n,j} z_{n,j} - p\right\|^2 \\
&\le \sum_{j=1}^{N_2} \nu_{n,j}\|z_{n,j} - p\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{3.6}$$

From the condition $\lim_{n\to\infty} \alpha_n = 0$, we may assume, with no loss of generality, that $\alpha_n < \frac{1}{\eta\|L\|}$ for all n. It follows from Lemma 2.9 that

$$\begin{aligned}
\|x_{n+1} - p\| &= \bigl\|\alpha_n \gamma f(x_n) + (I - \alpha_n \eta L)y_n - p\bigr\| \\
&= \bigl\|\alpha_n\bigl(\gamma f(x_n) - \eta L p\bigr) + (I - \alpha_n \eta L)(y_n - p)\bigr\| \\
&\le \alpha_n\bigl\|\gamma f(x_n) - \eta L p\bigr\| + \|I - \alpha_n \eta L\|\|y_n - p\| \\
&= \alpha_n\bigl\|\gamma f(x_n) - \gamma f(p) + \gamma f(p) - \eta L p\bigr\| + \|1 - \alpha_n \eta L\|\|y_n - p\|
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\gamma\rho\|x_n-p\|+\alpha_n\bigl\|\gamma f(p)-\eta Lp\bigr\|+(1-\alpha_n\eta\delta)\|x_n-p\| \\
&= \bigl[1-\alpha_n(\eta\delta-\gamma\rho)\bigr]\|x_n-p\|+\alpha_n\bigl\|\gamma f(p)-\eta Lp\bigr\| \\
&= \bigl[1-\alpha_n(\eta\delta-\gamma\rho)\bigr]\|x_n-p\|+\alpha_n(\eta\delta-\gamma\rho)\frac{\|\gamma f(p)-\eta Lp\|}{\eta\delta-\gamma\rho} \\
&\le \max\biggl\{\|x_n-p\|,\frac{\|\gamma f(p)-\eta Lp\|}{\eta\delta-\gamma\rho}\biggr\} \\
&\le \cdots \\
&\le \max\biggl\{\|x_1-p\|,\frac{\|\gamma f(p)-\eta Lp\|}{\eta\delta-\gamma\rho}\biggr\}.
\end{aligned}$$

That is, $\{x_n\}$ is bounded, and $\{y_n\}$, $\{z_{n,j}\}$, $\{\omega_{n,i}\}$, $\{f(x_n)\}$ and $\{S_j\omega_n\}$ are also bounded.

Next, we show $\omega_\omega(x_n)\subseteq\Gamma$. To see this, we take $q\in\omega_\omega(x_n)$ and assume that $x_{n_l}\rightharpoonup q$ as $l\to\infty$ for some subsequence $\{x_{n_l}\}$ of $\{x_n\}$. Observe that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \bigl\|\alpha_n\gamma f(x_n)+(I-\alpha_n\eta L)y_n-p\bigr\|^2 \\
&= \bigl\|\alpha_n\bigl(\gamma f(x_n)-\eta Lp\bigr)+(I-\alpha_n\eta L)(y_n-p)\bigr\|^2 \\
&= \alpha_n^2\bigl\|\gamma f(x_n)-\eta Lp\bigr\|^2+\bigl\|(I-\alpha_n\eta L)(y_n-p)\bigr\|^2 \\
&\quad+2\alpha_n\bigl\langle\gamma f(x_n)-\eta Lp,(I-\alpha_n\eta L)(y_n-p)\bigr\rangle \\
&\le \alpha_n^2\bigl\|\gamma f(x_n)-\eta Lp\bigr\|^2+(1-\alpha_n\eta\delta)^2\|y_n-p\|^2 \\
&\quad+2\alpha_n(1-\alpha_n\eta\delta)\bigl\|\gamma f(x_n)-\eta Lp\bigr\|\|y_n-p\| \\
&\le \alpha_n^2\bigl\|\gamma f(x_n)-\eta Lp\bigr\|^2+(1-\alpha_n\eta\delta)^2\|y_n-p\|^2 \\
&\quad+2\alpha_n(1-\alpha_n\eta\delta)\bigl\|\gamma f(x_n)-\eta Lp\bigr\|\|x_n-p\|.
\end{aligned}\tag{3.7}$$

From (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n^2\bigl\|\gamma f(x_n)-\eta Lp\bigr\|^2+(1-\alpha_n\eta\delta)^2\sum_{j=1}^{N_2}\nu_{n,j}\|z_{n,j}-p\|^2 \\
&\quad+2\alpha_n(1-\alpha_n\eta\delta)\bigl\|\gamma f(x_n)-\eta Lp\bigr\|\|x_n-p\| \\
&\le \alpha_n^2\bigl\|\gamma f(x_n)-\eta Lp\bigr\|^2+(1-\alpha_n\eta\delta)^2\sum_{j=1}^{N_2}\nu_{n,j}\bigl[\|\omega_n-p\|^2 \\
&\quad+(1-\beta_{n,j})(\kappa_j-\beta_{n,j})\|\omega_n-S_j\omega_n\|^2\bigr] \\
&\quad+2\alpha_n(1-\alpha_n\eta\delta)\bigl\|\gamma f(x_n)-\eta Lp\bigr\|\|x_n-p\| \\
&\le \alpha_n^2\bigl\|\gamma f(x_n)-\eta Lp\bigr\|^2+(1-\alpha_n\eta\delta)^2\sum_{j=1}^{N_2}\nu_{n,j}\bigl[\|x_n-p\|^2 \\
&\quad+\bigl(\xi_{n,i_n}^2\|A_{i_n}\|^2-\xi_{n,i_n}\bigr)\bigl\|\bigl(T_{r_{n,i_n}}^{F_{i_n},H_{i_n}}-I\bigr)A_{i_n}x_n\bigr\|^2 \\
&\quad+(1-\beta_{n,j})(\kappa_j-\beta_{n,j})\|\omega_n-S_j\omega_n\|^2\bigr] \\
&\quad+2\alpha_n(1-\alpha_n\eta\delta)\bigl\|\gamma f(x_n)-\eta Lp\bigr\|\|x_n-p\| \\
&= \alpha_n^2\bigl\|\gamma f(x_n)-\eta Lp\bigr\|^2+(1-\alpha_n\eta\delta)^2\|x_n-p\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha_n\eta\delta)^2\big(\xi_{n,i_n}^2\|A_{i_n}\|^2 - \xi_{n,i_n}\big)\big\|\big(T_{r_{n,i_n}}^{F_{i_n},H_{i_n}} - I\big)A_{i_n}x_n\big\|^2 \\
&+ (1-\alpha_n\eta\delta)^2\sum_{j=1}^{N_2}\nu_{n,j}(1-\beta_{n,j})(\kappa_j - \beta_{n,j})\|\omega_n - S_j\omega_n\|^2 \\
&+ 2\alpha_n(1-\alpha_n\eta\delta)\big\|\gamma f(x_n) - \eta Lp\big\|\|x_n - p\|. \qquad (3.8)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(1-\alpha_n\eta\delta)^2\big(\xi_{n,i_n} - \xi_{n,i_n}^2\|A_{i_n}\|^2\big)\big\|\big(T_{r_{n,i_n}}^{F_{i_n},H_{i_n}} - I\big)A_{i_n}x_n\big\|^2 \leq{}& \alpha_n^2\big\|\gamma f(x_n) - \eta Lp\big\|^2 \\
&- \|x_{n+1} - p\|^2 \\
&+ (1-\alpha_n\eta\delta)^2\|x_n - p\|^2 \\
&+ 2\alpha_n(1-\alpha_n\eta\delta) \\
&\times \big\|\gamma f(x_n) - \eta Lp\big\| \\
&\times \|x_n - p\| \qquad (3.9)
\end{aligned}
$$

and

$$
\begin{aligned}
(1-\alpha_n\eta\delta)^2\sum_{j=1}^{N_2}\nu_{n,j}(1-\beta_{n,j})(\beta_{n,j} - \kappa_j)\|\omega_n - S_j\omega_n\|^2 \leq{}& \alpha_n^2\big\|\gamma f(x_n) - \eta Lp\big\|^2 \\
&- \|x_{n+1} - p\|^2 \\
&+ (1-\alpha_n\eta\delta)^2\|x_n - p\|^2 \\
&+ 2\alpha_n(1-\alpha_n\eta\delta) \\
&\times \big\|\gamma f(x_n) - \eta Lp\big\| \\
&\times \|x_n - p\|. \qquad (3.10)
\end{aligned}
$$

Next, we analyze the inequality (3.9) and (3.10) by considering the following two cases.

Case 1. Assume that there exists n_0 large enough such that $\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$ for all $n \geq n_0$. Since $\|x_n - p\|^2$ is bounded, we see that $\lim_{n\to\infty}\|x_n - p\|^2$ exists. From the conditions (iii) and (v), we obtain

$$\big\|\big(T_{r_{n,i_n}}^{F_{i_n},H_{i_n}} - I\big)A_{i_n}x_n\big\| \to 0 \quad (n\to\infty)$$

and

$$\|\omega_n - S_j\omega_n\| \to 0 \quad (n\to\infty), \forall j \in \{1,2,\ldots,N_2\}.$$

Then we have

$$\|z_{n,j} - \omega_n\| = (1-\beta_{n,j})\|S_j\omega_n - \omega_n\| \to 0 \quad (n\to\infty), \forall j \in \{1,2,\ldots,N_2\}. \qquad (3.11)$$

Since $p = T_{r_{n,i}}^{f_i,h_i} p$ and $T_{r_{n,i}}^{f_i,h_i}$ is firmly nonexpansive, we obtain

$$\begin{aligned}
\|\omega_{n,i} - p\|^2 &= \bigl\| T_{r_{n,i}}^{f_i,h_i}\bigl(x_n + \xi_{n,i}A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr) - p\bigr\|^2 \\
&= \bigl\| T_{r_{n,i}}^{f_i,h_i}\bigl(x_n + \xi_{n,i}A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr) - T_{r_{n,i}}^{f_i,h_i} p\bigr\|^2 \\
&\le \bigl\langle \omega_{n,i} - p, x_n + \xi_{n,i}A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n - p\bigr\rangle \\
&= \frac{1}{2}\bigl\{\|\omega_{n,i} - p\|^2 + \bigl\|x_n + \xi_{n,i}A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n - p\bigr\|^2 \\
&\quad - \bigl\|\omega_{n,i} - x_n - \xi_{n,i}A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\|^2\bigr\} \\
&= \frac{1}{2}\bigl\{\|\omega_{n,i} - p\|^2 + \|x_n - p\|^2 + \xi_{n,i}^2\bigl\|A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\|^2 \\
&\quad + 2\xi_{n,i}\bigl\langle x_n - p, A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\rangle \\
&\quad - \bigl[\|\omega_{n,i} - x_n\|^2 + \xi_{n,i}^2\bigl\|A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\|^2 \\
&\quad - 2\xi_{n,i}\bigl\langle \omega_{n,i} - x_n, A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\rangle\bigr]\bigr\} \\
&= \frac{1}{2}\bigl\{\|\omega_{n,i} - p\|^2 + \|x_n - p\|^2 - \|\omega_{n,i} - x_n\|^2 \\
&\quad + 2\xi_{n,i}\bigl\langle A_i\omega_{n,i} - A_i p, \bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\rangle\bigr\} \\
&\le \frac{1}{2}\bigl\{\|\omega_{n,i} - p\|^2 + \|x_n - p\|^2 - \|\omega_{n,i} - x_n\|^2 \\
&\quad + 2\xi_{n,i}\|A_i\omega_{n,i} - A_i p\|\bigl\|\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\|\bigr\}.
\end{aligned}$$

Then we get

$$\|\omega_{n,i} - p\|^2 \le \|x_n - p\|^2 - \|\omega_{n,i} - x_n\|^2 + 2\xi_{n,i}\|A_i\omega_{n,i} - A_i p\|\bigl\|\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_i x_n\bigr\|.$$

From (3.8), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n^2\bigl\|\gamma f(x_n) - \eta L p\bigr\|^2 + (1 - \alpha_n\eta\delta)^2\sum_{j=1}^{N_2}\nu_{n,j}\bigl[\|\omega_n - p\|^2 \\
&\quad + (1 - \beta_{n,j})(\kappa_j - \beta_{n,j})\|\omega_n - S_j\omega_n\|^2\bigr] \\
&\quad + 2\alpha_n(1 - \alpha_n\eta\delta)\bigl\|\gamma f(x_n) - \eta L p\bigr\|\|x_n - p\| \\
&\le \alpha_n^2\bigl\|\gamma f(x_n) - \eta L p\bigr\|^2 + (1 - \alpha_n\eta\delta)^2\|\omega_n - p\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n\eta\delta)\bigl\|\gamma f(x_n) - \eta L p\bigr\|\|x_n - p\| \\
&\le \alpha_n^2\bigl\|\gamma f(x_n) - \eta L p\bigr\|^2 + (1 - \alpha_n\eta\delta)^2\bigl[\|x_n - p\|^2 - \|\omega_n - x_n\|^2 \\
&\quad + 2\xi_{n,i_n}\|A_{i_n}\omega_n - A_{i_n}p\|\bigl\|\bigl(T_{r_{n,i_n}}^{F_{i_n},H_{i_n}} - I\bigr)A_{i_n}x_n\bigr\|\bigr] \\
&\quad + 2\alpha_n(1 - \alpha_n\eta\delta)\bigl\|\gamma f(x_n) - \eta L p\bigr\|\|x_n - p\|.
\end{aligned}$$

Then

$$\begin{aligned}
(1 - \alpha_n\eta\delta)^2\|\omega_n - x_n\|^2 &\le \alpha_n^2\bigl\|\gamma f(x_n) - \eta L p\bigr\|^2 + (1 - \alpha_n\eta\delta)^2\|x_n - p\|^2 \\
&\quad - \|x_{n+1} - p\|^2 + 2(1 - \alpha_n\eta\delta)^2\xi_{n,i_n}
\end{aligned}$$

$$\begin{aligned}&\times \|A_{i_n}\omega_n - A_{i_n}p\| \big\| \big(T_{r_{n,i_n}}^{F_{i_n},H_{i_n}} - I\big)A_{i_n}x_n \big\| \\ &+ 2\alpha_n(1-\alpha_n\eta\delta)\big\|\gamma f(x_n) - \eta Lp\big\| \|x_n - p\|.\end{aligned}$$

Since $\alpha_n \to 0$, $\|(T_{r_{n,i_n}}^{F_{i_n},H_{i_n}} - I)A_{i_n}x_n\| \to 0$, as $n \to \infty$, and $\lim_{n\to\infty}\|x_n - p\|$ exists, we obtain

$$\|\omega_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.12}$$

From (3.11), we have

$$\|z_{n,j} - x_n\| \to 0 \quad (n \to \infty).$$

Let $z_n = \sum_{j=1}^{N_2} \nu_{n,j} z_{n,j}$, then $y_n = P_C(I - \lambda_n B)z_n$, and we have

$$\begin{aligned}\|z_n - x_n\| &\le \left\| \sum_{j=1}^{N_2} \nu_{n,j} z_{n,j} - x_n \right\| \\ &\le \sum_{j=1}^{N_2} \nu_{n,j} \|z_{n,j} - x_n\| \\ &\to 0 \quad (n \to \infty).\end{aligned} \tag{3.13}$$

Since B is σ-ism, we obtain

$$\begin{aligned}\|y_n - p\|^2 &= \big\|P_C(I - \lambda_n B)z_n - p\big\|^2 \\ &= \big\|P_C(I - \lambda_n B)z_n - P_C(I - \lambda_n B)p\big\|^2 \\ &\le \big\|(I - \lambda_n B)z_n - (I - \lambda_n B)p\big\|^2 \\ &= \|z_n - p\|^2 + \lambda_n^2\|Bz_n - Bp\|^2 - 2\lambda_n\langle z_n - p, Bz_n - Bp\rangle \\ &\le \|z_n - p\|^2 + \big(\lambda_n^2 - 2\lambda_n\sigma\big)\|Bz_n - Bp\|^2.\end{aligned}$$

From (3.7), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n^2\big\|\gamma f(x_n) - \eta Lp\big\|^2 + (1-\alpha_n\eta\delta)^2\|y_n - p\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n\eta\delta)\big\|\gamma f(x_n) - \eta Lp\big\|\|x_n - p\| \\ &\le \alpha_n^2\big\|\gamma f(x_n) - \eta Lp\big\|^2 + (1-\alpha_n\eta\delta)^2\big[\|z_n - p\|^2 \\ &\quad + \big(\lambda_n^2 - 2\lambda_n\sigma\big)\|Bz_n - Bp\|^2\big] \\ &\quad + 2\alpha_n(1-\alpha_n\eta\delta)\big\|\gamma f(x_n) - \eta Lp\big\|\|x_n - p\|.\end{aligned}$$

Then

$$\begin{aligned}(1-\alpha_n\eta\delta)^2\big(2\lambda_n\sigma - \lambda_n^2\big)\|Bz_n - Bp\|^2 &\le \alpha_n^2\big\|\gamma f(x_n) - \eta Lp\big\|^2 \\ &\quad + (1-\alpha_n\eta\delta)^2\|z_n - p\|^2 \\ &\quad - \|x_{n+1} - p\|^2 + 2\alpha_n(1-\alpha_n\eta\delta)\end{aligned}$$

$$\begin{aligned}
&\times \left\|\gamma f(x_n) - \eta Lp\right\| \|x_n - p\| \\
&\le \alpha_n^2 \left\|\gamma f(x_n) - \eta Lp\right\|^2 \\
&\quad + (1 - \alpha_n\eta\delta)^2 \|x_n - p\|^2 \\
&\quad - \|x_{n+1} - p\|^2 + 2\alpha_n(1 - \alpha_n\eta\delta) \\
&\quad \times \left\|\gamma f(x_n) - \eta Lp\right\| \|x_n - p\|.
\end{aligned}$$

Since $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\sigma$, $\alpha_n \to 0$, as $n \to \infty$, and $\lim_{n\to\infty} \|x_n - p\|$ exists, we obtain

$$\|Bz_n - Bp\| \to 0 \quad (n \to \infty).$$

Since P_C is firmly nonexpansive, we obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|P_C(I - \lambda_n B)z_n - p\right\|^2 \\
&= \left\|P_C(I - \lambda_n B)z_n - P_C(I - \lambda_n B)p\right\|^2 \\
&\le \left\langle y_n - p, (I - \lambda_n B)z_n - (I - \lambda_n B)p\right\rangle \\
&= \frac{1}{2}\Big\{\|y_n - p\|^2 + \left\|(I - \lambda_n B)z_n - (I - \lambda_n B)p\right\|^2 \\
&\quad - \left\|y_n - z_n + \lambda_n(Bz_n - Bp)\right\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|y_n - p\|^2 + \|z_n - p\|^2 + \lambda_n^2\|Bz_n - Bp\|^2 \\
&\quad - 2\lambda_n\langle z_n - p, Bz_n - Bp\rangle - \big[\|y_n - z_n\|^2 + \lambda_n^2\|Bz_n - Bp\|^2 \\
&\quad + 2\lambda_n\langle y_n - z_n, Bz_n - Bp\rangle\big]\Big\} \\
&= \frac{1}{2}\Big\{\|y_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n\|^2 \\
&\quad - 2\lambda_n\langle y_n - p, Bz_n - Bp\rangle\Big\} \\
&\le \frac{1}{2}\Big\{\|y_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\lambda_n\|y_n - p\| \|Bz_n - Bp\|\Big\}.
\end{aligned}$$

Then

$$\|y_n - p\|^2 \le \|z_n - p\|^2 - \|y_n - z_n\|^2 + 2\lambda_n\|y_n - p\| \|Bz_n - Bp\|.$$

From (3.7), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n^2\left\|\gamma f(x_n) - \eta Lp\right\|^2 + (1 - \alpha_n\eta\delta)^2\|y_n - p\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n\eta\delta)\left\|\gamma f(x_n) - \eta Lp\right\| \|x_n - p\| \\
&\le \alpha_n^2\left\|\gamma f(x_n) - \eta Lp\right\|^2 + (1 - \alpha_n\eta\delta)^2\big[\|z_n - p\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\lambda_n\|y_n - p\| \|Bz_n - Bp\|\big] + 2\alpha_n(1 - \alpha_n\eta\delta) \\
&\quad \times \left\|\gamma f(x_n) - \eta Lp\right\| \|x_n - p\|
\end{aligned}$$

$$\leq \alpha_n^2 \left\| \gamma f(x_n) - \eta Lp \right\|^2 + (1-\alpha_n\eta\delta)^2 \big[\|x_n - p\|^2 - \|y_n - z_n\|^2 + 2\lambda_n \|x_n - p\| \|Bz_n - Bp\| \big] + 2\alpha_n(1-\alpha_n\eta\delta) \times \left\| \gamma f(x_n) - \eta Lp \right\| \|x_n - p\|.$$

Then we have

$$(1-\alpha_n\eta\delta)^2 \|y_n - z_n\|^2 \leq \alpha_n^2 \left\| \gamma f(x_n) - \eta Lp \right\|^2 + (1-\alpha_n\eta\delta)^2 \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\lambda_n(1-\alpha_n\eta\delta)^2 \times \|x_n - p\| \|Bz_n - Bp\| + 2\alpha_n(1-\alpha_n\eta\delta) \times \left\| \gamma f(x_n) - \eta Lp \right\| \|x_n - p\|.$$

Since $\alpha_n \to 0$, $\|Bz_n - Bp\| \to 0$, as $n \to \infty$, and $\lim_{n\to\infty} \|x_n - p\|$ exists, we obtain

$$\|y_n - z_n\| \to 0 \quad (n \to \infty).$$

Since $x_{n_l} \rightharpoonup q$ as $l \to \infty$ for some subsequence $\{x_{n_l}\}$ of $\{x_n\}$. from (3.12), we have $\omega_{n_l} \rightharpoonup q$ as $l \to \infty$ for some subsequence $\{\omega_{n_l}\}$ of $\{\omega_n\}$. Again since $\lim_{n\to\infty} \|\omega_{n_l} - S_j\omega_{n_l}\| = 0$ and $S_j - I$ are demiclosed at 0, for each $j \in \{1, 2, \ldots, N_2\}$, it follows from Definition 2.1 that $q \in \bigcap_{i=1}^{N_2} F(S_j)$.

Since $y_n = P_C(I - \lambda_n B)z_n$ and $\lambda_n > 0$ is bounded, with no loss of generality, we may assume that

$$\lambda_{n_l} \to \lambda \quad (l \to \infty).$$

Then we have

$$\begin{aligned}
\left\| z_{n_l} - P_C(I - \lambda B)z_{n_l} \right\| &= \left\| z_{n_l} - P_C(I - \lambda_{n_l} B)z_{n_l} \right\| \\
&\quad + \left\| P_C(I - \lambda_{n_l} B)z_{n_l} - P_C(I - \lambda B)z_{n_l} \right\| \\
&\leq \|z_{n_l} - y_{n_l}\| + \left\| (I - \lambda_{n_l} B)z_{n_l} - (I - \lambda B)z_{n_l} \right\| \\
&= \|z_{n_l} - y_{n_l}\| + |\lambda_{n_l} - \lambda| \|Bz_{n_l}\| \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

From (3.13), we have $z_{n_l} \rightharpoonup q$ as $l \to \infty$ for some subsequence $\{z_{n_l}\}$ of $\{z_n\}$. Again since $P_C(I - \lambda B)$ is nonexpansive and we have Lemma 2.2, we obtain $q \in \mathrm{VI}(C, B)$. It follows from Lemma 2.8 that $q \in \bigcap_{l=1}^{N_3} \mathrm{VI}(C, B_l)$.

From (3.1) and (3.12), we have

$$\|\omega_{n,i} - x_n\| \to 0 \quad \text{as } n \to \infty, 1 \leq i \leq N_1,$$

and from (3.3), we obtain

$$\left\| \left(T_{r_{n,i}}^{F_i, H_i} - I \right) A_i x_n \right\| \to 0 \quad \text{as } n \to \infty, 1 \leq i \leq N_1. \tag{3.14}$$

Let $\omega_{n,i} = T_{r_{n,i}}^{f_i,h_i}\upsilon_{n,i}$, where $\upsilon_{n,i} = x_n + \xi_{n,i}A_i^*(T_{r_{n,i}}^{F_i,H_i} - I)A_ix_n$, and we have

$$\begin{aligned}\|\upsilon_{n,i} - x_n\| &= \bigl\|\xi_{n,i}A_i^*\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_ix_n\bigr\| \\ &\le \xi_{n,i}\|A_i\|\bigl\|\bigl(T_{r_{n,i}}^{F_i,H_i} - I\bigr)A_ix_n\bigr\| \\ &\to 0 \quad (n\to\infty).\end{aligned}$$

Then we have $\|\omega_{n,i} - \upsilon_{n,i}\| \to 0$ as $n\to\infty$, $1\le i\le N_1$.

Since $\omega_{n,i} = T_{r_{n,i}}^{f_i,h_i}\upsilon_{n,i}$, we have

$$f_i(\omega_{n,i},\omega) + h_i(\omega_{n,i},\omega) + \frac{1}{r_{n,i}}\langle\omega - \omega_{n,i},\omega_{n,i} - \upsilon_{n,i}\rangle \ge 0, \quad \forall\omega\in C,$$

then

$$h_i(\omega_{n,i},\omega) + \frac{1}{r_{n,i}}\langle\omega - \omega_{n,i},\omega_{n,i} - \upsilon_{n,i}\rangle \ge -f_i(\omega_{n,i},\omega) \ge f_i(\omega,\omega_{n,i}), \quad \forall\omega\in C.$$

Since $\|\omega_{n,i} - \nu_{n,i}\| \to 0$, $\omega_{n,i} \rightharpoonup q$, f_i is lower semicontinuous in the second argument and h_i is upper semicontinuous in the first argument, we obtain

$$h_i(q,\omega) \ge f_i(\omega,q), \quad \forall\omega\in C.$$

Then we have

$$f_i(\omega,q) + h_i(\omega,q) \le f_i(\omega,q) - h_i(q,\omega) \le 0, \quad \forall\omega\in C.$$

Let $\hat\omega = t\omega + (1-t)q \in C$, we have $\hat\omega\in C$ and $f_i(\hat\omega,q) + h_i(\hat\omega,q) \le 0$. Observe that

$$\begin{aligned}0 &= f_i(\hat\omega,\hat\omega) + h_i(\hat\omega,\hat\omega) \\ &= t\bigl[f_i(\hat\omega,\omega) + h_i(\hat\omega,\omega)\bigr] + (1-t)\bigl[f_i(\hat\omega,q) + h_i(\hat\omega,q)\bigr] \\ &\le t\bigl[f_i(\hat\omega,\omega) + h_i(\hat\omega,\omega)\bigr].\end{aligned}$$

Hence

$$f_i(\hat\omega,\omega) + h_i(\hat\omega,\omega) \ge 0, \quad \forall\omega\in C.$$

Since f_i is upper hemicontinuous and h_i is upper semicontinuous in the first argument, we have

$$f_i(q,\omega) + h_i(q,\omega) \ge 0, \quad \forall\omega\in C.$$

That is, $q\in \mathrm{GEP}(f_i,h_i)$, for $\forall i\in\{1,\dots,N_1\}$.

Next, we show that $A_iq\in \mathrm{GEP}(F_i,H_i)$. Since $x_{n_l}\rightharpoonup q$ and continuity of A_i, we have $A_ix_{n_l}\rightharpoonup A_iq$. Let $\vartheta_{n,i} = A_ix_n - T_{r_{n,i}}^{F_i,H_i}A_ix_n$, from (3.14), we have $\lim_{n\to\infty}\vartheta_{n,i} = 0$, for $\forall i\in\{1,\dots,N_1\}$. And since $T_{r_{n,i}}^{F_i,H_i}A_ix_n = A_ix_n - \tau_{n,i}$, for $\forall\varepsilon\in Q$, we have

$$F_i(A_ix_n - \vartheta_{n,i},\varepsilon) + H_i(A_ix_n - \vartheta_{n,i},\varepsilon) + \frac{1}{r_{n,i}}\bigl\langle\varepsilon - (A_ix_n - \vartheta_{n,i}),(A_ix_n - \vartheta_{n,i}) - A_ix_n\bigr\rangle \ge 0.$$

Since F_i and H_i are upper semicontinuous in the first argument, we have

$$F_i(A_iq,\varepsilon) + H_i(A_iq,\varepsilon) \geq 0, \quad \forall \varepsilon \in Q.$$

Then we obtain $A_iq \in \mathrm{GEP}(F_i,H_i)$, for $\forall i \in \{1,\dots,N_1\}$. Therefore we conclude that $q \in \Gamma$.

Next, we show that

$$\limsup_{n\to\infty}\langle(\eta L - \gamma f)x^*, x^* - x_n\rangle \leq 0. \tag{3.15}$$

Indeed, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\eta L - \gamma f)x^*, x^* - x_n\rangle = \lim_{j\to\infty}\langle(\eta L - \gamma f)x^*, x^* - x_{n_j}\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_j} \rightharpoonup \bar{x} \in \Gamma$. Then we obtain

$$\limsup_{n\to\infty}\langle(\eta L - \gamma f)x^*, x^* - x_n\rangle = \langle(\eta L - \gamma f)x^*, x^* - \bar{x}\rangle \leq 0.$$

Finally, we show that $x_n \to x^*(n \to \infty)$. From Lemma 2.9, Lemma 2.11 and (3.6), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\eta L)y_n - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - \eta Lx^*) + (I - \alpha_n\eta L)(y_n - x^*)\|^2 \\
&\leq (1 - \alpha_n\eta\delta)^2\|y_n - x^*\|^2 + 2\alpha_n\langle\gamma f(x_n) - \eta Lx^*, x_{n+1} - x^*\rangle \\
&= (1 - \alpha_n\eta\delta)^2\|y_n - x^*\|^2 + 2\alpha_n\langle\gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle\gamma f(x^*) - \eta Lx^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \alpha_n\eta\delta)^2\|y_n - x^*\|^2 + 2\alpha_n\gamma\rho\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\langle\gamma f(x^*) - \eta Lx^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \alpha_n\eta\delta)^2\|x_n - x^*\|^2 + \alpha_n\gamma\rho(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad + 2\alpha_n\langle\gamma f(x^*) - \eta Lx^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
(1 - \alpha_n\gamma\rho)\|x_{n+1} - x^*\|^2 &\leq [(1 - \alpha_n\eta\delta)^2 + \alpha_n\gamma\rho]\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(x^*) - \eta Lx^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

Since $\eta\delta > \gamma\rho$ and $0 < \alpha_n \leq \frac{1}{\eta\|L\|} \leq \frac{1}{\eta\delta}$, we have $1 - \alpha_n\gamma\rho > 1 - \alpha_n\eta\delta \geq 0$. Hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \frac{(1 - \alpha_n\eta\delta)^2 + \alpha_n\gamma\rho}{1 - \alpha_n\gamma\rho}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma\rho} \\
&\quad \times \langle\gamma f(x^*) - \eta Lx^*, x_{n+1} - x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&= \left[1 - \frac{2\alpha_n(\eta\delta - \gamma\rho)}{1 - \alpha_n\gamma\rho}\right]\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma\rho} \\
&\quad \times \langle \gamma f(x^*) - \eta L x^*, x_{n+1} - x^* \rangle + \frac{\alpha_n^2\eta^2\delta^2}{1 - \alpha_n\gamma\rho}\|x_n - x^*\|^2 \\
&\leq \left[1 - \frac{2\alpha_n(\eta\delta - \gamma\rho)}{1 - \alpha_n\gamma\rho}\right]\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n(\eta\delta - \gamma\rho)}{1 - \alpha_n\gamma\rho}\left(\frac{\langle \gamma f(x^*) - \eta L x^*, x_{n+1} - x^* \rangle}{\eta\delta - \gamma\rho} + \alpha_n M\right),
\end{aligned}$$

where M is the constant satisfying

$$M = \sup_{n\geq 0}\left\{\frac{\eta^2\delta^2}{2(\eta\delta - \gamma\rho)}\|x_n - x^*\|^2\right\}.$$

From the condition $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$ and (3.15), we have

$$\sum_{n=0}^{\infty}\frac{2\alpha_n(\eta\delta - \gamma\rho)}{1 - \alpha_n\gamma\rho} > \sum_{n=0}^{\infty}2\alpha_n(\eta\delta - \gamma\rho) = \infty$$

and

$$\limsup_{n\to\infty}\left(\frac{\langle \gamma f(x^*) - \eta L x^*, x_{n+1} - x^* \rangle}{\eta\delta - \gamma\rho} + \alpha_n M\right) \leq 0.$$

By Lemma 2.14, we obtain $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Case 2. Assume that there exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2$ for all $j \in N$. Then it follows from Lemma 2.13 that there exists a nondecreasing sequence $\{m_k\}$ of N, such that $\lim_{k\to\infty} m_k = \infty$, and the following inequalities hold for all $k \in N$:

$$\|x_{m_k} - p\|^2 \leq \|x_{m_k+1} - p\|^2 \quad \text{and} \quad \|x_k - p\|^2 \leq \|x_{m_k+1} - p\|^2. \tag{3.16}$$

Similarly, we get

$$\begin{aligned}
&\|\omega_{m_k,i} - x_n\| \to 0 \quad (k \to \infty), \forall i \in \{1, 2, \ldots, N_1\}, \\
&\|\omega_{m_k} - S_j\omega_{m_k}\| \to 0 \quad (k \to \infty), \forall j \in \{1, 2, \ldots, N_2\}, \\
&\|z_{m_k,j} - \omega_{m_k}\| \to 0 \quad (k \to \infty), \forall j \in \{1, 2, \ldots, N_2\}, \\
&\|z_{m_k} - x_{m_k}\| \to 0 \quad (k \to \infty), \\
&\|y_{m_k} - z_{m_k}\| \to 0 \quad (k \to \infty), \\
&\|\omega_{m_k,i} - \upsilon_{m_k,i}\| \to 0 \quad (k \to \infty), \forall i \in \{1, 2, \ldots, N_1\}, \\
&\left\|A_i x_{m_k} - T_{r_{m_k,i}}^{F_i,H_i} A_i x_{m_k}\right\| \to 0 \quad (k \to \infty), \forall i \in \{1, 2, \ldots, N_1\}.
\end{aligned}$$

By a similar argument to that in Case 1, we have $\omega_\omega(x_{m_k}) \subset \Gamma$.

Also, we obtain

$$\limsup_{k\to\infty}\langle(\eta L-\gamma f)x^*,x^*-x_{m_k}\rangle\le 0$$

and

$$\|x_{m_k+1}-x^*\|^2\le\left[1-\frac{2\alpha_{m_k}(\eta\delta-\gamma\rho)}{1-\alpha_{m_k}\gamma\rho}\right]\|x_{m_k}-x^*\|^2 + \frac{2\alpha_{m_k}(\eta\delta-\gamma\rho)}{1-\alpha_{m_k}\gamma\rho}\left(\frac{\langle\gamma f(x^*)-\eta Lx^*,x_{m_k+1}-x^*\rangle}{\eta\delta-\gamma\rho}+\alpha_{m_k}M\right),$$

where M is the constant satisfying

$$M=\sup_{k\ge 0}\left\{\frac{\eta^2\delta^2}{2(\eta\delta-\gamma\rho)}\|x_{m_k}-x^*\|^2\right\}.$$

By a similar argument to that in Case 1, we obtain $\|x_{m_k}-x^*\|\to 0$ as $k\to\infty$. By (3.16), we get $\|x_k-x^*\|\le\|x_{m_k}-x^*\|$, $\forall k\in N$. Therefore, $x_k\to x^*$ as $k\to\infty$. □

Remark 3.2 We present several corollaries of Theorem 3.1 and we can consider the following cases:

(i) $A_i=A$, for $1\le i\le N_1$;
(ii) $h_i=\varphi_i(\cdot,\cdot)$ and $H_i=\phi_i(\cdot,\cdot)$, where $\varphi_i:C\to R\cup\{+\infty\}$ and $\phi_i:Q\to R\cup\{+\infty\}$ is proper lower semicontinuous and convex functions, for $1\le i\le N_1$;
(iii) $h_i=0$ and $H_i=0$, for $1\le i\le N_1$;
(iv) $H_i=0$ and $F_i=0$, for $1\le i\le N_1$;
(v) $N_1=N_2=N_3=1$.
(vi) S_j is κ_j-strict pseudo-contractive for any j. From Lemma 2.2, we can remove the condition that S_j-I is demiclosed at 0 of Theorem 3.1;
(vii) S_j is a nonspreading mapping or an N-generalized hybrid mapping. We note that nonspreading mappings with nonempty fixed point set and N-generalized hybrid mappings with nonempty fixed point set are quasi-nonexpansive operators.

4 Numerical examples

In this section, we give two numerical examples to demonstrate the convergence of our algorithm. All codes were written in Matlab 2010b and run on Dell i-5 Dual-Core laptop.

Example 4.1 We consider the case that $N_1=N_2=N_3=1$.

Let $\mathcal{H}_1=\mathcal{H}_2=R$, $C=Q=[-20,10]$ and let $f_1,h_1:C\times C\to R$ and $F_1,H_1:Q\times Q\to R$ be defined by $f_1(z,y)=y^2+3zy-4z^2$, $h_1(z,y)=y^2-z^2$ and $F_1(z,y)=3y^2+2zy-5z^2$, $H_1(z,y)=0$. Define S_1,A_1,B_1,f,L as follows:

$$S_1(x)=\begin{cases}x, & x\in(-\infty,0),\\ -2x, & x\in[0,\infty).\end{cases}$$

$A_1x=-x$, $B_1x=\frac{1}{2}x$ and $f(x)=L(x)=2x$. Put $\alpha_n=\frac{1}{n+1}$, $\beta_{n,1}=\frac{1}{2}+\frac{1}{n+2}$, $\xi_{n,1}=1-\frac{1}{n+1}$, $\lambda_n=\frac{1}{5}$ and $\gamma=\frac{1}{2}$. It is easy to verify that f_1, h_1, F_1, H_1, S_1, A_1, B_1, f, L, α_n, $\beta_{n,1}$, $\xi_{n,1}$, λ_n, γ satisfy all the conditions of Theorem 3.1. Then, by Lemma 2.7, we see that $T_r^{f_1,h_1}$ and $T_r^{F_1,H_1}$ are single-value mappings on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Hence, for $r_n = r > 0$, $x \in \mathcal{H}_1$ and $x \in \mathcal{H}_2$, there exist $z_1 \in C$ and $z_2 \in Q$ such that

$$f_1(z_1,y) + h_1(z_1,y) + \frac{1}{r}\langle y - z_1, z_1 - x\rangle \geq 0, \quad \forall y \in C,$$

and

$$F_1(z_2,y) + H_1(z_2,y) + \frac{1}{r}\langle y - z_2, z_2 - x\rangle \geq 0, \quad \forall y \in Q.$$

We can reform the above inequalities to standard quadratic form in the variable y as follows:

$$P_1(y) = 2ry^2 + (2rz_1 + z_1 - x)y + \left(xz_1 - 5rz_1^2 - z_1^2\right) \geq 0, \quad \forall y \in C,$$

and

$$P_2(y) = 3ry^2 + (2rz_2 + z_2 - x)y + \left(xz_2 - 5rz_2^2 - z_2^2\right) \geq 0, \quad \forall y \in Q.$$

It is easy to check that the discriminants of the above two quadratic inequalities are nonnegative. And since $P_1(y) \geq 0$ for all $y \in C$ and $P_2(y) \geq 0$ for all $y \in Q$, we see that the discriminant must be zero. Then we obtain $z_1 = T_r^{f_1,h_1}(x) = \frac{x}{1+7r}$ and $z_2 = T_r^{F_1,H_1}(x) = \frac{x}{1+8r}$. It is clear that $\Gamma = \{0\}$.

In what follows, we observe the convergence of the Algorithm 3.1 by considering the following three cases:

Case 1. Taking the different initial point $x_1 = -5, 1, 5$ with $\eta = 1$, $r = 4$, Fig. 1 presents the convergence behaviors of $\{x_n\}$ for Algorithm 3.1.

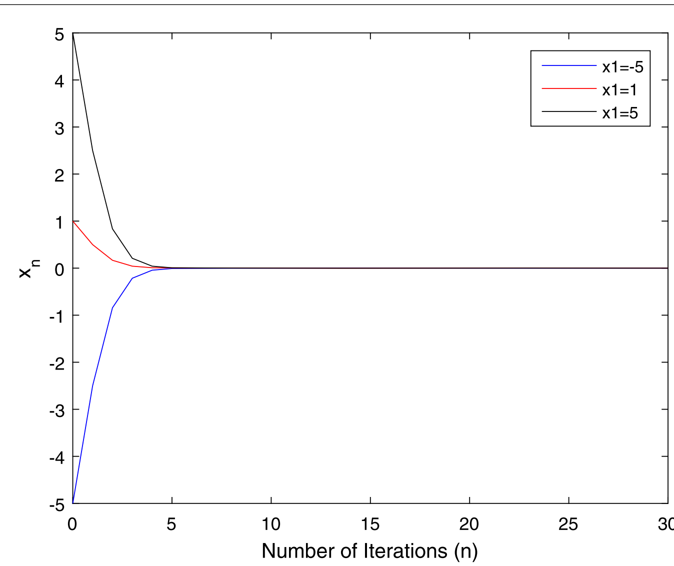

Figure 1 Behaviors of $\{x_n\}$ with initial points $x_1 = -5, 1, 5$, $\eta = 1$ and $r = 4$

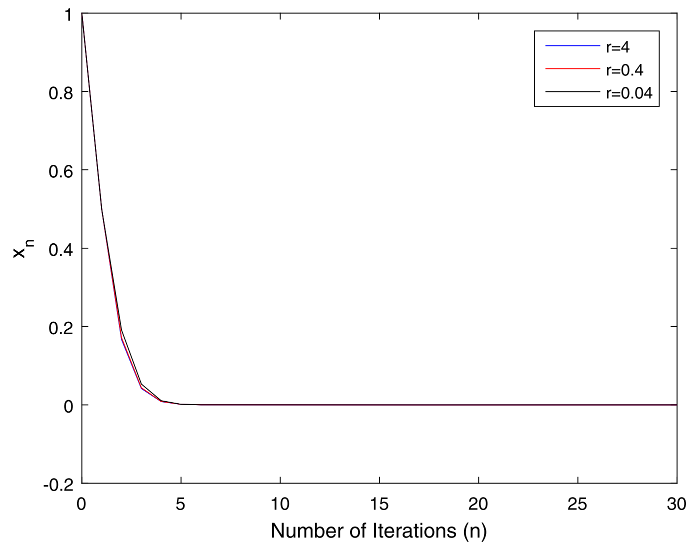

Figure 2 Behaviors of $\{x_n\}$ with different $r = 4, 0.4, 0.04$, $x_1 = 1$ and $\eta = 1$

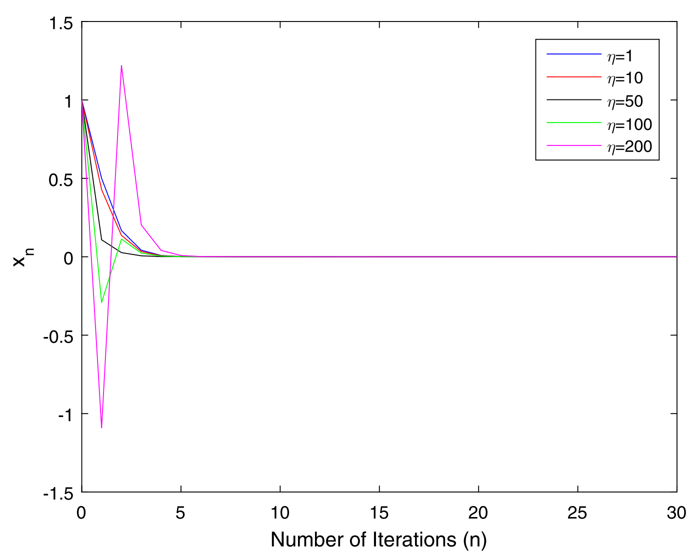

Figure 3 Behaviors of $\{x_n\}$ with different $\eta = 1, 10, 50, 100, 200$, $x_1 = 1$ and $r = 4$

Case 2. Taking the different $r = 4, 0.4, 0.04$ with $x_1 = 1$, $\eta = 1$, Fig. 2 presents the convergence behaviors of $\{x_n\}$ for Algorithm 3.1.

Case 3. Taking the different $\eta = 1, 10, 50, 100, 200$ with $x_1 = 1$, $r = 4$, Fig. 3 presents the convergence behaviors of $\{x_n\}$ for Algorithm 3.1.

Example 4.2 We consider the case that $N_1 = N_2 = N_3 = 2$.

Let $H_1 = H_2 = R$, $C = Q = [-20, 10]$ and let $f_i, h_i : C \times C \to R$ and $F_i, H_i : Q \times Q \to R$ be defined by $f_1(z, y) = y^2 + 3zy - 4z^2$, $h_1(z, y) = y^2 - z^2$, $F_1(z, y) = 3y^2 + 2zy - 5z^2$, $H_1(z, y) = 0$ and $f_2(z, y) = y^2 - z^2$, $h_2(z, y) = 0$, $F_2(z, y) = y^2 + 3zy - 4z^2$, $H_2(z, y) = \frac{1}{2}(y^2 - z^2)$. Let S_1, A_1, B_1, f,

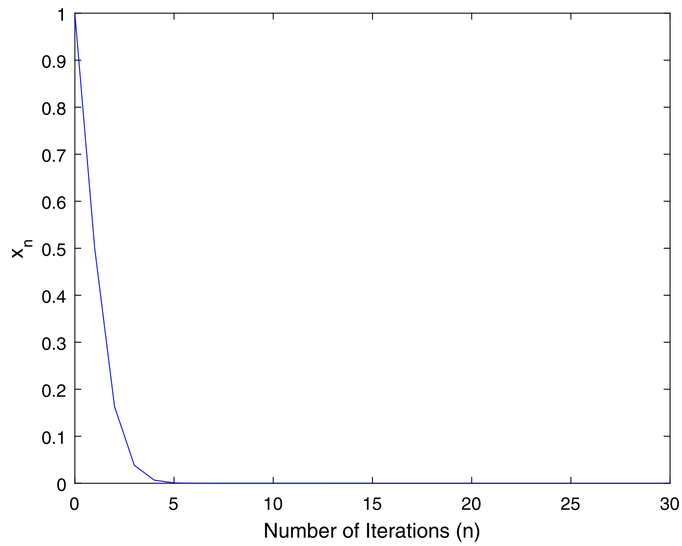

Figure 4 Behaviors of $\{x_n\}$ with $x_1 = 1$

L, α_n, $\beta_{n,1}$, $\xi_{n,1}$, λ_n, γ be the same as that of Example 4.1, and define S_2, A_2, B_2 by $S_2x = \frac{4}{5}x$, $A_2x = 2x$ and $B_2x = 2x$. Put $\beta_{n,2} = \frac{1}{2} - \frac{1}{n+2}$, $\xi_{n,2} = \frac{1}{2}(1 - \frac{1}{n+1})$, $\nu_{n,1} = \frac{1}{2} - \frac{1}{n+2}$, $\nu_{n,2} = \frac{1}{2} + \frac{1}{n+2}$, $r_{n,1} = r_{n,2} = r = 4$, $\mu_1 = \mu_2 = \frac{1}{2}$ and $\eta = 1$. It is easy to verify that they satisfy all the conditions of Theorem 3.1.

Following an argument similar to that of Example 4.1, we obtain $T_r^{f_1,h_1}(x) = \frac{x}{1+7r}$, $T_r^{F_1,H_1}(x) = \frac{x}{1+8r}$, $T_r^{f_2,h_2}(x) = \frac{x}{1+2r}$ and $T_r^{F_2,H_2}(x) = \frac{x}{1+6r}$ and $\Gamma = \{0\}$. Figure 4 presents the convergence behaviors of $\{x_n\}$ for Algorithm 3.1.

5 Conclusion

In this paper, we first propose a new parallel hybrid viscosity method for finding a common element of the set of solutions of a finite family of split generalized equilibrium problems, variational inequality problems and the set of common fixed points of a finite family of demicontractive operators in Hilbert spaces. And then we establish the corresponding strong convergence theorem under suitable conditions. We study more general split equilibrium problems and fixed point problems of operators than those in [18]. Our results in this paper improve and extend many recent results in the literature. Finally, we present numerical examples to demonstrate the effectiveness of our algorithm.

Acknowledgements

The author is most grateful to the referees and the editor for their helpful comments and advice which helped to improve the contents of this paper.

Funding

Not applicable.

Availability of data and materials

All data generated or analysed during this study are included in this published article.

Competing interests

The author declares to have no competing interests.

Authors' contributions

The author completed the work alone, and read and approved the final manuscript.

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Received: 27 February 2019 Accepted: 30 May 2019 Published online: 11 June 2019

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