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Inequalities for curvature integrals in Euclidean plane

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Abstract

Let γ be a closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, and $\tilde{A}_1$ denote the oriented area of the domain enclosed by the locus of curvature centers of γ. Pan and Xu conjectured that there exists a best constant C such that

$$L^2 - 4\pi A \leq C|\tilde{A}_1|,$$

with equality if and only if γ is a circle. In this paper, we give an affirmative answer to this question. Moreover, instead of working with the domain enclosed by the locus of curvature centers we consider the domain enclosed by the locus of width centers of γ, and we obtain some new reverse isoperimetric inequalities.

MSC: 52A10; 52A22

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1 Introductions and main results

The notion of curvature is one of the central concepts of differential geometry; it also plays a key role in physics. The magnitude of a force required to move an object at constant speed along a curved path is, according to Newton's laws, a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein, by the curvature of space-time. All sorts of shapes, from soap bubbles to red blood cells, seem to be determined by various curvatures.

A geometric inequality describes the relation among the invariants of geometric object in space. The isoperimetric inequality is one of the most ancient and beautiful geometric inequality. It states: For a closed curve γ in the Euclidean plane $\mathbb{R}^2$, its length L and area A enclosed by γ satisfies

$$L^2 - 4\pi A \geq 0, \tag{1.1}$$

where equality holds if and only if γ is a circle. It follows that the circle is the only curve of constant length L enclosing a maximum area. One can find more stronger isoperimetric inequalities in [1–15].

There are many well-known inequalities or equalities about curvature, such as the Ros theorem, the Fenchel theorem, and the Willmore theorem. There are also many interesting results involving integrals of curvature for a plane curve, one can refer to [16–19].

In [18], Pan and Zhang established a reverse isoperimetric inequality for the plane curves under some assumptions on curvature. If γ is a closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, then

$$L^2 - 4\pi A \leq 4\pi |\tilde{A}_1|, \tag{1.2}$$

where $\tilde{A}_1$ denotes the oriented area of the domain enclosed by the locus of curvature centers of γ, and the equality holds if and only if γ is a circle.

In [20], Pan and Xu posed the following question: Does there exist a best constant C such that

$$L^2 - 4\pi A \leq C|\tilde{A}_1|,$$

with equality if and only if γ is a circle?

In this paper, we give a new method to prove the reverse isoperimetric inequality (1.2). Meanwhile, we give an affirmative answer to Pan and Xu's question.

Theorem 1 *Let γ be a C^2 closed and strictly convex curve with length L in the Euclidean plane $\mathbb{R}^2$. Let A and $\tilde{A}_1$ denote the area of the domain enclosed by γ and the oriented area of the domain enclosed by the locus of curvature centers of γ, respectively. Then, for any $\varepsilon > 0$,*

$$L^2 - 4\pi A \leq (1+\varepsilon)\pi |\tilde{A}_1|,$$

where equality holds if and only if γ is a circle.

Let D be a domain bounded by γ, and the width $w(\theta)$ of D in direction $u(\theta) = (\cos\theta, \sin\theta)$ is defined to be the distance between two tangents to γ be perpendicular to $u(\theta)$. It is clearly that

$$w(\theta) = p(\theta) + p(\theta + \pi). \tag{1.3}$$

If the point has a distance of $w(\theta)/2$ from γ along the unit inward normal vector, we call it as the width center of γ. Let ζ denote the locus of width centers of γ, then $\zeta(\theta)$ can be given by

$$\zeta(\theta) = \gamma(\theta) + \frac{w(\theta)}{2} N(\theta), \tag{1.4}$$

where $N(\theta) = (-\cos\theta, -\sin\theta)$ is the unit inward normal vector along γ.

In Sect. 4, we research the domain enclosed by the locus of width centers instead of the domain enclosed by the locus of curvature centers of closed strictly convex curve γ, and establish the following reverse isoperimetric inequality.

Theorem 2 *Let γ be a C^2 closed and strictly convex curve with length L in the Euclidean plane $\mathbb{R}^2$. Let A denote the area of the domain enclosed by γ. Then, for any $\varepsilon > 0$,*

$$L^2 - 4\pi A \leq \frac{3+\varepsilon}{2}\pi\big(|\tilde{A}_1| - |\tilde{A}_2|\big).$$

The equality holds if and only if γ is a circle. Here $\tilde{A}_1$ denotes the oriented area of the domain enclosed by the locus of curvature centers of γ and $\tilde{A}_2$ denotes the oriented area of the domain enclosed by ζ.

2 Preliminaries

In this section, we recall some facts about Minkowski's support function and Fourier series. Without loss of generality, suppose γ is a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane. Take a point O inside γ as the origin of our frame. Let p be the oriented perpendicular distance from O to the tangent line at a point on γ, and θ the oriented angle from the positive x_1-axis to this perpendicular ray. Clearly, p is a single-valued periodic function of θ with period 2π and γ can be parameterized in terms of θ and $p(\theta)$ as follows:

$$\gamma(\theta) = \big(x_1(\theta), x_2(\theta)\big) = \big(p(\theta)\cos\theta - p'(\theta)\sin\theta, p(\theta)\sin\theta + p'(\theta)\cos\theta\big). \tag{2.1}$$

The couple $(\theta, p(\theta))$ is usually called the polar tangential coordinate on γ, and $p(\theta)$ is the support function of γ.

Then the curvature κ of γ can be calculated,

$$\kappa(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0.$$

The radius of the curvature ρ of γ is given by

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta).$$

Since the support function of a given convex curve γ is always continuous, bounded and 2π-periodic, it has the following Fourier series:

$$p(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta), \tag{2.2}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} p(\theta)\,d\theta,$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} p(\theta)\cos n\theta\,d\theta,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} p(\theta)\sin n\theta\,d\theta,$$

for $n \geq 1$.

Then we have

$$p'(\theta) = \sum_{n=1}^{\infty}(-na_n \sin n\theta + nb_n \cos n\theta), \tag{2.3}$$

$$p''(\theta) = \sum_{n=1}^{\infty}\bigl[-n^2(a_n \cos n\theta + b_n \sin n\theta)\bigr]. \tag{2.4}$$

Let L be the length of γ, and A be the area of the domain enclosed by γ, then

$$L = \int_\gamma ds = \int_0^{2\pi} \bigl(p(\theta) + p''(\theta)\bigr)\, d\theta = \int_0^{2\pi} p(\theta)\, d\theta, \tag{2.5}$$

$$A = \frac{1}{2}\int_\gamma p(\theta)\, ds = \frac{1}{2}\int_0^{2\pi} \bigl(p^2(\theta) - p'^2(\theta)\bigr)\, d\theta, \tag{2.6}$$

$$\int_\gamma \frac{1}{\kappa}\, ds = \int_0^{2\pi} \rho^2\, d\theta = \int_0^{2\pi} \bigl(p(\theta) + p''(\theta)\bigr)^2\, d\theta. \tag{2.7}$$

By (2.2), (2.3), (2.4) and the Parseval equality, the above inequalities (2.5), (2.6), (2.7) can be, respectively, rewritten as

$$L = \pi a_0, \tag{2.8}$$

$$A = \frac{1}{4}\pi a_0^2 + \frac{1}{2}\pi \sum_{n=1}^{\infty}\bigl(1 - n^2\bigr)\bigl(a_n^2 + b_n^2\bigr), \tag{2.9}$$

$$\int_\gamma \frac{1}{\kappa}\, ds = \frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty}\bigl(1 - n^2\bigr)^2\bigl(a_n^2 + b_n^2\bigr). \tag{2.10}$$

3 Formulas of curvature

In this section, we will give a new proof for the inequalities about curvature which obtained in [19] and [18]. Meanwhile, we obtain some new reverse isoperimetric inequalities for convex plane curves.

Lemma 1 ([19]) *Let γ be a C^2 closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with the length L and enclosing a domain D of area A. Let κ be the curvature of γ, then*

$$\frac{L^2}{2\pi} \le \int_\gamma \frac{1}{\kappa}\, ds,$$

where equality holds if and only if γ is a circle.

Proof By (2.8) and (2.10) we can easily get

$$\begin{aligned} L^2 &= \pi^2 a_0^2 \\ &\le 2\pi\left(\frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty}\bigl(1 - n^2\bigr)^2\bigl(a_n^2 + b_n^2\bigr)\right) \\ &= 2\pi \int_\gamma \frac{1}{\kappa}\, ds, \end{aligned}$$

where equality holds if and only if $a_n = b_n = 0$ for $n > 1$, from (2.2), this shows $p(\theta) = a_0/2 + a_1 \cos\theta + b_1 \sin\theta$, which together with (2.1) implies that γ is a circle with center (a_1, b_1) and radius of $a_0/2$. □

Lemma 2 ([19]) *Let γ be a C^2 closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ enclosing a domain D of area A. Let κ be the curvature of γ, then*

$$2A \le \int_\gamma \frac{1}{\kappa}\, ds,$$

where equality holds if and only if γ is a circle.

Proof By (2.10) and (2.9) we have

$$\begin{aligned}\int_\gamma \frac{1}{\kappa}\, ds &= \frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} \bigl(1-n^2\bigr)^2 \bigl(a_n^2 + b_n^2\bigr) \\ &= \frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} \bigl(1-n^2\bigr)\bigl(a_n^2 + b_n^2\bigr) + \pi \sum_{n=1}^{\infty} n^2\bigl(n^2-1\bigr)\bigl(a_n^2 + b_n^2\bigr) \\ &\ge \frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} \bigl(1-n^2\bigr)\bigl(a_n^2 + b_n^2\bigr) \\ &= 2A,\end{aligned}$$

where equality holds if and only if $a_n = b_n = 0$ for $n > 1$. By the same argument as in the proof of Lemma 1, we conclude that γ is a circle. □

In [18], Pan and Zhang investigated the properties of the locus of curvature centers of a closed strictly convex plane curve γ which is given by (2.1). Let β denote the locus of curvature centers of γ. Then $\beta(\theta) = (\bar{x}_1(\theta), \bar{x}_2(\theta))$ can be given by

$$\begin{aligned}\beta(\theta) &= \gamma(\theta) + \rho(\theta)N(\theta) \\ &= \bigl(-p'(\theta)\sin\theta - p''(\theta)\cos\theta, p'(\theta)\cos\theta - p''(\theta)\sin\theta\bigr),\end{aligned} \tag{3.1}$$

where $N(\theta) = (-\cos\theta, -\sin\theta)$ is the unit inward normal vector along γ.

We calculate the oriented area, denoted by $\tilde{A}_1$, of β by Green's formula. From (3.1), we get

$$\bar{x}_1\, d\bar{x}_2 - \bar{x}_2\, d\bar{x}_1 = p'(\theta)\bigl(p'(\theta) + p'''(\theta)\bigr)\, d\theta,$$

and thus $\tilde{A}_1$ is given by

$$\begin{aligned}\tilde{A}_1 &= \frac{1}{2}\int_\gamma \bar{x}_1\, d\bar{x}_2 - \bar{x}_2\, d\bar{x}_1 \\ &= \frac{1}{2}\int_0^{2\pi} p'(\theta)\bigl(p'(\theta) + p'''(\theta)\bigr)\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \bigl(p'^2(\theta) - p''^2(\theta)\bigr)\, d\theta.\end{aligned}$$

By (2.3), (2.4) and the Parseval equality, we have

$$\tilde{A}_1 = \frac{1}{2}\pi \sum_{n=1}^{\infty} n^2(1-n^2)(a_n^2+b_n^2) \leq 0, \tag{3.2}$$

where equality holds if and only if $a_n = b_n = 0$ for $n > 1$, namely γ is a circle.

Via (2.9), (3.2) and (2.10) we have

$$\begin{aligned} 2(A + |\tilde{A}_1|) &= \frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} (n^2-1)^2(a_n^2+b_n^2) \\ &= \int_\gamma \frac{1}{\kappa}\, ds. \end{aligned} \tag{3.3}$$

Then we can prove the following inequalities.

Lemma 3 *Let γ be a C^2 closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with the length L and enclosing a domain D of area A. Let κ be the curvature of γ, then*

$$L^2 - 4\pi A \leq 2\pi\left(\int_\gamma \frac{1}{\kappa}\, ds - \frac{L^2}{2\pi}\right), \tag{3.4}$$

$$L^2 - 4\pi A \leq 2\pi\left(\int_\gamma \frac{1}{\kappa}\, ds - 2A\right), \tag{3.5}$$

each equality holds if and only if γ is a circle.

Proof Via (2.8), (2.9) and (2.10), we have

$$\begin{aligned} 2\pi \int_\gamma \frac{1}{\kappa}\, ds - L^2 &= 2\pi^2 \sum_{n=1}^{\infty} (1-n^2)^2(a_n^2+b_n^2) \\ &\geq 2\pi^2 \sum_{n=1}^{\infty} (n^2-1)(a_n^2+b_n^2) \\ &= L^2 - 4\pi A. \end{aligned}$$

Hence, we get (3.4), and equality holds if and only if $a_n = b_n = 0$ for $n > 1$, namely γ is a circle. (3.5) is an immediate consequence according to (3.4) and the isoperimetric inequality (1.1). □

The inequality (3.4) is new. Substituting (3.3) into (3.5), we obtain the following reverse isoperimetric inequality as shown in [18].

Corollary 1 ([18]) *Let γ be a C^2 closed and strictly convex curve with length L in the Euclidean plane, κ be the curvature of γ, A the area enclosed by γ and $\tilde{A}_1$ the oriented area enclosed by β. Then*

$$L^2 - 4\pi A \leq 4\pi |\tilde{A}_1|,$$

where equality holds if and only if γ is a circle.

By (3.3) and (3.4), it immediately yields the following.

Corollary 2 *Let γ be a C^2 closed and strictly convex curve with length L in the Euclidean plane, κ be the curvature of γ, A the area enclosed by γ and $\tilde{A}_1$ the oriented area enclosed by β. Then*

$$L^2 - 4\pi A \leq 2\pi |\tilde{A}_1|,$$

where equality holds if and only if γ is a circle.

We are now in the position to prove Theorem 1.

Proof of Theorem 1 Because of $\varepsilon > 0$, so $n^2(1+\varepsilon)/4 > 1$ for $n \geq 2$. Together with this fact and (2.8), (2.9) and (3.2), we have

$$\begin{aligned} L^2 - 4\pi A &= 2\pi^2 \sum_{n=1}^{\infty} (n^2-1)(a_n^2+b_n^2) \\ &\leq 2\pi^2 \sum_{n=1}^{\infty} \frac{1+\varepsilon}{4} n^2 (n^2-1)(a_n^2+b_n^2) \\ &= \frac{1+\varepsilon}{2}\pi^2 \sum_{n=1}^{\infty} n^2(n^2-1)(a_n^2+b_n^2) \\ &= (1+\varepsilon)\pi |\tilde{A}_1|, \end{aligned}$$

equality holds if and only if $a_n = b_n = 0$ for $n > 1$, namely γ is a circle. □

Remark 1 The proof of Theorem 1 shows that the coefficient $(1+\varepsilon)\pi$ of $|\tilde{A}_1|$ is the best constant for the equality condition. In fact: suppose there exists a constant $\lambda < (1+\varepsilon)\pi$, such that

$$L^2 - 4\pi A \leq \lambda |\tilde{A}_1|,$$

with the equality holds if and only if γ is a circle. Since ε is arbitrary, $\lambda \leq \pi$. If $\lambda = \pi$, the equality holds in Theorem 1 if and only if $a_n = b_n = 0$ for $n > 2$. That means there exists a γ that is not circle when the equality holds.

4 Stronger reverse isoperimetric inequality

From the definition of the width (1.3), (1.4) and (2.1), $\zeta(\theta) = (\tilde{x}_1(\theta), \tilde{x}_2(\theta))$ can be given by

$$\begin{cases} \tilde{x}_1(\theta) = \frac{p(\theta)-p(\theta+\pi)}{2}\cos\theta - p'(\theta)\sin\theta, \\ \tilde{x}_2(\theta) = \frac{p(\theta)-p(\theta+\pi)}{2}\sin\theta + p'(\theta)\cos\theta. \end{cases} \tag{4.1}$$

Lemma 4 *Let γ be a C^2 closed and strictly convex curve in the Euclidean plane, D be a domain bounded by γ, w be the width function of D, A be the area enclosed by γ and $\tilde{A}_2$ be the oriented area enclosed by ζ. Then $\tilde{A}_2 > 0$, $\tilde{A}_2 = 0$ or $\tilde{A}_2 < 0$.*

Proof Using Green's formula and (4.1), we have

$$\begin{aligned}
&\tilde{x}_1\,d\tilde{x}_2 - \tilde{x}_2\,d\tilde{x}_1 \\
&\quad = \frac{1}{2}\biggl(p(\theta)p''(\theta) + p'^2(\theta) + p'(\theta)p'(\theta+\pi) - p''(\theta)p(\theta+\pi) \\
&\qquad + \frac{1}{2}\bigl(p(\theta) - p(\theta+\pi)\bigr)^2\biggr)\,d\theta,
\end{aligned}$$

and thus $\tilde{A}_2$ is given by

$$\begin{aligned}
\tilde{A}_2 &= \frac{1}{2}\int_\gamma \tilde{x}_1\,d\tilde{x}_2 - \tilde{x}_2\,d\tilde{x}_1 \\
&= \frac{1}{4}\int_0^{2\pi}\biggl[p(\theta)p''(\theta) + p'^2(\theta) + p'(\theta)p'(\theta+\pi) \\
&\quad - p''(\theta)p(\theta+\pi) + \frac{1}{2}\bigl(p(\theta) - p(\theta+\pi)\bigr)\biggr]^2\,d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\biggl(2p'(\theta)p'(\theta+\pi) + \frac{1}{2}\bigl[p(\theta) - p(\theta+\pi)\bigr]^2\biggr)\,d\theta.
\end{aligned}$$

By (2.2) we get

$$p(\theta+\pi) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(-1)^n(a_n\cos n\theta + b_n\sin n\theta), \tag{4.2}$$

$$p'(\theta+\pi) = \sum_{n=1}^{\infty}(-1)^n(-na_n\sin n\theta + nb_n\cos n\theta). \tag{4.3}$$

Via the Parseval equality, we have

$$\begin{aligned}
\tilde{A}_2 &= \frac{\pi}{2}\sum_{n=1}^{\infty}(-1)^n n^2\bigl(a_n^2 + b_n^2\bigr) + \frac{\pi}{8}\sum_{n=1}^{\infty}\bigl(1-(-1)^n\bigr)^2\bigl(a_n^2 + b_n^2\bigr) \\
&= \frac{\pi}{2}\sum_{k=1}^{\infty}\Bigl((2k)^2\bigl(a_{2k}^2 + b_{2k}^2\bigr) + \bigl[1-(2k-1)^2\bigr]\bigl(a_{2k-1}^2 + b_{2k-1}^2\bigr)\Bigr) \\
&= B_1 + B_2,
\end{aligned} \tag{4.4}$$

where

$$B_1 = \frac{\pi}{2}\sum_{k=1}^{\infty}(2k)^2\bigl(a_{2k}^2 + b_{2k}^2\bigr) \geq 0, \tag{4.5}$$

$$B_2 = \frac{\pi}{2}\sum_{k=1}^{\infty}\bigl(1-(2k-1)^2\bigr)\bigl(a_{2k-1}^2 + b_{2k-1}^2\bigr) \leq 0. \tag{4.6}$$

(I) If $a_0 = 20$, $a_2 = b_2 = 1$ and $a_n = b_n = 0$ ($n \geq 1$ and $n \neq 2$). Then $p(\theta) = 10 + \cos 2\theta + \sin 2\theta$ is Minkowski's support function of a convex domain D, since $p(\theta) + p''(\theta) = 10 - 3(\cos 2\theta + \sin 2\theta) > 0$. From (4.4), $\tilde{A}_2 = 2\pi > 0$.

(II) If $a_0 = 40$, $a_2 = \sqrt{2}$, $a_3 = 1$, $a_n = 0$ ($n \geq 1$ and $n \neq 2, 3$) and $b_n = 0$ ($n \geq 1$). Then $p(\theta) = 20 + \sqrt{2}\cos 2\theta + \cos 3\theta$ is Minkowski's support function of a convex domain D, since $p(\theta) + p''(\theta) = 20 - 3\sqrt{2}\cos 2\theta - 8\cos 3\theta > 0$. From (4.4), $\tilde{A}_2 = 0$.

(III) If $a_0 = 20$, $a_3 = 1$, $a_n = 0$ ($n \geq 1$ and $n \neq 3$) and $b_n = 0$ ($n \geq 1$). Then $p(\theta) = 10 + \cos 3\theta$ is Minkowski's support function of a convex domain D, since $p(\theta) + p''(\theta) = 10 - 8\cos 3\theta > 0$. From (4.4), $\tilde{A}_2 = -4\pi < 0$. □

Next, we consider two special cases of γ and get the following lemma.

Lemma 5 *Let γ be a C^2 closed and strictly convex curve in the Euclidean plane, D be a domain bounded by γ, w be the width function of D, A be the area enclosed by γ and $\tilde{A}_2$ be the oriented area enclosed by ζ.*

(i) *If D is symmetrical about the origin, then $\tilde{A}_2 \geq 0$ and equality can occur if and only if D is a disc.*

(ii) *If D has constant width, then $\tilde{A}_2 \leq 0$ and equality can occur if and only if D is a disc.*

Proof (i) If D is symmetrical about the origin, i.e., $p(\theta) = p(\theta + \pi)$. By (2.2) and (4.2) we have $a_{2k-1} = b_{2k-1} = 0$, for $k \geq 1$, from (4.4), this yields $\tilde{A}_2 = B_1 \geq 0$, the equality holds if and only if D is a disc.

(ii) When D is a convex domain with constant width, $\omega(\theta) = p(\theta) + p(\pi + \theta) =$ constant, so that $\omega'(\theta) = p'(\theta) + p'(\pi + \theta) = 0$. By (2.3) and (4.3), we have

$$
\begin{aligned}
0 = \omega'(\theta) &= p'(\theta) + p'(\pi + \theta) \\
&= \sum_{n=1}^{\infty} \bigl(1 + (-1)^n\bigr)(-na_n \sin n\theta + nb_n \cos n\theta) \\
&= \sum_{k=1}^{\infty} 2(-2ka_{2k} \sin 2k\theta + 2kb_{2k} \cos 2k\theta).
\end{aligned}
$$

We can easily get $a_{2k} = b_{2k} = 0$, for $k \geq 1$.

From (4.4), (4.5) and (4.6), we have

$$
\tilde{A}_2 = B_2 \leq 0,
$$

where equality holds if and only if $a_{2k-1} = b_{2k-1} = 0$ for $k \geq 2$, namely D is a disc. □

It is obvious that the sign of $\tilde{A}_2$ is more complex than $\tilde{A}_1$, so we consider the absolute value of $\tilde{A}_2$ throughout this paper, and obtain a reverse isoperimetric inequality (Theorem 2).

Proof of Theorem 2 (I) If $\tilde{A}_2 \geq 0$. Let $f_1(x) = \frac{x^2(x^2-2)}{x^2-1}$, $x \geq 2$, then $f_1'(x) = \frac{x^3(3x^2-4)+4x}{(x^2-1)^2} > 0$ for $x \geq 2$, which yields $f_1(x)$ to be a monotonically increasing function. So for any real $\varepsilon > 0$, $(2k)^2[(2k)^2 - 2] > \frac{8}{3+\varepsilon}[(2k)^2 - 1]$ for $k \geq 1$, and $[(2k-1)^2 + 1] \geq 10 > \frac{8}{3+\varepsilon}$ for $k > 1$.

By (3.2), (4.4), (2.8) and (2.9), we have

$$\begin{aligned}
4\pi\left(|\tilde{A}_1|-|\tilde{A}_2|\right) &= 2\pi^2\sum_{n=1}^{\infty}n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right)-2\pi^2\sum_{k=1}^{\infty}(2k)^2\left(a_{2k}^2+b_{2k}^2\right)\\
&\quad+2\pi^2\sum_{k=1}^{\infty}\left[(2k-1)^2-1\right]\left(a_{2k-1}^2+b_{2k-1}^2\right)\\
&= 2\pi^2\sum_{k=1}^{\infty}(2k)^2\left[(2k)^2-2\right]\left(a_{2k}^2+b_{2k}^2\right)\\
&\quad+2\pi^2\sum_{k=1}^{\infty}\left[(2k-1)^2+1\right]\left[(2k-1)^2-1\right]\left(a_{2k-1}^2+b_{2k-1}^2\right)\\
&\geq 2\pi^2\sum_{n=1}^{\infty}\frac{8}{3+\varepsilon}\left(n^2-1\right)\left(a_n^2+b_n^2\right)\\
&= \frac{8}{3+\varepsilon}\left(L^2-4\pi A\right),
\end{aligned}$$

where equality holds if and only if $a_n=b_n=0$ for $n>1$, namely γ is a circle.

(II) If $\tilde{A}_2<0$. Let $f_2(x)=\frac{x^4}{x^2-1}$, $x\geq 2$, then $f_2'(x)=\frac{2x^3(x^2-2)}{(x^2-1)^2}>0$ for $x\geq 2$, which yields $f_2(x)$ to be a monotonically increasing function. So for any real $\varepsilon>0$, $(2k)^4>\frac{16}{3+\varepsilon}[(2k)^2-1]>\frac{8}{3+\varepsilon}[(2k)^2-1]$ for $k\geq 1$, and $(2k-1)^2-1\geq 8>\frac{8}{3+\varepsilon}$ for $k>1$.

By (3.2), (4.4), (2.8) and (2.9) we have

$$\begin{aligned}
4\pi\left(|\tilde{A}_1|-|\tilde{A}_2|\right) &= 2\pi^2\sum_{n=1}^{\infty}n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right)+2\pi^2\sum_{k=1}^{\infty}(2k)^2\left(a_{2k}^2+b_{2k}^2\right)\\
&\quad+2\pi^2\sum_{k=1}^{\infty}\left[1-(2k-1)^2\right]\left(a_{2k-1}^2+b_{2k-1}^2\right)\\
&= 2\pi^2\sum_{k=1}^{\infty}(2k)^4\left(a_{2k}^2+b_{2k}^2\right)\\
&\quad+2\pi^2\sum_{k=1}^{\infty}\left[(2k-1)^2-1\right]\left[(2k-1)^2-1\right]\left(a_{2k-1}^2+b_{2k-1}^2\right)\\
&\geq 2\pi^2\sum_{n=1}^{\infty}\frac{8}{3+\varepsilon}\left(n^2-1\right)\left(a_n^2+b_n^2\right)\\
&= \frac{8}{3+\varepsilon}\left(L^2-4\pi A\right),
\end{aligned}$$

where equality holds if and only if $a_n=b_n=0$ for $n>1$, namely γ is a circle.

Therefore we obtain Theorem 2 and complete the proof. □

Letting $\varepsilon=5$ in Theorem 2, we get a stronger lower bound of the integral of radius of curvature by using $\tilde{A}_2$.

Theorem 3 *Let γ be a C^2 closed and strictly convex curve with length L in the Euclidean plane, κ be the curvature of γ, w be the width function of D, A be the area enclosed by γ*

and $\tilde{A}_2$ be the oriented area enclosed by ζ. Then

$$\frac{L^2}{2\pi} + 2|\tilde{A}_2| \le \int_\gamma \frac{1}{\kappa}\,ds,$$

where equality holds if and only if γ is a circle.

Proof Let $\varepsilon = 5$ in Theorem 2, by (3.3), we have

$$L^2 + 4\pi|\tilde{A}_2| \le 4\pi\left(|\tilde{A}_1| + A\right) = 2\pi\int_\gamma \frac{1}{\kappa}\,ds,$$

i.e.,

$$\frac{L^2}{2\pi} + 2|\tilde{A}_2| \le \int_\gamma \frac{1}{\kappa}\,ds,$$

where equality holds if and only if γ is a circle. □

Via the isoperimetric inequality (1.1) and Theorem 3 we can easily get the following inequality.

Corollary 3 *Let γ be a C^2 closed and strictly convex curve in the Euclidean plane, w be the width function of D, A be the area enclosed by γ and $\tilde{A}_2$ be the oriented area enclosed by ζ. Then*

$$2\left(A + |\tilde{A}_2|\right) \le \int_\gamma \frac{1}{\kappa}\,ds,$$

where equality holds if and only if γ is a circle.

By the isoperimetric inequality (1.1) and (3.3) we get an upper bound for the integral of radius of curvature.

Corollary 4 *Let γ be a C^2 closed and strictly convex curve in the Euclidean plane, κ be the curvature of γ, A be the area enclosed by γ and $\tilde{A}_1$ be the oriented area enclosed by β. Then*

$$\int_\gamma \frac{1}{\kappa}\,ds \le \frac{L^2}{2\pi} + 2|\tilde{A}_1|.$$

where equality holds if and only if γ is a circle.

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