Wang *Journal of Inequalities and Applications* (2019) 2019:157
https://doi.org/10.1186/s13660-019-2111-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A modified singular Trudinger–Moser inequality

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $W_0^{1,2}(\Omega)$ be the standard Sobolev space. Assuming certain conditions on a function $g: \mathbb{R} \to \mathbb{R}$, we derive a modified singular Trudinger–Moser inequality, which was originally established by Adimurthi and Sandeep (Nonlinear Differ. Equ. Appl. 13:585–603, 2007), namely,

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega (1+g(u)) \frac{e^{4\pi(1-\gamma)u^2}}{|x|^{2\gamma}} \, dx, \tag{1}$$

where $0 < \gamma < 1$. Following Yang and Zhu (J. Funct. Anal. 272:3347–3374, 2017), we prove that the extremal functions for the supremum in (1) exist. The proof is based on a blow-up analysis.

MSC: 46E35

Keywords: Singular Trudinger–Moser inequality; Blow-up analysis; Extremal function

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$, and $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm $\|u\|_{W_0^{1,2}(\Omega)} = (\int_\Omega |\nabla u|^2 \, dx)^{1/2}$. For $1 \le p < 2$, the standard Sobolev embedding theorem states that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 < q \le 2p/(2-p)$; while if $p > 2$, we have $W_0^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$. As a borderline of the Sobolev embeddings, the classical Trudinger–Moser inequality [21–23, 26, 33] says

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega e^{\alpha u^2} \, dx < +\infty, \quad \forall \alpha \le 4\pi. \tag{2}$$

Moreover, these integrals are still finite for any $\alpha > 4\pi$, but the supremum is infinity. Here and in the sequel, for any real number $q \ge 1$, $\|\cdot\|_q$ denotes the $L^q(\Omega)$-norm with respect to the Lebesgue measure.

A function u_0 is called an extremal function for the Trudinger–Moser inequality (2) if u_0 belongs to $W_0^{1,2}(\Omega)$, $\|\nabla u_0\|_2 \le 1$ and

$$\int_\Omega e^{\alpha u_0^2} \, dx = \sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega e^{\alpha u^2} \, dx.$$

An interesting question on Trudinger–Moser inequalities is whether or not extremal functions exist. The existence of extremal functions for (2) was obtained by Carleson–Chang [5] when Ω is a unit ball, and by Struwe [24] when Ω is close to the ball in the sense of measure. Then Flucher [12] extended this result when Ω is a general bounded smooth domain in $\mathbb{R}^2$. Later, Lin [16] generalized the existence result when Ω is an arbitrary dimensional domain. For recent developments, we refer the reader to Yang [28].

Using a rearrangement argument and a change of variables, Adimurthi–Sandeep [2] generalized the Trudinger–Moser inequality (1) to a singular version as follows:

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\le 1}\int_\Omega \frac{e^{4\pi(1-\gamma)u^2}}{|x|^{2\gamma}}\,dx<\infty. \tag{3}$$

This inequality is also sharp in the sense that all integrals are still finite when $\alpha > 1-\gamma$, but the supremum is infinity. Clearly, if $\gamma = 0$, (3) reduces to (1). Following the lines of Flucher [12], in Csato and Roy [9], they adopt the concentration–compactness alternative by Lions [17] and deduced that the existence of extremals for such singular functionals. Later, (3) was extend to the entire $\mathbb{R}^N$ by Adimurthi and Yang [4]. Meanwhile, Souza and do Ó modified the singular to another version in $\mathbb{R}^N$ in [10]. When Ω is the unit ball $\mathbb{B}$, (3) was improved by Yuan and Zhu [32]. Similarly, an analog is also be proved by Yuan and Huang by using the method of symmetrization in [31]. Such singular Trudinger–Moser inequalities play an important role in the study of partial differential equations and conformal geometry; see [2, 4, 10, 14, 27] and [6] for details.

Recently, using a method of energy estimates in [19], Mancini–Martinazzi [20] reproved Carleson–Chang's result. For applications of this method, we refer the reader to Yang [29]. Using the same idea, they proved that the supremum

$$\sup_{u\in W_0^{1,2}(\mathbb{B}),\|\nabla u\|_2\le 1}\int_{\mathbb{B}}\big(1+g(u)\big)e^{4\pi u^2}\,dx \tag{4}$$

can be achieved for certain smooth function $g:\mathbb{R}\to\mathbb{R}$, where $\mathbb{B}$ is a unit ball. On the other hand, in Yang and Zhu [30], one studied the following singular form:

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_{1,\alpha}\le 1}\int_\Omega \frac{e^{\beta u^2}}{|x|^{2\gamma}}\,dx, \tag{5}$$

and they verified there exists some function u_0 to achieve this supremum for any $\beta < 4\pi(1-\gamma)$, where

$$\|u\|_{1,\alpha}=\left(\int_\Omega|\nabla u|^2\,dx-\alpha\int_\Omega u^2\,dx\right)^{1/2},$$

and α satisfies

$$\alpha<\inf_{u\in W_0^{1,2}(\Omega),u\not\equiv 0}\frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

Motivated by the above results, in this paper, we make a combination of (4) and (5) under the case $\alpha = 0$ to discuss a new version of the singular Trudinger–Moser inequality.

We are aim to prove two main results: One is to explain the new supremum is finite; the other is to discuss the existence of extremals for such functionals. In our proof, unlike the previous energy estimate procedure in [19, 20, 29], we mainly employ the method of blow-up analysis as in [11, 14, 15, 18] to prove the supremum in the following (9) can be achieved. Based on Mancini–Martinazzi [20] (see pages 3 and 4), we assume the function g in (9) satisfies

$$\begin{aligned} &g \in C^1(\mathbb{R}), \qquad \inf_{\mathbb{R}} g > -1, \qquad g(-t) = g(t), \\ &\lim_{|t|\to\infty} t^2 g(t) = 0, \qquad g'(t) > 0 \quad (\forall t > 0). \end{aligned} \tag{6}$$

In the proof, we derive

$$-\Delta u_\varepsilon = \frac{1}{\lambda_\varepsilon}\left(1 + g(u_\varepsilon) + \frac{g'(u_\varepsilon)}{8\pi(1-\gamma-\varepsilon)u_\varepsilon}\right) u_\varepsilon e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2} = \frac{1}{\lambda_\varepsilon}\big(1 + h(u_\varepsilon)\big) u_\varepsilon e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}$$

for some $\lambda_\varepsilon \in \mathbb{R}$, where we set

$$h(t) := g(t) + \frac{g'(t)}{8\pi(1-\gamma-\varepsilon)t}, \quad t \in \mathbb{R} \setminus \{0\}. \tag{7}$$

We further assume

$$\inf_{(0,+\infty)} h(t) > -1, \qquad \sup_{(0,+\infty)} h(t) < +\infty, \quad \text{and} \quad \lim_{t\to\infty} t^2 h(t) = 0. \tag{8}$$

Comparing the conditions on the function g in Mancini–Martinazzi [20], one can see some differences. In this note, we assume $g'(t) > 0$ $(\forall t > 0)$, which is used in the Lemma 4. Moreover, the assumptions in (6) and (8) implies that $\lim_{|t|\to\infty} g(t) = 0$ in [20]. Our main conclusion can be stated as the following two theorems, respectively.

Theorem 1 *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $W_0^{1,2}(\Omega)$ be the usual Sobolev space. Let $0 < \gamma < 1$ be fixed. Suppose $g \in C^1(\mathbb{R})$ satisfies the hypotheses in (6) and (8). Then the supremum*

$$\Lambda_{4\pi(1-\gamma)} := \sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega \big(1 + g(u)\big) \frac{e^{4\pi(1-\gamma)u^2}}{|x|^{2\gamma}}\, dx < \infty. \tag{9}$$

Theorem 2 *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $W_0^{1,2}(\Omega)$ be the usual Sobolev space. Let $0 < \gamma < 1$ be fixed. Suppose $g \in C^1(\mathbb{R})$ satisfies the hypotheses in (6) and (8). Then, for any $\beta \le 4\pi(1-\gamma)$, the supremum*

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega \big(1 + g(u)\big) \frac{e^{\beta u^2}}{|x|^{2\gamma}}\, dx$$

can be attained by some function $u_0 \in W_0^{1,2}(\Omega) \cap C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}) \cap C^0(\overline{\Omega})$.

In order to prove the critical singular Trudinger–Moser inequality, we firstly discuss the existence of extremal functions for a subcritical one, which is based on a direct method variation. We derive a different Euler–Lagrange equation on which the analysis is performed. The essential problem is the presence of the function g. To meet the necessary of our proof, we assume g satisfies certain conditions. Then following Yang and Zhu [30], we define maximizing sequences of functions by using a more delicate scaling. The existence of singular term $|x|^{-2\gamma}$ with $0<\gamma<1$ causes exact asymptotic behavior of certain maximizing sequence near the blow-up point. Unlike in [28], we employ two different classification theorems of Chen and Li [7, 8] to get the desired bubble. And our method in dealing with the bubble is also different from Yang–Zhu [30] because of the function g. We refer to Adimurthi and Druet [1], Carleson–Chang [5], Li [15], Struwe [24], Adimurthi and Struwe [3], Iula and Mancini [13], Yang [28], Lu and Yang [18], respectively.

2 Proof of Theorem 1

We divide the proof into several steps as follows.

2.1 Existence of maximizers for $\Lambda_{4\pi(1-\gamma-\varepsilon)}$ and the Euler–Lagrange equation

In this subsection, we shall prove that maximizers for the subcritical singular Trudinger–Moser functionals exist.

Proposition 3 *For any* $0<\varepsilon<1-\beta$, *there exists some* $u_\varepsilon \in W_0^{1,2}(\Omega)\cap C^1_{\mathrm{loc}}(\overline{\Omega}\setminus\{0\})\cap C^0(\overline{\Omega})$ *satisfying* $\|\nabla u\|_2=1$ *and*

$$\int_\Omega \bigl(1+g(u_\varepsilon)\bigr)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx=\Lambda_{4\pi(1-\gamma-\varepsilon)}:=\sup_{\substack{u\in W_0^{1,2}(\Omega),\\ \|\nabla u\|_2\le 1}}\int_\Omega \bigl(1+g(u)\bigr)\frac{e^{4\pi(1-\gamma-\varepsilon)u^2}}{|x|^{2\gamma}}\,dx. \quad (10)$$

Proof This is based on a direct method of variation. For any $0<\beta<1$, let $0<\varepsilon<1-\gamma$ be fixed. We take a sequence of functions $u_j\in W_0^{1,2}(\Omega)$ satisfying $\|\nabla u_j\|_2\le 1$ and, as $j\to\infty$,

$$\lim_{j\to\infty}\int_\Omega \bigl(1+g(u_j)\bigr)\frac{e^{4\pi(1-\gamma-\varepsilon)u_j^2}}{|x|^{2\gamma}}\,dx=\Lambda_{4\pi(1-\gamma-\varepsilon)}. \quad (11)$$

Since u_j is bounded in $W_0^{1,2}(\Omega)$, there exists some $u_\varepsilon\in W_0^{1,2}(\Omega)$ such that up to a subsequence, assuming

$$\begin{aligned} &u_j \rightharpoonup u_\varepsilon \quad \text{weakly in } W_0^{1,2}(\Omega),\\ &u_j \to u_\varepsilon \quad \text{strongly in } L^p(\Omega), \forall p\ge 1,\\ &u_j\to u_\varepsilon \quad \text{a.e. in } \Omega. \end{aligned}$$

Since

$$0\le \int_\Omega |\nabla u_\varepsilon|^2\,dx \le \limsup_{j\to\infty}\left(\int_\Omega |\nabla u_\varepsilon|^2\,dx\right)^{\frac12}\left(\int_\Omega |\nabla u_j|^2\,dx\right)^{\frac12},$$

we have $0\le\|\nabla u_\varepsilon\|_2\le 1$. Note that

$$\begin{aligned}\int_\Omega \bigl|\nabla(u_\varepsilon-u_j)\bigr|^2\,dx &= \int_\Omega|\nabla u_\varepsilon|^2\,dx-\int_\Omega|\nabla u_j|^2+o_j(1)\\ &\le 1-\int_\Omega|\nabla u_\varepsilon|^2+o_j(1). \end{aligned}\quad (12)$$

Following Hölder's inequality, for any $1 < p \leq \frac{1}{\gamma}$, $\delta > 0$, $w > 1$ and $w' = w/(w-1)$, we have

$$\int_\Omega \bigl(1+g(u_j)\bigr)^p \frac{1}{|x|^{2\gamma p}} e^{4\pi(1-\gamma-\varepsilon)pu_j^2}\,dx \leq C\biggl(\int_\Omega \frac{1}{|x|^{2\gamma p}} e^{4\pi(1-\gamma-\varepsilon)p(1+\delta)w(u_j-u_\varepsilon)^2}\,dx\biggr)^{\frac{1}{w}} \times \biggl(\int_\Omega \frac{1}{|x|^{2\gamma p}} e^{4\pi(1-\gamma-\varepsilon)p(1+\frac{1}{4\delta})w'u_\varepsilon^2}\,dx\biggr)^{\frac{1}{w'}}. \tag{13}$$

When p, $1+\delta$ and s are sufficiently close to 1, we have

$$(1-\gamma-\varepsilon)p(1+\delta)w + \gamma wp < 1. \tag{14}$$

Combining (12), (13) and (14), we have by the singular Trudinger–Moser inequality (3)

$$\bigl(1+g(u_\varepsilon)\bigr)|x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2} \quad \text{is bounded in } L^p(\Omega),$$

for some $p > 1$. Note that

$$\begin{aligned} &\biggl|\bigl(1+g(u_j)\bigr)\frac{e^{4\pi(1-\gamma-\varepsilon)u_j^2}}{|x|^{-2\gamma}} - \bigl(1+g(u_\varepsilon)\bigr)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{-2\gamma}}\biggr| \\ &\quad\leq C|x|^{-2\gamma}\bigl(e^{4\pi(1-\gamma-\varepsilon)u_j^2} + e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\bigr)\bigl|u_j^2 - u_\varepsilon^2\bigr|, \\ &\qquad + |x|^{-2\gamma}\max\bigl\{g'(u_j), g'(u_\varepsilon)\bigr\}|u_j - u_\varepsilon| e^{4\pi(1-\gamma-\varepsilon)u_j^2}. \end{aligned} \tag{15}$$

Since $u_j \to u_\varepsilon$ strongly in $L^p(\Omega)$ for any $p > 1$, in view of (6) and (8), we can conclude from (15) that

$$\int_\Omega \bigl(1+g(u_j)\bigr)|x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_j^2}\,dx \to \int_\Omega \bigl(1+g(u_\varepsilon)\bigr)|x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx,$$

as $j \to \infty$. This together with (11) immediately leads to (10). Obviously $u_\varepsilon \not\equiv 0$. If $\|\nabla u_\varepsilon\|_2 < 1$, set $\widetilde{u}_\varepsilon = \frac{u_\varepsilon}{\|\nabla u_\varepsilon\|_2}$, then we obtain $\|\nabla \widetilde{u}_\varepsilon\|_2 = 1$. Since $0 \leq u_\varepsilon < \widetilde{u}_\varepsilon$ and $u_\varepsilon \not\equiv 0$, it follows from (6) that

$$\int_\Omega \bigl(1+g(u_\varepsilon)\bigr)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx < \int_\Omega \bigl(1+g(\widetilde{u}_\varepsilon)\bigr)\frac{e^{4\pi(1-\gamma-\varepsilon)\widetilde{u}_\varepsilon^2}}{|x|^{2\gamma}}\,dx \leq \Lambda_{4\pi(1-\gamma-\varepsilon)},$$

which contradicts (10). Consequently, $\|\nabla u_\varepsilon\|_2 = 1$ holds. Furthermore, one can also check that $|u_\varepsilon|$ attains the supremum $\Lambda_{4\pi(1-\gamma-\varepsilon)}$. Thus, u_ε can be chosen so that $u_\varepsilon \geq 0$. It is not difficult to see that u_ε satisfies the following Euler–Lagrange equation:

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon^{-1}|x|^{-2\gamma}(1+h(u_\varepsilon))u_\varepsilon e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_\varepsilon \geq 0, \qquad \|\nabla u_\varepsilon\|_2 = 1 & \text{in } \Omega \subset \mathbb{R}^2, \\ \lambda_\varepsilon = \int_\Omega |x|^{-2\gamma}(1+h(u_\varepsilon))u_\varepsilon^2 e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx, \end{cases} \tag{16}$$

where $h(x)$ is defined as in (7). □

2.1.1 The case when u_ε is uniformly bounded in Ω

The proof of Theorem 2 will be ended if we can find some $u_0 \in W_0^{1,2}(\Omega) \cap C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}) \cap C^0(\overline{\Omega})$ satisfying $\|\nabla u_0\|_2 = 1$ and

$$\int_\Omega \bigl(1 + g(u_0)\bigr) \frac{e^{4\pi(1-\gamma)u_0^2}}{|x|^{2\gamma}}\, dx = \sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega \bigl(1 + g(u)\bigr) \frac{e^{4\pi(1-\gamma)u^2}}{|x|^{2\gamma}}\, dx. \tag{17}$$

Since u_ε is bounded in $W_0^{1,2}(\Omega)$, we assume without loss of generality

$$\begin{aligned} &u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } W_0^{1,2}(\Omega), \\ &u_\varepsilon \to u_0 \quad \text{strongly in } L^p(\Omega), \forall p \ge 1, \\ &u_\varepsilon \to u_0 \quad \text{a.e. in } \Omega. \end{aligned} \tag{18}$$

Let $c_\varepsilon = u_\varepsilon(x_\varepsilon) = \max_\Omega u_\varepsilon$. If c_ε is bounded, for any $u \in W_0^{1,2}(\Omega)$ with $u \ge 0$, $\|\nabla u_0\|_2 = 1$, together with Lebesgue dominated convergence theorem gives

$$\begin{aligned} \int_\Omega \bigl(1 + g(u)\bigr) \frac{e^{4\pi(1-\gamma)u^2}}{|x|^{2\gamma}}\, dx &= \lim_{\varepsilon \to 0} \int_\Omega \bigl(1 + g(u_\varepsilon)\bigr) \frac{e^{4\pi(1-\gamma-\varepsilon)u^2}}{|x|^{2\gamma}}\, dx \\ &\le \lim_{\varepsilon \to 0} \int_\Omega \bigl(1 + g(u_\varepsilon)\bigr) \frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\, dx \\ &= \int_\Omega \bigl(1 + g(u_0)\bigr) \frac{e^{4\pi(1-\gamma)u_0^2}}{|x|^{2\gamma}}\, dx. \end{aligned} \tag{19}$$

By the arbitrariness of $u \in W_0^{1,2}(\Omega)$, we conclude that u_0 is the desired maximizer when u_ε is uniformly bounded in Ω. Applying elliptic estimates to its Euler–Lagrange equation, one can deduce that $u_0 \in W_0^{1,2}(\Omega) \cap C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}) \cap C^0(\overline{\Omega})$. And then (17) follows immediately.

2.2 Blowing up analysis

In this subsection, as in [1, 17], we will use the blow-up analysis to understand the asymptotic behavior of the maximizers u_ε. Assume $c_\varepsilon = u_\varepsilon(x_\varepsilon) \to \infty$ and we distinguish two cases to proceed.

Case 1. If $u_0 \not\equiv 0$, the supremum in (9) can be attained by u_0 without difficulty. And the proof will just be divided into several simple steps.

Step 1. A similar estimate as in (13), one can easily check that $\frac{(1+g(u_\varepsilon))}{|x|^{2\gamma}} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}$ is bounded in $L^p(\Omega)$ $(p > 1)$.

Step 2. By the mean value theorem and the Hölder inequality, we have

$$\lim_{\varepsilon \to 0} \int_\Omega |x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\, dx = \int_\Omega |x|^{-2\gamma} e^{4\pi(1-\gamma)u_0^2}\, dx.$$

Step 3. Based on the above steps, one can easily check that

$$\begin{aligned}
&\int_\Omega \left|\big(1+g(u_\varepsilon)\big)|x|^{-2\gamma}e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}-\big(1+g(u_0)\big)|x|^{-2\gamma}e^{4\pi(1-\gamma)u_0^2}\right| dx \\
&\quad\le \left|g(u_0)+1\right| \int_\Omega \big(|x|^{-2\gamma}e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}-|x|^{-2\gamma}e^{4\pi(1-\gamma)u_0^2}\big)\,dx \\
&\qquad+\int_\Omega |x|^{-2\gamma}e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\left|g(u_\varepsilon)-g(u_0)\right| dx \\
&\quad= o_\varepsilon(1).
\end{aligned}$$

Thus, we arrive at the conclusion that

$$\lim_{\varepsilon\to0}\int_\Omega \big(1+g(u_\varepsilon)\big)|x|^{-2\gamma}e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx=\int_\Omega \big(1+g(u_0)\big)|x|^{-2\gamma}e^{4\pi(1-\gamma)u_0^2}\,dx.$$

This together with (17) gives the desired result.

Case 2. If $u_0\equiv0$, in view of Eq. (16), it is important to figure out whether λ_ε has a positive lower bound or not. For this purpose, we have the following.

Lemma 4 *Let λ_ε be as in* (16). *Then we have* $\liminf_{\varepsilon\to0}\lambda_\varepsilon>0$.

Proof By an inequality $e^{t^2}\le 1+t^2e^{t^2}$ for $t\ge0$, it follows from (6) and (7) that

$$\begin{aligned}
\lambda_\varepsilon &\ge \frac{1}{4\pi(1-\gamma-\varepsilon)}\int_\Omega \big(1+h(u_\varepsilon)\big)\frac{(e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}-1)}{|x|^{2\gamma}}\,dx \\
&\ge \frac{1}{4\pi(1-\gamma-\varepsilon)}\Bigg(\int_\Omega \big(1+g(u_\varepsilon)\big)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx-\int_\Omega \frac{(1+g(u_\varepsilon))}{|x|^{2\gamma}}\,dx \\
&\quad+\int_\Omega \frac{g'(u_\varepsilon)}{8\pi(1-\gamma-\varepsilon)|x|^{2\gamma}u_\varepsilon}\big(e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}-1\big)\,dx\Bigg) \\
&\ge \frac{1}{4\pi(1-\gamma-\varepsilon)}\Bigg(\int_\Omega \big(1+g(u_\varepsilon)\big)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx-\int_\Omega \frac{(1+g(u_\varepsilon))}{|x|^{2\gamma}}\,dx\Bigg).
\end{aligned}$$

This together with (10) leads to

$$\liminf_{\varepsilon\to0}\lambda_\varepsilon\ge\frac{1}{4\pi(1-\gamma)}\left(\Lambda_{4\pi(1-\gamma)}-\int_\Omega\frac{(1+g(0))}{|x|^{2\gamma}}\,dx\right)>0.$$

Or equivalently, we have

$$\frac{1}{\lambda_\varepsilon}\le C. \tag{20}$$

Therefore, $\frac{1}{\lambda_\varepsilon}$ is uniformly bounded in Ω. This ends the proof of the lemma. □

2.2.1 Energy concentration phenomenon

Using the same argument as the one in step 2 of [28], we get the following concentration phenomenon, which is crucial in our blow-up analysis.

Proposition 5 *For the function sequence $\{u_\varepsilon\}$, we have $u_\varepsilon \rightharpoonup 0$ weakly in $W_0^{1,2}(\Omega)$ and $u_\varepsilon \to 0$ strongly in $L^q(\Omega)$ for any $q > 1$. Moreover, $|\nabla u_\varepsilon|^2\,dx \rightharpoonup \delta_0$ weakly in a sense of measure, where δ_0 is the usual Dirac measure centered at the point* 0.

Proof Since $\|\nabla u_\varepsilon\|_2 = 1$, we have the same assumptions as in (18). Observe that

$$\int_\Omega \left|\nabla(u_\varepsilon - u_0)\right|^2 dx = 1 - \int_\Omega |\nabla u_0|^2\,dx + o(1). \tag{21}$$

Suppose $u_0 \not\equiv 0$. In view of (21) and an obvious analog of (13), it follows that

$$\bigl(1 + g(u_\varepsilon)\bigr)|x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2} \quad \text{is bounded in } L^q(\Omega),$$

for some $q > 1$. Then applying elliptic estimates to (18), one can deduce that u_ε is bounded in $W_0^{2,q}(\Omega)$. Together with Sobolev embedding results, we conclude u_ε is bounded in $C^0(\overline{\Omega})$, which contradicts $c_\varepsilon \to \infty$. Therefore $u_0 \equiv 0$ and (21) becomes

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx = 1 + o_\varepsilon(1). \tag{22}$$

We next prove $|\nabla u_\varepsilon|^2\,dx \rightharpoonup \delta_{x_0}$. If the statements were false, suppose $|\nabla u_\varepsilon|^2\,dx \rightharpoonup \eta$ in a sense of measure. In view of $\eta \neq \delta_{x_0}$, there exists $r_0 > 0$ such that

$$\lim_{\varepsilon\to 0} \int_{B_{r_0}(x_0)} |\nabla u_\varepsilon|^2\,dx \leq \frac{\eta + 1}{2} < 1.$$

In view of (22) and $u_0 \equiv 0$, we can choose a cut-off function $\phi \in C_0^1(B_{r_0}(x_0))$, which is equal to 1 on $B_{r_0/2}(x_0)$, then it follows that

$$\limsup_{\varepsilon\to 0} \int_{B_{r_0}(x_0)} \left|\nabla(\phi u_\varepsilon)\right|^2 dx < 1.$$

By the singular Trudinger–Moser inequality (3), one sees that $(1 + g(\phi u_\varepsilon))\frac{e^{4\pi(1-\gamma-\varepsilon)(\phi u_\varepsilon)^2}}{|x|^{2\gamma}}$ is bounded in $L^r(B_{r_0}(x_0))$ for some $r > 1$. Applying elliptic estimates to (16), one gets u_ε is uniformly bounded in Ω, which contradicts $c_\varepsilon \to \infty$ again. Therefore $|\nabla u_\varepsilon|^2\,dx \rightharpoonup \delta_{x_0}$. Moreover, we get $u_\varepsilon \to 0$ in $C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{0, x_0\}) \cap C^0_{\mathrm{loc}}(\overline{\Omega} \setminus \{x_0\})$.

In fact, we have $x_0 = 0$. Set $r_0 = |x_0|/2$. Note that $\lambda_\varepsilon^{-1}|x|^{-2\gamma}(1 + h(u_\varepsilon))u_\varepsilon e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}$ is bounded in $L^{q_1}(B_{r_0}(0))$ for some $q_1 > 1$. When $|x| > r_0$, by the classical Trudinger–Moser inequality (2), we recognize $\lambda_\varepsilon^{-1}|x|^{-2\gamma}(1 + h(u_\varepsilon))u_\varepsilon e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}$ is bounded in $L^{q_2}(\Omega \setminus B_{r_0}(0))$ for some $q_2 > 1$. Choose $q = \min\{q_1, q_2\} > 1$, and we conclude $\lambda_\varepsilon^{-1}|x|^{-2\gamma}(1 + h(u_\varepsilon))u_\varepsilon e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}$ is bounded in $L^q(\Omega)$. Then the elliptic estimate on the Euler–Lagrange equation (16) implies that c_ε is bounded, which also makes a contradiction. Thus, we complete the proof of the proposition. □

2.2.2 Asymptotic behavior of u_ε near the concentration point

Let

$$r_\varepsilon = \sqrt{\lambda_\varepsilon}\, c_\varepsilon^{-1} e^{-2\pi(1-\gamma-\varepsilon)c_\varepsilon^2}. \tag{23}$$

For any $0 < \delta < 1-\gamma$, in view of (8), we have by using the Hölder inequality and the singular Trudinger–Moser inequality (3),

$$\begin{aligned}\lambda_\varepsilon &= \int_\Omega |x|^{-2\gamma}\big(1+h(u_\varepsilon)\big)u_\varepsilon^2 e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx \\ &\le e^{4\pi\delta c_\varepsilon^2}\int_\Omega |x|^{-2\gamma}\big(1+h(u_\varepsilon)\big)u_\varepsilon^2 e^{4\pi(1-\gamma-\varepsilon-\delta)u_\varepsilon^2}\,dx \\ &\le Ce^{4\pi\delta c_\varepsilon^2}\end{aligned}$$

for some constant C depending only on δ. This leads to

$$r_\varepsilon^2 e^{4\pi\mu c_\varepsilon^2} \le Cc_\varepsilon^{-2}e^{4\pi(\delta+\mu)}e^{-4\pi(1-\gamma-\varepsilon)c_\varepsilon^2} \to 0, \quad \text{for } \forall 0<\mu<1-\gamma, \tag{24}$$

as $\varepsilon \to 0$. To characterize the blow-up behavior more exactly, we need to divide the process into two cases as in [30].

Case 1. $r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon \le C$.

Let $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : x_\varepsilon + r_\varepsilon^{1/(1-\gamma)}x \in \Omega\}$. Define two blow-up sequences of function on Ω_ε as

$$\zeta_\varepsilon(x) = c_\varepsilon^{-1}u_\varepsilon\big(x_\varepsilon + r_\varepsilon^{1/(1-\gamma)}x\big), \qquad \vartheta_\varepsilon(x) = c_\varepsilon\big(u_\varepsilon\big(x_\varepsilon + r_\varepsilon^{1/(1-\gamma)}x\big) - c_\varepsilon\big).$$

A direct computation shows

$$-\Delta\zeta_\varepsilon(x) = c_\varepsilon^{-2}\big|x + r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon\big|^{-2\gamma}\big(1+h(u_\varepsilon)\big)\zeta_\varepsilon e^{4\pi(1-\gamma-\varepsilon)(u_\varepsilon^2-c_\varepsilon^2)} \quad \text{in } \Omega_\varepsilon, \tag{25}$$

$$-\Delta\vartheta_\varepsilon(x) = \big|x + r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon\big|^{-2\gamma}\big(1+h(u_\varepsilon)\big)\zeta_\varepsilon e^{4\pi(1-\gamma-\varepsilon)(1+\zeta_\varepsilon)\vartheta_\varepsilon} \quad \text{in } \Omega_\varepsilon. \tag{26}$$

We now investigate the convergence behavior of $\zeta_\varepsilon(x)$ and $\vartheta_\varepsilon(x)$. Assume $\lim_{\varepsilon\to 0} r_\varepsilon^{-1/(1-\gamma)} \times x_\varepsilon = -\bar{x}$. From (24), we have $r_\varepsilon \to 0$ obviously. Thus $\Omega_\varepsilon \to \mathbb{R}^2$ as $\varepsilon \to 0$. In view of $|\zeta_\varepsilon(x)| \le 1$ and $\Delta\zeta_\varepsilon(x) \to 0$ in $x \in \Omega_\varepsilon \setminus \{\bar{x}\}$ as $\varepsilon \to 0$, we have by elliptic estimates that $\zeta_\varepsilon(x) \to \zeta(x)$ in $C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{\bar{x}\}) \cap C^0_{\text{loc}}(\mathbb{R}^2)$, where ζ is a bounded harmonic function in $\mathbb{R}^2$. Observe that $\zeta(x) \le \limsup_{\varepsilon\to 0}\zeta_\varepsilon(x) \le 1$ and $\zeta(0) = 1$. It follows from the Liouville theorem that $\zeta \equiv 1$ on $\mathbb{R}^2$. Thus, we have

$$\zeta_\varepsilon \to 1 \quad \text{in } C^1_{\text{loc}}\big(\mathbb{R}^2 \setminus \{\bar{x}\}\big) \cap C^0_{\text{loc}}\big(\mathbb{R}^2\big) \tag{27}$$

as $\varepsilon \to 0$. Note also that

$$\vartheta_\varepsilon(x) \le \vartheta_\varepsilon(0) = 0 \quad \text{for all } x \in \Omega_\varepsilon(x).$$

In view of (27), we conclude by applying elliptic estimates to (26) that

$$\vartheta_\varepsilon \to \vartheta \quad \text{in } C^1_{\text{loc}}\big(\mathbb{R}^2 \setminus \{\bar{x}\}\big) \cap C^0_{\text{loc}}\big(\mathbb{R}^2\big), \tag{28}$$

where ϑ is a distributional solution to

$$-\Delta\vartheta = |x-\bar{x}|^{-2\gamma}e^{8\pi(1-\gamma)\vartheta} \quad \text{in } \mathbb{R}^2 \setminus \{\bar{x}\}.$$

Observe that

$$\zeta_\varepsilon(x) = \frac{u_\varepsilon(x_\varepsilon + r_\varepsilon^{1/(1-\gamma)}x)}{c_\varepsilon} \to 1 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{B}_R \setminus \mathbb{B}_{1/R}), \tag{29}$$

as $\varepsilon \to 0$. Set $y = x_\varepsilon + r_\varepsilon^{1/(1-\gamma)}x$ with $|x - \bar{x}| \le R$, and then we have

$$|y| \le r_\varepsilon^{1/(1-\gamma)}|x - \bar{x}| + \left|x_\varepsilon + r_\varepsilon^{1/(1-\gamma)}\bar{x}\right| \le 2Rr_\varepsilon^{1/(1-\gamma)}.$$

Since $r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon \le C$, choose R big enough such that

$$\left|x - r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon\right| \le R.$$

This together with (29) leads to

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\| \frac{u_\varepsilon(r_\varepsilon^{1/(1-\gamma)}x)}{c_\varepsilon} \right\|_{L^\infty(\mathbb{B}_R\setminus\mathbb{B}_{1/R}(\bar{x}))} \\
&\quad = \lim_{\varepsilon\to 0}\left\| \frac{u_\varepsilon(x_\varepsilon + r_\varepsilon^{1/(1-\gamma)}(x - r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon))}{c_\varepsilon} \right\|_{L^\infty(\mathbb{B}_R\setminus\mathbb{B}_{1/R}(\bar{x}))} \\
&\quad = 1.
\end{aligned}$$

Combining with Fatou's lemma, we obtain

$$\begin{aligned}
&\int_{\mathbb{B}_R\setminus\mathbb{B}_{1/R}(\bar{x})} |x - \bar{x}|^{-2\gamma} e^{8\pi(1-\gamma)\vartheta}\, dx \\
&\quad \le \limsup_{\varepsilon\to 0} \int_{\mathbb{B}_R\setminus\mathbb{B}_{1/R}(\bar{x})} \left|x + r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon\right|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)(1+\zeta_\varepsilon)\vartheta_\varepsilon}\, dx \\
&\quad \le \limsup_{\varepsilon\to 0} \frac{1}{\lambda_\varepsilon} \int_{\mathbb{B}_{2Rr_\varepsilon^{1/(1-\gamma)}}\setminus\mathbb{B}_{\frac{1}{2}Rr_\varepsilon^{-1/(1-\gamma)}}(0)} \bigl(1 + h(u_\varepsilon)\bigr) \frac{u_\varepsilon^2(y)}{|y|^{2\gamma}} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2(y)}\, dy \\
&\quad \le 1.
\end{aligned} \tag{30}$$

Passing to the limit $R \to \infty$, we have

$$\int_{\mathbb{R}^2} |x - \bar{x}|^{-2\gamma} e^{8\pi(1-\gamma)\vartheta}\, dx \le 1.$$

The uniqueness theorem obtained in [3] implies that

$$\vartheta(x) = -\frac{1}{4\pi(1-\gamma)} \log\left(1 + \frac{1}{1-\gamma}|x - \bar{x}|^{2(1-\gamma)}\right). \tag{31}$$

Let $x = 0$, and then

$$\vartheta(0) = \lim_{\varepsilon\to 0} \vartheta_\varepsilon(0) = 0.$$

Thus, it follows from (31) that $\bar{x} = 0$. Namely,

$$\vartheta(x) = -\frac{1}{4\pi(1-\gamma)} \log\left(1 + \frac{1}{1-\gamma}|x|^{2(1-\gamma)}\right). \tag{32}$$

Furthermore, we can get

$$\int_{\mathbb{R}^2} |x|^{-2\gamma} e^{8\pi(1-\gamma)\vartheta}\, dx = 1. \tag{33}$$

Case 2. $r_\varepsilon^{-1/(1-\gamma)} x_\varepsilon \to +\infty$. Set

$$\widetilde{\Omega}_\varepsilon = \left\{x \in \mathbb{R}^2 : x_\varepsilon + r_\varepsilon |x_\varepsilon|^\gamma x \in \Omega\right\}.$$

Denote the blowing up functions on $\overline{\Omega}_\varepsilon$

$$\alpha_\varepsilon(x) = c_\varepsilon^{-1} u_\varepsilon\left(x_\varepsilon + r_\varepsilon |x_\varepsilon|^\gamma x\right), \qquad \beta_\varepsilon(x) = c_\varepsilon\left(u_\varepsilon\left(x_\varepsilon + r_\varepsilon |x_\varepsilon|^\gamma x\right) - c_\varepsilon\right).$$

In view of (16), $\alpha_\varepsilon(x)$ is a distributional solution to the equation

$$-\Delta\alpha_\varepsilon(x) = f_\varepsilon(u) \quad \text{in } \overline{\Omega}_\varepsilon, \tag{34}$$

where

$$f_\varepsilon = c_\varepsilon^{-2} |x_\varepsilon|^{2\gamma} \left|x_\varepsilon + r_\varepsilon |x_\varepsilon|^\gamma x\right|^{-2\gamma} \left(1 + h(u_\varepsilon)\right)\alpha_\varepsilon e^{4\pi(1-\gamma-\varepsilon)c_\varepsilon^2(\alpha_\varepsilon^2 - 1)}.$$

Since $r_\varepsilon^{-1/(1-\gamma)} x_\varepsilon \to +\infty$, we have $|x_\varepsilon|^{2\gamma}|x_\varepsilon + r_\varepsilon|x_\varepsilon|^\gamma x|^{-2\gamma} = 1 + o_\varepsilon(1)$ clearly. Since $|\alpha_\varepsilon(x)| \leq 1$, we obtain f_ε is bounded in L^p ($p > 1$) according to (8). Elliptic estimates and embedding theorem lead to $\alpha_\varepsilon \to \alpha$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where α satisfies

$$-\Delta\alpha(x) = 0 \quad \text{in } \mathbb{R}^2.$$

Note that $\alpha \leq 1$ and $\alpha(0) = 1$. Thus, together with the Liouville theorem, we obtain $\alpha \equiv 1$. Also we have

$$-\Delta\beta_\varepsilon = |x_\varepsilon|^{2\gamma} \left|x_\varepsilon + r_\varepsilon |x_\varepsilon|^\gamma x\right|^{-2\gamma} \left(1 + h(u_\varepsilon)\right)\alpha_\varepsilon e^{4\pi(1-\gamma-\varepsilon)\beta_\varepsilon(\alpha_\varepsilon + 1)} \quad \text{in } \overline{\Omega}_\varepsilon. \tag{35}$$

Applying elliptic estimates to (35), we conclude that $\beta_\varepsilon \to \beta$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where β is a distributional solution to

$$\begin{cases} \beta(0) = 0 = \sup\beta, \\ \Delta\beta = -e^{8\pi(1-\gamma)\beta} \quad \text{in } \mathbb{R}^2. \end{cases} \tag{36}$$

For $0 < \beta < 1$, (36) follows from Chen and Li [6] that β satisfies

$$\int_{\mathbb{R}^2} e^{8\pi(1-\gamma)\beta}\, dx \geq \frac{1}{1-\beta} > 1.$$

Using a suitable change of variable $y = x_\varepsilon + r_\varepsilon |x_\varepsilon|^\gamma x$, for any $R > 0$, we have

$$\begin{aligned}\int_{\mathbb{B}_R(\bar{x})} e^{8\pi(1-\gamma)\beta}\,dx &= \lim_{\varepsilon\to 0}\int_{\mathbb{B}_R(0)} \big(1+h(u_\varepsilon)\big)e^{4\pi(1-\gamma-\varepsilon)(u_\varepsilon^2(x_\varepsilon+r_\varepsilon|x_\varepsilon|^\gamma x)-c_\varepsilon^2)}\,dx \\ &\le \lim_{\varepsilon\to 0}\frac{1}{\lambda_\varepsilon}\int_{\mathbb{B}_{Rr_\varepsilon|x_\varepsilon|^\gamma}(x_\varepsilon)} \big(1+h(u_\varepsilon)\big)\frac{u_\varepsilon^2(y)}{|y|^{2\gamma}}e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2(y)}\,dy \\ &\le 1, \end{aligned} \tag{37}$$

which leads to a contradiction. Thus, it is impossible for *Case 2* to happen.

2.2.3 Convergence away from the concentration point

To understand the convergence behavior away from the blow-up point $x_0 = 0$, we need to investigate how $c_\varepsilon u_\varepsilon$ converges. Similar to [1, 15], define $u_{\varepsilon,\tau} = \min\{\tau c_\varepsilon, u_\varepsilon\}$, then we have the following.

Lemma 6 *For any* $0 < \tau < 1$, *we have*

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\tau}|^2\,dx = \tau.$$

Proof Observe that $u_\varepsilon/c_\varepsilon = 1 + o_\varepsilon(1)$ in $B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)$. For any $0 < \tau < 1$, it follows from Eq. (16) and the divergence theorem that

$$\begin{aligned}\int_\Omega |\nabla u_{\varepsilon,\tau}|^2\,dx &= \frac{1}{\lambda_\varepsilon}\int_\Omega \frac{u_{\varepsilon,\tau}u_\varepsilon}{|x|^{2\gamma}}\big(1+h(u_\varepsilon)\big)e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx \\ &\ge \frac{1}{\lambda_\varepsilon}\int_{B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \frac{u_{\varepsilon,\tau}u_\varepsilon}{|x|^{2\gamma}}\big(1+h(u_\varepsilon)\big)e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx + o_\varepsilon(1) \\ &= \tau\int_{B_R(0)} \frac{(1+h(u_\varepsilon))e^{4\pi(1-\gamma-\varepsilon)(u_\varepsilon^2(x_\varepsilon+r_\varepsilon^{1/(1-\gamma)}y)-c_\varepsilon^2)}}{|y+r_\varepsilon^{-1/(1-\gamma)}x_\varepsilon|^{2\gamma}}\,dy + o_\varepsilon(1). \end{aligned}$$

Hence

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\tau}|^2\,dx \ge \tau\int_{B_R(0)} e^{8\pi(1-\gamma)\vartheta}\,dy, \quad \forall R > 0.$$

In view of (33), passing to the limit $R \to +\infty$, we obtain

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_{\varepsilon,\tau}|^2\,dx \ge \tau. \tag{38}$$

Note that

$$\big|\nabla(u_\varepsilon - \tau c_\varepsilon)^+\big|^2 = \nabla(u_\varepsilon - \tau c_\varepsilon)^+ \cdot \nabla u_\varepsilon \quad \text{on } \Omega$$

and

$$(u_\varepsilon - \tau c_\varepsilon)^+ = \big(1 + o_\varepsilon(1)\big)(1-\tau)c_\varepsilon \quad \text{in } B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_0).$$

Testing Eq. (16) by $(u_\varepsilon - \tau c_\varepsilon)^+$, for any fixed $R > 0$, simple computation shows that

$$\begin{aligned}\int_\Omega \left|\nabla(u_\varepsilon - \tau c_\varepsilon)^+\right|^2 dx &= \int_\Omega (u_\varepsilon - \tau c_\varepsilon)^+ \frac{u_\varepsilon}{\lambda_\varepsilon |x|^{2\gamma}} \left(1 + h(u_\varepsilon)\right) e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\, dx \\ &\geq \int_{B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} (u_\varepsilon - \tau c_\varepsilon)^+ \frac{u_\varepsilon(1 + h(u_\varepsilon))}{\lambda_\varepsilon |x|^{2\gamma}} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\, dx \\ &= \left(1 + o_\varepsilon(1)\right)(1-\tau) \int_{B_{R(0)}} \zeta_\varepsilon \left(1 + h(u_\varepsilon)\right) e^{4\pi(1-\gamma-\varepsilon)\vartheta_\varepsilon^2}\, dx.\end{aligned}$$

By passing to the limit $\varepsilon \to 0$, we get

$$\liminf_{\varepsilon\to 0} \int_\Omega \left|\nabla(u_\varepsilon - \tau c_\varepsilon)^+\right|^2 dx \geq (1-\tau) \int_{B_{R(0)}} e^{8\pi(1-\gamma)\vartheta}\, dx = 1 - \tau. \tag{39}$$

Since $|\nabla u_{\varepsilon,\tau}|^2 + |\nabla(u_\varepsilon - \tau c_\varepsilon)^+|^2 = |\nabla u_\varepsilon|^2$ almost everywhere, it follows that

$$\int_\Omega \left|\nabla(u_\varepsilon - \tau c_\varepsilon)^+\right|^2 dx + \int_\Omega |\nabla u_{\varepsilon,\tau}|^2\, dx = \int_\Omega |\nabla u_\varepsilon|^2\, dx = 1 + o_\varepsilon(1). \tag{40}$$

Therefore, we end the proof of this lemma together with (38), (39) and (40). □

The following estimate is a byproduct of Lemma 6 and will be employed in the next section.

Lemma 7 *We have*

$$\lim_{\varepsilon\to 0} \int_\Omega |x|^{-2\gamma} \left(1 + g(u_\varepsilon)\right) e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\, dx = \left(1 + g(0)\right) \int_\Omega |x|^{-2\gamma}\, dx + \lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{41}$$

Proof Let $0 < \tau < 1$ be fixed. By the definition of $u_{\varepsilon,\tau}$, we can get

$$\begin{aligned}&\int_{u_\varepsilon \leq \tau c_\varepsilon} \left(1 + g(u_\varepsilon)\right) \frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\, dx - \left(1 + g(0)\right) \int_\Omega \frac{1}{|x|^{2\gamma}}\, dx \\ &\quad\leq \int_\Omega \left(1 + g(u_{\varepsilon,\tau})\right) \frac{e^{4\pi(1-\gamma-\varepsilon)u_{\varepsilon,\tau}^2}}{|x|^{2\gamma}}\, dx - \left(1 + g(0)\right) \int_\Omega \frac{1}{|x|^{2\gamma}}\, dx \\ &\quad\leq \int_\Omega \left|g(u_{\varepsilon,\tau}) - g(0)\right| \frac{e^{4\pi(1-\gamma-\varepsilon)u_{\varepsilon,\tau}^2}}{|x|^{2\gamma}}\, dx + \left|1 + g(0)\right| \int_\Omega \frac{(e^{4\pi(1-\gamma-\varepsilon)u_{\varepsilon,\tau}^2} - 1)}{|x|^{2\gamma}}\, dx.\end{aligned} \tag{42}$$

Combining Lemma 6 and Proposition 5, we see that $u_{\varepsilon,\sigma}$ converges to 0 in $C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\})$ as $\varepsilon \to 0$. Then from (3), one can deduce that

$$\int_\Omega \frac{e^{4\pi(1-\gamma-\varepsilon)u_{\varepsilon,\tau}^2}}{|x|^{2\gamma}} \left|g(u_{\varepsilon,\tau}) - g(0)\right| dx = o_\varepsilon(1). \tag{43}$$

According to the Hölder inequality and the Lagrange theorem, we have

$$\int_\Omega \frac{1}{|x|^{2\gamma}} \left(e^{4\pi(1-\gamma-\varepsilon)u_{\varepsilon,\tau}^2} - 1\right) dx = o_\varepsilon(1). \tag{44}$$

Inserting (43) and (44) into (42), one has

$$\lim_{\varepsilon\to 0}\int_{u_\varepsilon\le\tau c_\varepsilon}\big(1+g(u_\varepsilon)\big)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx=\big(1+g(0)\big)\int_\Omega\frac{1}{|x|^{2\gamma}}\,dx. \tag{45}$$

Moreover, we calculate

$$\begin{aligned}&\int_{u_\varepsilon>\tau c_\varepsilon}\big(1+g(u_\varepsilon)\big)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx\\&\quad\le\frac{1}{\tau^2}\int_{u_\varepsilon>\tau c_\varepsilon}\frac{u_\varepsilon^2}{c_\varepsilon^2}\big(1+g(u_\varepsilon)\big)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx\\&\quad\le\frac{1}{\tau^2}\frac{\lambda_\varepsilon^2}{c_\varepsilon^2}.\end{aligned} \tag{46}$$

Combining (45) and (46), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega\frac{(1+g(u_\varepsilon))e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx\le\big(1+g(0)\big)\int_\Omega\frac{1}{|x|^{2\gamma}}\,dx+\frac{1}{\tau^2}\liminf_{\varepsilon\to 0}\frac{\lambda_\varepsilon^2}{c_\varepsilon^2}.$$

It follows by letting $\tau\to 1$ that

$$\lim_{\varepsilon\to 0}\int_\Omega\frac{(1+g(u_\varepsilon))e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx-\big(1+g(0)\big)\int_\Omega\frac{1}{|x|^{2\gamma}}\,dx\le\liminf_{\varepsilon\to 0}\frac{\lambda_\varepsilon^2}{c_\varepsilon^2}. \tag{47}$$

On the other hand, in view of (16), we estimate

$$\begin{aligned}&\int_\Omega\big(1+g(u_\varepsilon)\big)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\,dx-\big(1+g(0)\big)\int_\Omega\frac{1}{|x|^{2\gamma}}\,dx\\&\quad\ge\int_\Omega\frac{u_\varepsilon^2}{c_\varepsilon^2}\left(\big(1+g(u_\varepsilon)\big)\frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}-\big(1+g(0)\big)\frac{1}{|x|^{2\gamma}}\right)dx\\&\quad=\frac{\lambda_\varepsilon}{c_\varepsilon^2}-\frac{1}{c_\varepsilon^2}\int_\Omega\frac{(1+g(0))u_\varepsilon^2}{|x|^{2\gamma}}\,dx-\frac{1}{c_\varepsilon^2}\int_\Omega\frac{u_\varepsilon g'(u_\varepsilon)}{8\pi(1-\gamma-\varepsilon)|x|^{2\gamma}}e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx.\end{aligned}$$

Thus, by Proposition 5 and (6), (8), one can check that

$$\limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon^2}{c_\varepsilon^2}\le\lim_{\varepsilon\to 0}\int_\Omega|x|^{-2\gamma}\big(1+g(u_\varepsilon)\big)e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx-\big(1+g(0)\big)\int_\Omega|x|^{-2\gamma}\,dx. \tag{48}$$

In view of (47) and (48), we complete the proof of Lemma 7. □

Corollary 8 *If* $\theta<2$, *then* $\frac{\lambda_\varepsilon}{c_\varepsilon^\theta}\to\infty$ *as* $\varepsilon\to 0$.

Proof In contrast, we have $\lambda_\varepsilon/c_\varepsilon^2\to 0$ as $\varepsilon\to 0$. For any $v\in W_0^{1,2}(\Omega)$ with $\|\nabla v\|_2\le 1$, clearly, it is impossible for (41) to hold since $v\not\equiv 0$. □

Lemma 9 *For any function* $\phi\in C_0^1(\Omega)$, *we have*

$$\lim_{\varepsilon\to 0}\int_\Omega\big(1+h(u_\varepsilon)\big)\lambda_\varepsilon^{-1}c_\varepsilon u_\varepsilon|x|^{-2\gamma}e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\phi\,dx=\phi(0). \tag{49}$$

Proof To see this, let $\phi \in C_0^1(\Omega)$ be fixed. Write for simplicity

$$\omega_\varepsilon = \big(1 + h(u_\varepsilon)\big)\lambda_\varepsilon^{-1} c_\varepsilon u_\varepsilon |x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}.$$

Clearly

$$\begin{aligned}\int_\Omega \omega_\varepsilon \phi\, dx = {}& \int_{\{u_\varepsilon < \tau c_\varepsilon\}} \omega_\varepsilon \phi\, dx + \int_{\{u_\varepsilon \geq \tau c_\varepsilon\} \setminus B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \omega_\varepsilon \phi\, dx \\ & + \int_{\{u_\varepsilon \geq \tau c_\varepsilon\} \cap B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \omega_\varepsilon \phi\, dx. \end{aligned} \tag{50}$$

Given $0 < \tau < 1$, we estimate the three integrals on the right hand of (50), respectively. Note that $u_\varepsilon \to 0$ in L^q ($\forall q > 1$). This together with Lemma 6 and Corollary 8 gives

$$\begin{aligned}\int_{\{u_\varepsilon < \tau c_\varepsilon\}} \omega_\varepsilon \phi\, dx &\leq \lambda_\varepsilon^{-1} c_\varepsilon \Big(\sup_\Omega \big|\phi\big(1 + h(u_\varepsilon)\big)\big|\Big) \int_{\{u_\varepsilon < \tau c_\varepsilon\}} u_\varepsilon |x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_{\varepsilon,\tau}^2}\, dx \\ &\leq C\lambda_\varepsilon^{-1} c_\varepsilon \int_{\{u_\varepsilon < \tau c_\varepsilon\}} u_\varepsilon |x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_{\varepsilon,\tau}^2}\, dx \\ &= o_\varepsilon(1). \end{aligned} \tag{51}$$

Now we consider in $B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon) \subset \{x \in \Omega \mid u_\varepsilon \geq \tau c_\varepsilon\}$ for sufficiently small $\varepsilon > 0$. One can deduce from (33) that

$$\begin{aligned}\int_{\{u_\varepsilon \geq \tau c_\varepsilon\} \cap B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \omega_\varepsilon \phi\, dx &= \phi(0)\big(1 + o_\varepsilon(1)\big) \int_{B_R \setminus 1/R(0)} |x|^{-2\gamma} e^{8\pi\vartheta}\, dx \\ &= \phi(0)\big(1 + o_\varepsilon(1) + o_R(1)\big). \end{aligned} \tag{52}$$

On the other hand, we calculate

$$\begin{aligned}\int_{\{u_\varepsilon \geq \tau c_\varepsilon\} \setminus B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \omega_\varepsilon \phi\, dx &\leq \frac{C}{\tau}\left(1 - \int_{B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \frac{u_\varepsilon^2}{\lambda_\varepsilon} \frac{e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}}{|x|^{2\gamma}}\, dx\right) \\ &= \frac{C}{\tau}\left(1 - \int_{B_R(0)} \frac{e^{8\pi(1-\gamma)\vartheta}}{|x|^{2\gamma}}\, dx\right). \end{aligned}$$

Hence, we derive by (33) that

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{\{u_\varepsilon \geq \tau c_\varepsilon\} \setminus B_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \omega_\varepsilon \phi\, dx = 0. \tag{53}$$

Inserting (51)–(53) to (50), we conclude (49) finally. □

In particular, we propose, by letting $\phi = 1$,

$$\omega_\varepsilon(x) = \big(1 + h(u_\varepsilon)\big)\lambda_\varepsilon^{-1} c_\varepsilon u_\varepsilon |x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2} \quad \text{is bounded in } L^1(\Omega), \tag{54}$$

which will be used in the following proof.

We now prove that $c_\varepsilon u_\varepsilon$ converges to a Green function in distributional sense when $\varepsilon \to 0$, where δ_0 stands for the Dirac measure centered at 0. More precisely, we have

Lemma 10 *$c_\varepsilon u_\varepsilon \to G$ in $C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\})$ and weakly in $W_0^{1,q}(\Omega)$ for all $1 < q < 2$, where $G \in C^1(\overline{\Omega} \setminus \{0\})$ is a distributional solution satisfying the following*:

$$\begin{cases} -\Delta G = \delta_0 & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega. \end{cases} \tag{55}$$

Moreover, G takes the form

$$G(x) = -\frac{1}{2\pi} \log|x| + A_0 + \xi(x), \tag{56}$$

where $\xi(x) \in C^1(\overline{\Omega})$ and A_0 is a constant depending on 0.

Proof By Eq. (16), $c_\varepsilon u_\varepsilon$ is a distributional solution to

$$-\Delta(c_\varepsilon u_\varepsilon) = \omega_\varepsilon \quad \text{in } \Omega. \tag{57}$$

It follows from (54) that ω_ε is bounded in $L^1(\Omega)$. Using the argument in Struwe ([25], Theorem 2.2), one concludes that $c_\varepsilon u_\varepsilon$ is bounded in $W_0^{1,q}(\Omega)$ for all $1 < q < 2$. Hence, we can assume, for any $1 < q < 2$, $r > 1$, that

$$\begin{aligned} c_\varepsilon u_\varepsilon &\rightharpoonup G \quad \text{weakly in } W_0^{1,q}(\Omega), \\ c_\varepsilon u_\varepsilon &\to G \quad \text{strongly in } L^r(\Omega). \end{aligned}$$

Testing (57) by $\phi \in C_0^1(\Omega)$, we deduce

$$\int_\Omega \nabla(c_\varepsilon u_\varepsilon)\nabla\phi\, dx = \int_\Omega \phi\lambda_\varepsilon^{-1} c_\varepsilon u_\varepsilon \bigl(1 + h(u_\varepsilon)\bigr)|x|^{-2\gamma} e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}.$$

Let $\varepsilon \to 0$ and it yields by (55)

$$\int_\Omega \nabla G \nabla\phi\, dx = \phi(0),$$

which implies that $-\Delta G = \delta_0$ in a distributional sense. Since $\Delta(G + \frac{1}{2\pi}\log|x|) \in L^p(\Omega)$ for any $p > 2$, (56) follows from the elliptic solution immediately. Applying elliptic estimates to Eq. (57), we arrive at the conclusion

$$c_\varepsilon u_\varepsilon \to G \quad \text{in } C^1_{\mathrm{loc}}\bigl(\overline{\Omega} \setminus \{0\}\bigr). \tag{58}$$

Thus, the two assertions holds. □

2.3 Upper bound calculates by means of capacity estimate

In this subsection, we aim to derive an upper bound of the integrals $\int_\Omega (1+g(u_\varepsilon))|x|^{-2\gamma} \times e^{4\pi(1-\gamma-\varepsilon)u_\varepsilon^2}\,dx$. Analogous to the one obtained in [15], we mainly use the capacity estimate. Now choose a proper δ to ensure that $B_{2\delta} \subset \Omega$, and then construct a new function space

$$\mathcal{M}_\varepsilon(\rho_\varepsilon,\sigma_\varepsilon) = \bigl\{u | u \in W^{1,2}\bigl(\mathbb{B}_\delta(x_\varepsilon) \setminus \mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)\bigr) : u|_{\partial\mathbb{B}_\delta(x_\varepsilon)} = \rho_\varepsilon, u|_{\partial\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} = \sigma_\varepsilon\bigr\}$$

where

$$\rho_\varepsilon = \sup_{\partial\mathbb{B}_\delta(x_\varepsilon)} u_\varepsilon, \qquad \sigma_\varepsilon = \inf_{\partial\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} u_\varepsilon.$$

Define

$$\Lambda_\varepsilon = \inf_{u\in\mathcal{M}_\varepsilon(\rho_\varepsilon,\sigma_\varepsilon)} \int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla u|^2\,dx.$$

Clearly, the infimum Λ_ε can be attained by the sequence $u_k \in \mathcal{M}$ as $k \to \infty$. By the proof of the Poincaré inequality, we infer that u_k is bounded in $W_0^{1,2}(\Omega)$. Without loss of generality, there exists some function $t \in W^{1,2}(\Omega)$ such that up to a subsequence. As $k \to \infty$, we have $u_k \rightharpoonup t$ weakly in $W^{1,2}(\Omega)$, $u_k \to t$ in $L^p_{\mathrm{loc}}(\Omega)$ for any $p > 0$ and $u_k \to t$ a.e. in Ω. Besides, for $t \in \mathcal{M}_\varepsilon(\rho_\varepsilon,\sigma_\varepsilon)$, we have

$$\int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla t|^2\,dx \le \lim_{k\to\infty} \int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla u_k|^2\,dx = \Lambda_\varepsilon$$

and

$$\Lambda_\varepsilon \le \int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla t|^2\,dx.$$

Through the method of variation, we see that there exists some harmonic function $t(x)$ to reach the Λ_ε which satisfies the following:

$$\begin{cases} \Delta t = 0 \quad \text{in } \mathbb{B}_\delta(x_\varepsilon) \setminus \mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon), \\ t|_{\partial\mathbb{B}_\delta(x_\varepsilon)} = \rho_\varepsilon, \\ t|_{\partial\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} = \sigma_\varepsilon. \end{cases} \tag{59}$$

Obviously, the solution of (59) can be expressed as

$$t(x) = a\log|x - x_0| + b.$$

One can check that

$$\begin{cases} a = \frac{\sigma_\varepsilon - \rho_\varepsilon}{\log\delta - \log Rr_\varepsilon^{1/(1-\gamma)}}, \\ b = \frac{\sigma_\varepsilon \log Rr_\varepsilon^{1/(1-\gamma)} - \rho_\varepsilon\log\delta}{\log Rr_\varepsilon^{1/(1-\gamma)} - \log\delta}. \end{cases} \tag{60}$$

Thus, $t(x)$ can be expressed as

$$t(x) = \frac{\sigma_\varepsilon(\log\delta - \log|x - x_\varepsilon|) - \rho_\varepsilon(\log Rr_\varepsilon^{1/(1-\gamma)} - \log|x - x_\varepsilon|)}{\log\delta - \log Rr_\varepsilon^{1/(1-\gamma)}}.$$

With a direct computation, it is easy to check that

$$\int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla t|^2\,dx = \frac{2\pi(\sigma_\varepsilon - \rho_\varepsilon)^2}{\log\delta - \log Rr_\varepsilon^{1/(1-\gamma)}}. \tag{61}$$

According to (23), we have

$$\log\delta - \log Rr_\varepsilon^{1/(1-\gamma)} = \log\delta - \log R + \frac{2\pi(1-\gamma-\varepsilon)c_\varepsilon^2}{1-\gamma} - \frac{1}{2(1-\gamma)}\log\frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{62}$$

Furthermore, Lemma 10 and (31) show that

$$\sigma_\varepsilon = c_\varepsilon + \frac{1}{c_\varepsilon}\left(-\frac{1}{4\pi(1-\gamma)}\log\left(1 + \frac{\pi}{1-\gamma}R^{2(1-\gamma)}\right) + o(1)\right) \tag{63}$$

and

$$\rho_\varepsilon = \frac{1}{c_\varepsilon}\left(-\frac{1}{2\pi}\log\delta + A_0 + o(1)\right), \tag{64}$$

where $o(1) \to 0$ by letting $\varepsilon \to 0$ and $\delta \to 0$ in succession. Set $u_\varepsilon^* = \max\{\rho_\varepsilon, \min\{u_\varepsilon, \sigma_\varepsilon\}\}$. From $u_\varepsilon^* \in \mathcal{M}_\varepsilon(\rho_\varepsilon, \sigma_\varepsilon)$, one can easily check that

$$\int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla t|^2\,dx = \Lambda_\varepsilon \le \int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \left|\nabla u_\varepsilon^*\right|^2 dx. \tag{65}$$

Observe that $|\nabla u_\varepsilon^*| \le |\nabla u_\varepsilon|$ a.e. in $\mathbb{B}_\delta(x_\varepsilon) \setminus \mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)$ if ε is sufficiently small. Thus, it follows

$$\int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \left|\nabla u_\varepsilon^*\right|^2 dx \le \int_{\mathbb{B}_\delta(x_\varepsilon)\setminus\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla u_\varepsilon|^2\,dx. \tag{66}$$

In view of (61), (65) and (66), it can be inferred that

$$\begin{aligned} 2\pi(\sigma_\varepsilon - \rho_\varepsilon)^2 \le{}& \left(1 - \int_{\Omega\setminus\mathbb{B}_\delta(x_\varepsilon)} |\nabla u_\varepsilon|^2\,dx - \int_{\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla u_\varepsilon|^2\,dx\right) \\ &\times \left(\log\delta - \log Rr_\varepsilon^{1/(1-\gamma)}\right). \end{aligned} \tag{67}$$

Since $c_\varepsilon u_\varepsilon \to G$ in $C^1_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$, we obtain the conclusion through integrating by parts:

$$\begin{aligned}\int_{\Omega\backslash\mathbb{B}_\delta(x_\varepsilon)} |\nabla u_\varepsilon|^2\,dx &= \frac{1}{c_\varepsilon^2}\int_{\Omega\backslash\mathbb{B}_\delta(x_\varepsilon)} |\nabla G_\varepsilon|^2\,dx \\ &= -\frac{1}{c_\varepsilon^2}\left(\int_{\Omega\backslash\mathbb{B}_\delta(x_\varepsilon)} G\Delta G\,dx + \int_{\partial\mathbb{B}_\delta(x_\varepsilon)} G\frac{\partial G}{\partial\nu}\,ds\right) \\ &= -\frac{1}{c_\varepsilon^2}\left(\frac{1}{2\pi}\log\delta - A_0 + o_\varepsilon(1) + o_\delta(1)\right). \end{aligned} \tag{68}$$

Observe that $\vartheta_\varepsilon \to \vartheta$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\{0\})$, and

$$u_\varepsilon = \frac{\vartheta_\varepsilon(x)}{c_\varepsilon} + c_\varepsilon \quad \text{in } \mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon). \tag{69}$$

A direct computation shows that

$$\begin{aligned}\int_{\mathbb{B}_R(0)} |\nabla\vartheta|^2\,dx &= \int_0^R \frac{2\pi}{4(1-\gamma)^2(1+\frac{\pi}{1-\gamma}|r|^{2(1-\gamma)})^2} r^{-4\gamma}\,dr \\ &= \frac{1}{4\pi(1-\gamma)}\log\frac{\pi}{1-\gamma} + \frac{1}{2\pi}\log R - \frac{1}{4\pi(1-\gamma)} + O\left(\frac{1}{R^{2(1-\gamma)}}\right). \end{aligned} \tag{70}$$

Then it follows from (69) and (70) that

$$\begin{aligned}\int_{\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} |\nabla u_\varepsilon|^2\,dx &= \frac{1}{c_\varepsilon^2}\int_{\mathbb{B}_{Rr_\varepsilon^{1/(1-\gamma)}}(x_\varepsilon)} \left|\nabla\vartheta_\varepsilon(x)\right|^2 dx \\ &= \frac{1}{c_\varepsilon^2}\left(\int_{\mathbb{B}_R(0)} \left|\nabla\vartheta(y)\right|^2 dy + o_\varepsilon(1)\right) \\ &= \frac{1}{4\pi c_\varepsilon^2(1-\gamma)}\log\frac{\pi}{1-\gamma} + \frac{1}{2\pi c_\varepsilon^2}\log R - \frac{1}{4\pi c_\varepsilon^2(1-\gamma)} + \frac{o(1)}{c_\varepsilon^2}. \end{aligned}$$

This together with (62)–(64) and (68), we obtain

$$-2\pi A_0 - \frac{\log(1+\frac{\pi}{1-\gamma}R^{2(1-\gamma)})}{1-\gamma} \le -2\log R + \frac{(1-\log\frac{\lambda_\varepsilon}{c_\varepsilon^2} - \log\frac{\pi}{1-\gamma})}{2(1-\gamma)} + o(1).$$

Hence,

$$\limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} \le \frac{\pi}{1-\gamma}e^{4\pi(1-\gamma)A_0+1}.$$

In view of Lemma 7, we arrive at the conclusion

$$\Lambda_{4\pi(1-\gamma)} \le \left(1+g(0)\right)\int_\Omega |x|^{-2\gamma}\,dx + \frac{\pi}{1-\gamma}e^{4\pi(1-\gamma)A_0+1}. \tag{71}$$

2.4 Completion of the proof of Theorem 1

As a consequence, if $c_\varepsilon \to \infty$, it follows from (71) that $\Lambda_{4\pi(1-\gamma)}$ is bounded. Otherwise, we can find the extremal function u_0 which satisfies (17). Therefore, necessarily

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\le 1}\int_\Omega \bigl(1+g(u)\bigr)\frac{e^{4\pi(1-\gamma)u^2}}{|x|^{2\gamma}}\,dx<\infty.$$

3 Proof of Theorem 2

3.1 Test function computation

Similar to [30], we construct a blow-up sequence $\phi_\varepsilon \in W_0^{1,2}(\Omega)$ with $\|\nabla\phi_\varepsilon\|_2 = 1$. For sufficiently small $\varepsilon > 0$, there exists

$$\int_\Omega |x|^{-2\gamma}\bigl(1+g(\phi_\varepsilon)\bigr)e^{4\pi(1-\gamma)\phi_\varepsilon^2}\,dx > \bigl(1+g(0)\bigr)\int_\Omega |x|^{-2\gamma}\,dx + \frac{\pi}{1-\gamma}e^{4\pi(1-\gamma)A_0+1}. \tag{72}$$

Then we will find (72) is a contradiction to (71), so that c_ε has to be bounded, which means the blow-up cannot take place. Furthermore, Theorem 2 follows immediately from what we have proved according to the elliptic estimates. For this purpose we set

$$\phi_\varepsilon(x)=\begin{cases} c+\frac{1}{c}(-\frac{1}{4\pi(1-\gamma)}\log(1+\frac{\pi}{1-\gamma}\frac{|x|^{2(1-\gamma)}}{\varepsilon^{2(1-\gamma)}})+b), & \text{for } x\in\overline{\mathbb{B}}_{R\varepsilon},\\ \frac{G-\xi\eta}{c}, & \text{for } x\in \mathbb{B}_{2R\varepsilon}\setminus\overline{\mathbb{B}}_{R\varepsilon},\\ \frac{G}{c}, & \text{for } x\in\Omega\setminus\mathbb{B}_{2R\varepsilon}, \end{cases} \tag{73}$$

where $\eta \in C_0^1(\mathbb{B}_{2R\varepsilon})$ is a cut-off function satisfying $\eta = 1$ on $\mathbb{B}_{R\varepsilon}$, and $|\nabla\eta| \le \frac{2}{R\varepsilon}$. And G is given as in (56). b and c are constants which depend only on ε, to be determined later. To ensure $\phi_\varepsilon \in W_0^{1,2}(\Omega)$, we let

$$c+\frac{1}{c}\biggl(-\frac{1}{4\pi(1-\gamma)}\log\biggl(1+\frac{\pi}{1-\gamma}\frac{|x|^{2(1-\gamma)}}{\varepsilon^{2(1-\gamma)}}\biggr)+b\biggr)=\frac{1}{c}\biggl(-\frac{1}{2\pi}\log R\varepsilon + A_0\biggr),$$

which leads to

$$2\pi c^2 = -\log\varepsilon - 2\pi b + 2\pi A_0 + \frac{1}{2(1-\gamma)}\log\frac{\pi}{1-\gamma} + O\biggl(\frac{1}{R^{2(1-\gamma)}}\biggr). \tag{74}$$

Now we calculate

$$\begin{aligned}\int_{\mathbb{B}_{R\varepsilon}}|\nabla\phi_\varepsilon|^2\,dx &= \int_{\mathbb{B}_R}\frac{|x|^{2-4\gamma}}{4c^2(1-\gamma)^2(1+\frac{\pi}{1-\gamma}|x|^{2-4\gamma})^2}\,dx\\ &= \int_0^{\frac{\pi}{1-\gamma}R^{2-2\gamma}}\frac{t\,dt}{4\pi c^2(1-\gamma)(1+t)^2}\,dt\\ &= \frac{1}{4\pi c^2(1-\gamma)}\biggl(\log\frac{\pi}{1-\gamma}-1+\log R^{2-2\gamma}+O\biggl(\frac{1}{R^{2-2\gamma}}\biggr)\biggr). \end{aligned} \tag{75}$$

On the other hand

$$\begin{aligned}\int_{\Omega\setminus\mathbb{B}_{R\varepsilon}}|\nabla\phi_\varepsilon|^2\,dx &= \frac{1}{c^2}\biggl(\int_{\Omega\setminus\mathbb{B}_{R\varepsilon}}|\nabla G|^2\,dx+\int_{\mathbb{B}_{2R\varepsilon}\setminus\mathbb{B}_{R\varepsilon}}\bigl|\nabla(\xi\eta)\bigr|^2\,dx\\ &\quad -2\int_{\mathbb{B}_{2R\varepsilon}\setminus\mathbb{B}_{R\varepsilon}}\nabla G\nabla(\xi\eta)\,dx\biggr)\\ &= \frac{1}{c^2}\biggl(-\int_{\Omega\setminus\mathbb{B}_{R\varepsilon}}G\Delta G\,dx-\int_{\partial\mathbb{B}_{R\varepsilon}}G\frac{\partial G}{\partial\nu}\,ds\\ &\quad +\int_{\mathbb{B}_{2R\varepsilon}\setminus\mathbb{B}_{R\varepsilon}}\bigl|\nabla(\xi\eta)\bigr|^2\,dx-2\int_{\mathbb{B}_{2R\varepsilon}\setminus\mathbb{B}_{R\varepsilon}}\nabla G\nabla(\xi\eta)\,dx\biggr).\end{aligned}$$

Observe that $\xi(x)=O(|x|)$ as $x\to0$. Since η is a cut-off function, it yields $|\nabla(\xi\eta)|=O(1)$ as $\varepsilon\to0$. Then we have

$$\int_{\mathbb{B}_{2R\varepsilon}\setminus\mathbb{B}_{R\varepsilon}}\bigl|\nabla(\xi\eta)\bigr|^2\,dx=O\bigl(R^2\varepsilon^2\bigr),\qquad \int_{\mathbb{B}_{2R\varepsilon}\setminus\mathbb{B}_{R\varepsilon}}\nabla G\nabla(\xi\eta)\,dx=O(R\varepsilon),$$

which together with (56) leads to

$$\int_{\Omega\setminus\mathbb{B}_{R\varepsilon}}|\nabla\phi_\varepsilon|^2\,dx=\frac{1}{c^2}\biggl(-\frac{1}{2\pi}\log(R\varepsilon)+A_0+O(R\varepsilon)\biggr). \tag{76}$$

Combining (75) and (76), a delicate but straightforward calculation shows

$$\int_\Omega|\nabla\phi_\varepsilon|^2\,dx=\frac{1}{c^2}\biggl(-\frac{\log\varepsilon}{2\pi}-\frac{1}{4\pi(1-\gamma)}+\frac{1}{4\pi(1-\gamma)}\log\frac{\pi}{1-\gamma}+A_0+O\biggl(\frac{1}{R^{2-2\gamma}}\biggr)\biggr).$$

Put $\|\nabla\phi_\varepsilon\|_2=1$. It yields

$$c^2=A_0-\frac{1}{2\pi}\log\varepsilon+\frac{1}{4\pi(1-\gamma)}\log\frac{\pi}{1-\gamma}-\frac{1}{4\pi(1-\gamma)}+O\biggl(\frac{1}{R^{2-2\gamma}}\biggr). \tag{77}$$

Together with (74) and (77), we are led to

$$b=\frac{1}{4\pi(1-\gamma)}+O\biggl(\frac{1}{R^{2-2\gamma}}\biggr). \tag{78}$$

For all $x\in\mathbb{B}_{R\varepsilon}$, it follows from (77) and (78) that

$$\begin{aligned}4\pi(1-\gamma)\phi_\varepsilon^2 &\ge 4\pi(1-\gamma)c^2+8\pi(1-\gamma)b-2\log\biggl(1+\frac{\pi|x|^{2(1-\gamma)}}{(1-\gamma)\varepsilon^{2(1-\gamma)}}\biggr)\\ &= 1+4\pi(1-\gamma)A_0+\log\frac{\pi}{1-\gamma}-2(1-\gamma)\log\varepsilon\\ &\quad -2\log\biggl(1+\frac{\pi|x|^{2(1-\gamma)}}{(1-\gamma)\varepsilon^{2(1-\gamma)}}\biggr)+O\biggl(\frac{1}{R^{2-2\gamma}}\biggr).\end{aligned} \tag{79}$$

Note that $\|\frac{\phi_\varepsilon(x)}{c}\|_{L^\infty(B_{R\varepsilon})}\to1$ by passing to the limit $\varepsilon\to0$. When $r\le R\varepsilon$, there exists

$$\biggl|\frac{\phi_\varepsilon(x)}{c}\biggr|=\biggl|1+\frac{-\log(1+\pi\frac{r^2}{\varepsilon^2})+b}{c^2}\biggr|\to1.$$

as $\varepsilon \to 0$. Since $\phi_\varepsilon(x) \sim c$ in $\mathbb{B}_{R\varepsilon}$ and $g(c) = o(\frac{1}{c^2})$, we conclude $g(\phi_\varepsilon(\xi_\varepsilon)) = o(\frac{1}{c^2})$ as $\varepsilon \to 0$, where $\xi_\varepsilon \in \mathbb{B}_{R\varepsilon}$. Combining with the mean value theorem, it follows from (79) that

$$\begin{aligned}\int_{\mathbb{B}_{R\varepsilon}} \big(1 + g(\phi_\varepsilon)\big) \frac{e^{4\pi(1-\gamma)\phi_\varepsilon^2}}{|x|^{2\gamma}}\, dx &= \big(1 + g\big(\phi_\varepsilon(\xi_\varepsilon)\big)\big) \int_{\mathbb{B}_{R\varepsilon}} \frac{e^{4\pi(1-\gamma)\phi_\varepsilon^2}}{|x|^{2\gamma}}\, dx \\ &\geq \left(1 + o\left(\frac{1}{c^2}\right)\right) \frac{\pi}{1-\gamma} e^{1+4\pi(1-\gamma)A_0 + O(\frac{1}{R^{2-2\gamma}})} \\ &\quad \times \int_0^R \frac{2\pi r^{1-2\gamma}}{(1 + \frac{\pi}{1-\gamma} r^{2-2\gamma})^2}\, dr \\ &= \frac{\pi}{(1-\gamma)} e^{1+4\pi(1-\gamma)A_0} + O\left(\frac{1}{R^{2-2\gamma}}\right) + o\left(\frac{1}{c^2}\right). \end{aligned} \tag{80}$$

Furthermore, $\frac{G}{c_\varepsilon} \geq 0$ a.e. in $\Omega \setminus \mathbb{B}_{R\varepsilon}$, by using the inequality $e^t \geq t + 1$, $\forall t \geq 0$, we estimate

$$\begin{aligned}&\int_{\Omega \setminus \mathbb{B}_{R\varepsilon}} \big(1 + g(\phi_\varepsilon)\big) \frac{e^{4\pi(1-\gamma)\phi_\varepsilon^2}}{|x|^{2\gamma}}\, dx \\ &\quad \geq \int_{\Omega \setminus \mathbb{B}_{2R\varepsilon}} \big(1 + g(\phi_\varepsilon)\big) \frac{1 + 4\pi(1-\gamma)\phi_\varepsilon^2}{|x|^{2\gamma}}\, dx \\ &\quad \geq \int_\Omega \big(1 + g(0)\big) |x|^{-2\gamma}\, dx + O\big((R\varepsilon)^{2-2\gamma} \log^2(R\varepsilon)\big) \\ &\qquad + \frac{4\pi(1-\gamma)}{c^2} \int_\Omega \big(1 + g(0)\big) |x|^{-2\gamma} G^2\, dx + O\big((R\varepsilon)^{2-2\gamma}\big). \end{aligned} \tag{81}$$

Observe that

$$O\big((R\varepsilon)^{2-2\gamma}\big) = O\big((R\varepsilon)^{2-2\gamma} \log^2(R\varepsilon)\big) = O\left(\frac{1}{R^{2-2\gamma}}\right).$$

This together with (80) and (81) yields

$$\begin{aligned}&\int_\Omega \big(1 + g(\phi_\varepsilon)\big) \frac{e^{4\pi(1-\gamma)\phi_\varepsilon^2}}{|x|^{2\gamma}}\, dx \\ &\quad \geq \big(1 + g(0)\big) \int_\Omega |x|^{-2\gamma}\, dx + \frac{\pi}{(1-\gamma)} e^{4\pi(1-\gamma)A_0+1} \\ &\qquad + \frac{4\pi(1-\gamma)}{c^2} \int_\Omega \frac{(1 + g(0))G^2}{|x|^{2\gamma}}\, dx + O\left(\frac{1}{R^{2-2\gamma}}\right) + o\left(\frac{1}{c^2}\right). \end{aligned} \tag{82}$$

Recalling (77) and the choice $R = -\log \varepsilon^{1/(1-\gamma)}$, one can deduce that $\frac{1}{R^{2-2\gamma}} = o(\frac{1}{c^2})$. Therefore, we conclude from (82) that

$$\int_\Omega \big(1 + g(\phi_\varepsilon)\big) \frac{e^{4\pi(1-\gamma)\phi_\varepsilon^2}}{|x|^{2\gamma}}\, dx > \big(1 + g(0)\big) \int_\Omega |x|^{-2\gamma}\, dx + \frac{\pi}{1-\gamma} e^{4\pi(1-\gamma)A_0+1}.$$

for sufficiently small $\varepsilon > 0$.

3.2 Completion of the proof of Theorem 2

Comparing (71) with (72), we arrive at the final conclusion that c_ε must be bounded. Then applying elliptic estimates to (16), we can get the desired extremal function. This ends the proof of Theorem 2.

Funding
Not applicable.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 December 2018 Accepted: 28 May 2019 Published online: 04 June 2019

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