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Weighted inequalities for fractional Hardy operators and commutators

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Abstract

In this paper, we introduce a fractional maximal operators N_α on $(0,\infty)$ associated to the fractional Hardy operator P_α and its dual Q_α, $0 \le \alpha < 1$, and obtain some characterizations for the one-weight and two-weight inequalities for N_α. We also give some A_p type sufficient conditions for the two-weight inequalities for the fractional Hardy operators, the dual operators and the commutators of the fractional Hardy operators with CMO functions.

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1 Introduction

Let P_α and Q_α be the fractional Hardy operator and its adjoint on $(0,\infty)$,

$$P_\alpha f(x) = \frac{1}{x^{1-\alpha}} \int_0^x f(y)\,dy, \qquad Q_\alpha f(x) = \int_x^\infty \frac{f(y)}{y^{1-\alpha}}\,dy,$$

where $0 \le \alpha < 1$.

When $\alpha = 0$, we denote P_0 as P and Q_0 as Q. P and Q are the Hardy operator and its adjoint. Hardy [6, 7] established the Hardy integral inequalities

$$\int_0^\infty |Pf(y)|^p\,dy \le p'^p \int_0^\infty |f(y)|^p\,dy, \quad p > 1,$$

and

$$\int_0^\infty |Qf(y)|^p\,dy \le p^p \int_0^\infty |f(y)|^p\,dy, \quad p > 1,$$

where $p' = p/(p-1)$.

The two inequalities above go by the name of Hardy's integral inequalities. For earlier development of this kind of inequality and many applications in analysis, see [2, 8, 13].

The fractional Calderón operator S_α is defined as $S_\alpha = P_\alpha + Q_\alpha$. When $\alpha = 0$, S_0 is denoted S, and S is the Calderón operator, which plays a significant role in the theory of real interpolation; see [1]. Next, we introduce the fractional maximal operators N_α related to the fractional Calderón operator. Given a measurable function f on $(0,\infty)$, the fractional maximal operator N_α is defined as

$$N_\alpha f(x) = \sup_{t>x} \frac{1}{t^{1-\alpha}} \int_0^t \left|f(y)\right| dy, \quad x > 0.$$

Notice that $N_\alpha f$ is a decreasing function, and that $|P_\alpha f| \le N_\alpha f \le S_\alpha(|f|)$, for any f. Indeed, for any f and $t > x$ we have

$$\left|P_\alpha f(x)\right| \le \sup_{t>x} \frac{1}{t^{1-\alpha}} \int_0^t \left|f(y)\right| dy \le N_\alpha f(x),$$
$$\frac{1}{t^{1-\alpha}} \int_0^t \left|f(y)\right| dy \le \frac{1}{x^{1-\alpha}} \int_0^x \left|f(y)\right| dy + \int_x^t \frac{|f(y)|}{y^{1-\alpha}} dy \le S_\alpha f(x).$$

Notice that $N_\alpha f \le S_\alpha f$ for nonnegative f.

For $\alpha = 0$, N_0 is denoted as N. Duoandikoetxea, Martin-Reyes and Ombrosi in [3] introduced the maximal operator N related to the Calderón operator and studied the weighted inequalities for N. Li, Zhang and Xue [9] obtained some two-weight inequalities for N.

For $1 < p \le q < \infty$, we say a weight w satisfies the $A_{p,q,0}$ condition, denoted as $w \in A_{p,q,0}$, if

$$[w]_{p,q,0} = \sup_{t>0}\left(\frac{1}{t}\int_0^t w(y)^q\, dy\right)^{1/q}\left(\frac{1}{t}\int_0^t w(y)^{-p'}\, dy\right)^{1/p'} < \infty.$$

For $p = 1 < q < \infty$, we write $A_{1,q,0}$ for the class of nonnegative functions w such that

$$[w]_{1,q,0} = \sup_{t>0}\left(\frac{1}{t}\int_0^t w(y)^q\, dy\right)^{1/q}\left(\operatorname{ess\,sup}_{x\in(0,t)} \frac{1}{w(y)}\right) < \infty.$$

For $1 < p \le q < \infty$, we say a pair of weights (u, v) satisfies the two-weight $A_{p,q}$ condition, denoted $(u, v) \in A_{p,q}$, if

$$[u,v]_{p,q} = \sup_{t>0}\left(\frac{1}{t}\int_0^t u(y)^q\, dy\right)^{1/q}\left(\frac{1}{t}\int_0^t v(y)^{-p'}\, dy\right)^{1/p'} < \infty.$$

For $p = 1$ we write $(u, v) \in A_{1,q}$, if

$$[u,v]_{1,q} = \sup_{t>0}\left(\frac{1}{t}\int_0^t u(y)^q\, dy\right)^{1/q}\left(\operatorname{ess\,sup}_{x\in(0,t)} \frac{1}{v(y)}\right) < \infty.$$

Let $1 \le p < \infty$, we say b is a one-side dyadic CMO^p function, if

$$\sup_{j\in\mathbb{Z}}\left(\frac{1}{2^j}\int_0^{2^j} \left|b(y) - b_{(0,2^j]}\right|^p dy\right)^{1/p} = \|b\|_{\mathrm{CMO}^p} < \infty,$$

where $b_{(0,2^j]} = \frac{1}{2^j}\int_0^{2^j} f(x)\, dx$, we then say that $b \in \mathrm{CMO}^p$.

It is easy to see $\mathrm{BMO}(0,\infty) \subsetneqq \mathrm{CMO}^p$, where $1 \le p < \infty$. $\mathrm{CMO}^q \subsetneqq \mathrm{CMO}^p$ for $1 \le p < q < \infty$.

Let b be a locally integrable function on $(0,\infty)$, we define the commutators of the fractional Calderón operator S_α with b as $S_\alpha^b = P_\alpha^b + Q_\alpha^b$, where

$$P_\alpha^b f(x) = \frac{1}{x^{1-\alpha}}\int_0^x \bigl(b(x)-b(y)\bigr)f(y)\,dy, \qquad Q_\alpha^b f(x) = \int_x^\infty \frac{(b(x)-b(y))f(y)}{y^{1-\alpha}}\,dy.$$

For $\alpha = 0$, P_α^b and Q_α^b be denoted as P^b and Q^b, respectively. Long and Wang [10] established Hardy's integral inequalities for commutators generated by P and Q with one-sided dyadic CMO functions, and for $0<\alpha<1$ proved that the two commutators of P_α^b and Q_α^b are bounded from $L^p(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+)$ with the function b in one-side dyadic $\mathrm{CMO}^{\max(q,p')}$, where $1<p<q<\infty$, $\frac{1}{q}=\frac{1}{p}-\alpha$. Fu [4], Zheng and Fu [14] showed some boundedness properties for P_α^b and Q_α^b, respectively. Li, Zhang and Xue [9] obtained some A_p type sufficient conditions such that the two-weight inequalities are true for P, Q, P^b and Q^b. The commutator estimates for Hardy operator are actual in view of applications in PDE; see Mamedov and Brahimov [11].

In this paper, we discuss the one-weight and two-weight inequalities of operators N_α, P_α, Q_α, P_α^b, Q_α^b, and get the following results.

Theorem 1.1 *For $1\le p<\frac{1}{\alpha}$, $0\le\alpha<1$, $\frac{1}{q}=\frac{1}{p}-\alpha$, N_α is bounded from $L^p(w^p)$ to $L^{q,\infty}(w^q)$ if and only if $w\in A_{p,q,0}$. More precisely,*

$$\sup_{\lambda>0}\lambda\left(\int_{\{x:N_\alpha f(x)>\lambda\}} w(y)^q\,dy\right)^{1/q} \le [w]_{p,q,0}\|f\|_{L^p(w^p)}. \tag{1}$$

For $1<p<\frac{1}{\alpha}$, $0\le\alpha<1$, $\frac{1}{q}=\frac{1}{p}-\alpha$, N_α is bounded from $L^p(w^p)$ to $L^q(w^q)$ if and only if $w\in A_{p,q,0}$. Moreover,

$$\|N_\alpha f\|_{L^q(w^q)} \le C[w]_{p,q,0}^{(1-\alpha)p'q}\|f\|_{L^p(w^p)}. \tag{2}$$

Theorem 1.2 *For $1\le p<\frac{1}{\alpha}$, $0\le\alpha<1$, $\frac{1}{q}=\frac{1}{p}-\alpha$, N_α is bounded from $L^p(v^p)$ to $L^{q,\infty}(u^q)$ if and only if $(u,v)\in A_{p,q}$. More precisely,*

$$\sup_{\lambda>0}\lambda\left(\int_{\{x:N_\alpha f(x)>\lambda\}} u(y)^q\,dy\right)^{1/q} \le [u,v]_{p,q}\|f\|_{L^p(v^p)}.$$

Theorem 1.3 *For $1<p<\infty$, $0\le\alpha<1$, $0<q<\infty$, N_α is bounded from $L^p(v^p)$ to $L^q(u^q)$, if and only if, for any $t>0$, (u,v) satisfies*

$$\left(\int_0^t \bigl[N_\alpha\bigl(v^{-p'}\chi_{(0,t)}\bigr)(y)\bigr]^q u(y)^q\,dy\right)^{1/q} \le C\left(\int_0^t v(y)^{-p'}\,dy\right)^{1/p} < \infty. \tag{3}$$

But for $1<p<\frac{1}{\alpha}$, $0\le\alpha<1$, $\frac{1}{q}=\frac{1}{p}-\alpha$, N_α is not bounded from $L^p(v^p)$ to $L^q(u^q)$ if $(u,v)\in A_{p,q}$, the proof is similar to the case for the Hardy–Littlewood maximal function on $\mathbb{R}^n$; see [5]. Notice that $|P_\alpha f|\le N_\alpha f\le S_\alpha(|f|)$, by Theorem 1.1, we see that $(u,v)\in A_{p,q}$, is necessary but not sufficient for S_α to be bounded from $L^p(v^p)$ to $L^q(u^q)$.

Theorem 1.4 *Let $1<p<q<\infty$, $0\le\alpha<1$.*

(a) *Let (u,v) be a pair of weights for which there exists $r>1$ such that, for every $t>0$,*

$$t^{(1/q+\alpha-1/p)}\left(\frac{1}{t}\int_0^t u(y)^q\,dy\right)^{1/q}\left(\frac{1}{t}\int_0^t v(y)^{-rp'}\,dy\right)^{1/rp'}\le C<\infty. \tag{4}$$

Then

$$\left(\int_0^\infty \left|P_\alpha f(x)\right|^q u(x)^q\,dx\right)^{1/q}\le C\left(\int_0^\infty \left|f(x)\right|^p v(x)^p\,dx\right)^{1/p}. \tag{5}$$

(b) *Let (u,v) be a pair of weights for which there exists $r>1$ such that, for every $t>0$,*

$$t^{(1/q+\alpha-1/p)}\left(\frac{1}{t}\int_0^t u(y)^{rq}\,dy\right)^{1/rq}\left(\frac{1}{t}\int_0^t v(y)^{-p'}\,dy\right)^{1/p'}\le C<\infty. \tag{6}$$

Then

$$\left(\int_0^\infty \left|Q_\alpha f(x)\right|^q u(x)^q\,dx\right)^{1/q}\le C\left(\int_0^\infty \left|f(x)\right|^p v(x)^p\,dx\right)^{1/p}. \tag{7}$$

Theorem 1.5 *Let $1<p<q<\infty$, $0\le\alpha<1$, $b\in CMO^{r'\max\{q,p'\}}$, and (u,v) be a pair of weights for which there exists $r>1$ such that, for every $t>0$,*

$$t^{(1/q+\alpha-1/p)}\left(\frac{1}{t}\int_0^t u(y)^{rq}\,dy\right)^{1/rq}\left(\frac{1}{t}\int_0^t v(y)^{-rp'}\,dy\right)^{1/rp'}\le C<\infty. \tag{8}$$

Then

$$\left(\int_0^\infty \left|P_\alpha^b f(x)\right|^q u(x)^q\,dx\right)^{1/q}\le C\left(\int_0^\infty \left|f(x)\right|^p v(x)^p\,dx\right)^{1/p} \tag{9}$$

and

$$\left(\int_0^\infty \left|Q_\alpha^b f(x)\right|^q u(x)^q\,dx\right)^{1/q}\le C\left(\int_0^\infty \left|f(x)\right|^p v(x)^p\,dx\right)^{1/p}. \tag{10}$$

2 The proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3

In order to prove the theorems, we need the fractional maximal operator N_α^g associated to a fixed positive measurable function g. For $0\le\alpha<1$, we defined N_α^g as

$$N_\alpha^g f(x)=\sup_{t>x}\frac{\int_0^t |f(y)|g(y)\,dy}{(\int_0^t g(y)\,dy)^{1-\alpha}}.$$

Mamedov and Zeren [12] obtained the two-weight inequalities for this maximal operator in the Lebesgue spaces with variable exponent. When $\alpha=0$, we denote N_α^g as N^g. Duoandikoetxea, Martin-Reyes and Ombrosi in [3] obtained the following lemma for N^g.

Lemma 2.1 *Let $0\le\alpha<1$ and g be a nonnegative measurable function such that $0<g(0,t)=\int_0^t g(x)\,dx<\infty$, for all $t>0$.*

(i) N_α^g *is of weak type* $(1, \frac{1}{1-\alpha})$ *with respect to the measure* $g(t)\,dt$. *Actually,*

$$\left(\int_{\{x:N_\alpha^g f(x)>\lambda\}} g(y)\,dy\right)^{1-\alpha} \le \frac{1}{\lambda}\int_{\{x:N_\alpha^g f(x)>\lambda\}} \big|f(y)\big|g(y)\,dy, \tag{11}$$

for all $\lambda > 0$ *and all measurable functions* f.

(ii) N_α^g *is of strong type* (p,q), $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, *with respect to the measure* $g(t)\,dt$. *More precisely,*

$$\left(\int_0^\infty \big|N_\alpha^g f(y)\big|^q g(y)\,dy\right)^{1/q} \le C(p,\alpha)\left(\int_0^\infty \big|f(y)\big|^p g(y)\,dy\right)^{1/p},$$

in which the constant $C(p,\alpha)$ *independent of* f *and* g.

Proof By standard interpolation arguments, it suffices to prove (i) since by the Hölder inequality, we have

$$\begin{aligned}\frac{\int_0^t |f(x)|g(x)\,dx}{(\int_0^t g(x)\,dx)^{1-\alpha}} &\le \frac{(\int_0^t |f(x)|^{1/\alpha} g(x)\,dx)^\alpha (\int_0^t g(x)\,dx)^{1-\alpha}}{(\int_0^t g(x)\,dx)^{1-\alpha}} \\ &= \left(\int_0^t \big|f(x)\big|^{1/\alpha} g(x)\,dx\right)^\alpha \\ &\le \|f\|_{L^{1/\alpha}(g)}.\end{aligned}$$

Thus we obtain $\|N_\alpha^g f\|_\infty \le \|f\|_{L^{1/\alpha}(g)}$.

Observe that $N_\alpha^g f$ is decreasing and continuous. Therefore, if $\{t : N_\alpha^g(f)(t) > \lambda\}$ is not empty, then it is either a bounded interval $(0,d)$ or all of $(0,\infty)$. In the first case

$$\lambda\left(\int_0^d g(x)\,dx\right)^{1-\alpha} = \int_0^d \big|f(x)\big|g(x)\,dx, \tag{12}$$

whereas in the second case we have

$$\lambda\left(\int_0^\infty g(x)\,dx\right)^{1-\alpha} \le \int_0^\infty \big|f(x)\big|g(x)\,dx.$$

Thus we obtain (11). Notice that if $g(0,\infty) = \int_0^\infty g(x)\,dx = +\infty$ and f is integrable with respect to g, only the first case is possible and the equality holds. □

Proof of Theorem 1.1 Let us prove first the necessity of $A_{p,q,0}$ for the weak-type inequality.

(i) For $1 < p < \frac{1}{\alpha}$, let $E_k = \{x : w(x) > 1/k\}$ and $w_k = w\chi_{E_k}$. Take $f = w_k^{-p'}\chi_{(0,t)}$. Then

$$N_\alpha f(x) \ge \frac{1}{t^{1-\alpha}}\int_0^t w_k^{-p'}, \tag{13}$$

for $0 < x < t$. Thus, $(0,t) \subset \{x : N_\alpha f(x) > \lambda\}$ taking as λ the right-hand side of (13). By N_α is bounded from $L^p(w^p)$ to $L^{q,\infty}(w^q)$, we have

$$\left(\int_0^t w^q\right)^{1/q}\left(\frac{1}{t}\int_0^t w_k^{-p'}\right) = \frac{1}{t^\alpha}\left(\int_0^t w^q\right)^{1/q} \cdot \lambda$$

$$\le \frac{1}{t^\alpha}\left(\int_{\{x:N_\alpha f(x)>\lambda\}} w^q\right)^{1/q}\cdot\lambda$$
$$\le \frac{C}{t^\alpha}\left(\int_0^t w_k^{-p'p}w^p\right)^{1/p}$$
$$= \frac{C}{t^\alpha}\left(\int_0^t w_k^{-p'}\right)^{1/p}.$$

Thus we obtain

$$\left(\frac{1}{t}\int_0^t w^q\right)^{1/q}\left(\frac{1}{t}\int_0^t w_k^{-p'}\right)^{1/p'} \le C.$$

Letting k tend to infinity, $w \in A_{p,q,0}$ follows.

(ii) For $p = 1$, let $(0, s)$ be any interval, for any interval $(t_1, t_2) \subset (0, s)$. Taking $f = \chi_{(t_1,t_2)}$. Then

$$N_\alpha f(x) \ge \frac{1}{s^{1-\alpha}}\int_0^s |\chi_{(t_1,t_2)}|\,dy = \frac{t_2 - t_1}{s^{1-\alpha}}, \tag{14}$$

for $0 < x < s$. Thus, $(0, s) \subset \{x : N_\alpha f(x) > \lambda\}$ taking as λ the right-hand side of (14). By N_α is bounded from $L^p(w^p)$ to $L^{q,\infty}(w^q)$, we have

$$\lambda\left(\int_0^s w^q\right)^{1/q} \le \lambda\left(\int_{\{x:N_\alpha f(x)>\lambda\}} w^q\right)^{1/q} \le C\int_{t_1}^{t_2} w.$$

Thus we have

$$\left(\frac{1}{s}\int_0^s w^q\right)^{1/q} \le \frac{C}{t_2 - t_1}\int_{t_1}^{t_2} w.$$

By the Lebesgue differentiation theorem, for any $y \in (0, s)$, we have

$$\left(\frac{1}{s}\int_0^s w^q\right)^{1/q} \le Cw(y).$$

Thus, $w \in A_{1,q,0}$ follows.

For the sufficient, arguing as in the proof of the Lemma 2.1, we have (12) with $g \equiv 1$, that is,

$$d^{1-\alpha}\lambda = \int_0^d \left|f(y)\right|\,dy.$$

Then

$$\lambda\left(\int_{\{x:N_\alpha f(x)>\lambda\}} w(y)^q\,dy\right)^{1/q}$$
$$= \lambda\left(\int_0^d w(y)^q\,dy\right)^{1/q}$$
$$\le \frac{1}{d^{1-\alpha}}\left(\int_0^d \left|f(y)\right|^p w(y)^p\,dy\right)^{1/p}\left(\int_0^d w(y)^{-p'}\,dy\right)^{1/p'}\left(\int_0^d w(y)^q\,dy\right)^{1/q}$$

$$\leq [w]_{p,q,0}\left(\int_0^d \left|f(y)\right|^p w(y)^p\,dy\right)^{1/p},$$

and (1) follows.

Now we prove (2). If $w \in A_{p,q,0}$, $0 < x < t$, $\sigma = w^{-p'}$ and $\delta = w^q$, we have

$$\begin{aligned}\left(\frac{1}{t^{1-\alpha}}\int_0^t f(y)\,dy\right)^{\frac{1}{(1-\alpha)p'}} &\leq [w]_{p,q,0}\left(\frac{\int_0^t f(y)\,dy}{(\int_0^t \sigma(y)\,dy)^{(1-\alpha)}}\right)^{\frac{1}{(1-\alpha)p'}}\frac{(\int_0^t dy)^{1/q}}{(\int_0^t \delta(y)\,dy)^{1/q}}\\ &\leq [w]_{p,q,0}\frac{(\int_0^t |N_\alpha^\sigma(f\sigma^{-1})(y)|^{\frac{q}{(1-\alpha)p'}}\,dy)^{1/q}}{(\int_0^t \delta(y)\,dy)^{1/q}}\\ &\leq [w]_{p,q,0}\left|N^\delta\left(\delta^{-1}\left|N_\alpha^\sigma\left(f\sigma^{-1}\right)\right|^{\frac{q}{(1-\alpha)p'}}\right)(x)\right|^{1/q}.\end{aligned}$$

Therefore,

$$\left|N_\alpha f(x)\right|^q \leq [w]_{p,q,0}^{(1-\alpha)p'q}\left|N^\delta\left(\delta^{-1}\left|N_\alpha^\sigma\left(f\sigma^{-1}\right)\right|^{\frac{q}{(1-\alpha)p'}}\right)(x)\right|^{(1-\alpha)p'}.$$

Consequently, using Lemma 2.1,

$$\begin{aligned}\int_0^\infty \left|N_\alpha f(y)\right|^q w(y)^q\,dy &\leq C[w]_{p,q,0}^{(1-\alpha)p'q}\int_0^\infty \left|N_\alpha^\sigma\left(f\sigma^{-1}\right)(y)\right|^q\sigma(y)\,dy\\ &\leq C[w]_{p,q,0}^{(1-\alpha)p'q}\left(\int_0^\infty \left|f(y)\sigma^{-1}(y)\right|^p\sigma(y)\,dy\right)^{q/p}\\ &\leq C[w]_{p,q,0}^{(1-\alpha)p'q}\left(\int_0^\infty \left|f(y)\right|^p w(y)^p\,dy\right)^{q/p}.\end{aligned}$$

This ends the proof of the theorem. □

Proof of Theorem 1.2 For $1 \leq p < \frac{1}{\alpha}$, the proof for the necessity of $A_{p,q}$ for the weak-type inequality is standard and similar to the proof of Theorem 1.1, we omitted here. For the sufficient, we observe that $N_\alpha f$ is decreasing and continuous. Therefore, if $\{x : N_\alpha f(x) > \lambda\}$ is not empty, then it is a bounded interval $(0, d)$, thus

$$d^{1-\alpha}\lambda = \int_0^d \left|f(y)\right|\,dy.$$

Then

$$\begin{aligned}&\lambda\left(\int_{\{x:N_\alpha f(x)>\lambda\}} u(y)^q\,dy\right)^{1/q}\\ &\quad= \lambda\left(\int_0^d u(y)^q\,dy\right)^{1/q}\\ &\quad\leq \frac{1}{d^{1-\alpha}}\left(\int_0^d \left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\left(\int_0^d v(y)^{-p'}\,dy\right)^{1/p'}\left(\int_0^d u(y)^q\,dy\right)^{1/q}\\ &\quad\leq [u,v]_{p,q}\left(\int_0^d \left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}.\end{aligned}$$

This ends the proof. □

Proof of Theorem 1.3 Denote $\sigma = v^{-p'}$. The necessity of (3) follows by a standard argument if we substitute $f = \sigma\chi_{(0,t)}$ into $\|N_\alpha f\|_{L^q(u^q)} \leq C\|f\|_{L^p(v^p)}$.

To show that (3) is sufficient, fix a bounded nonnegative function f with compact support. Since $N_\alpha f$ is decreasing and continuous, for each $k \in \mathbb{Z}$, if $\{x \in (0,\infty) : N_\alpha f(x) > 2^k\}$ is not empty, then there exists d_k such that $\{x \in (0,\infty) : N_\alpha f(x) > 2^k\} = (0, d_k)$. Thus $0 < d_{k+1} \leq d_k$, $\Omega_k = \{x \in (0,\infty) : 2^k < N_\alpha f(x) \leq 2^{k+1}\} = [d_{k+1}, d_k)$ and

$$2^k d_k^{1-\alpha} = \int_0^{d_k} f(y)\,dy.$$

Fix a large integer $K > 0$, which will go to infinity later, and let $\Lambda_K = \{k \in \mathbb{Z} : |k| \leq K\}$. We have

$$\begin{aligned}
\mathcal{I}_K &= \int_{\bigcup_{k=-K}^{K}\Omega_K} \bigl(N_\alpha f(y)\bigr)^q u(y)^q\,dy \leq \sum_{k=-K}^{K} 2^{(k+1)q} \int_{d_{k+1}}^{d_k} u(y)^q\,dy \\
&= 2^q \sum_{k=-K}^{K} \int_{d_{k+1}}^{d_k} u(y)^q\,dy \biggl(\frac{1}{d_k^{1-\alpha}} \int_0^{d_k} f(y)\,dy\biggr)^q \\
&= 2^q \sum_{k=-K}^{K} \int_{d_{k+1}}^{d_k} u(y)^q\,dy \biggl(\frac{1}{d_k^{1-\alpha}} \int_0^{d_k} \sigma(y)\,dy\biggr)^q \biggl(\frac{\int_0^{d_k} (f\sigma^{-1})(y)\sigma(y)\,dy}{\int_0^{d_k} \sigma(y)\,dy}\biggr)^q \\
&= 2^q \int_{\mathbb{Z}} T_K\bigl(f\sigma^{-1}\bigr)^q\,d\nu,
\end{aligned}$$

where ν is the measure on $\mathbb{Z}$ given by

$$\nu(k) = \int_{d_{k+1}}^{d_k} u(y)^q\,dy \biggl(\frac{1}{d_k^{1-\alpha}} \int_0^{d_k} \sigma(y)\,dy\biggr)^q,$$

and, for every measurable function h, the operator T_K is defined by

$$T_K h(k) = \frac{\int_0^{d_k} h(y)\sigma(y)\,dy}{\int_0^{d_k} \sigma(y)\,dy} \chi_{\Lambda_K}(k).$$

If we prove that T_K is uniformly bounded from $L^p((0,\infty),\sigma)$ to $L^q(\mathbb{Z},\nu)$ independently of K, we shall obtain

$$\begin{aligned}
\mathcal{I}_K &\leq C \int_{\mathbb{Z}} T_K\bigl(f\sigma^{-1}\bigr)^q\,d\nu \\
&\leq C\biggl(\int_0^\infty \bigl[\bigl(f\sigma^{-1}\bigr)(y)\bigr]^p \sigma(y)\,dy\biggr)^{q/p} = C\biggl(\int_0^\infty f(y)^p v(y)^p\,dy\biggr)^{q/p}.
\end{aligned}$$

The uniformity in K of this estimate and the monotone convergence theorem will lead to the desired inequality.

Now we prove that T_K is a bounded operator from $L^p((0,\infty),\sigma)$ to $L^q(\mathbb{Z},\nu)$. It is clear that $T_K : L^\infty((0,\infty),\sigma) \to L^\infty(\mathbb{Z},\nu)$ with constant less than or equal to 1. The Marcinkiewicz interpolation theorem says that it is enough to prove the uniform boundedness of the operators T_K from $L^1((0,\infty),\sigma)$ to $L^{q/p,\infty}(\mathbb{Z},\nu)$. For this purpose, fix $h \geq 0$, a bounded function with compact support, and put $F_\lambda = \{k \in \mathbb{Z} : T_K h(k) > \lambda\} = \{|k| \le K : T_K h(k) > \lambda\}$, and $k_0 = \min\{k : k \in F_\lambda\}$. Using (3), we have

$$\begin{aligned}
\nu(F_\lambda) &= \sum_{k\in F_\lambda} \int_{d_{k+1}}^{d_k} \left(\frac{1}{d_k^{1-\alpha}} \int_0^{d_k} \sigma(y)\,dy \right)^q u(x)^q\,dx \\
&\le \sum_{k\in F_\lambda} \int_{d_{k+1}}^{d_k} \big(N_\alpha(\sigma\chi_{(0,d_k)})\big)(x)^q u(x)^q\,dx \\
&\le \sum_{k\in F_\lambda} \int_{d_{k+1}}^{d_k} \big(N_\alpha(\sigma\chi_{(0,d_{k_0})})\big)(x)^q u(x)^q\,dx \\
&\le \int_0^{d_{k_0}} \big(N_\alpha(\sigma\chi_{(0,d_{k_0})})\big)(x)^q u(x)^q\,dx \\
&\le C\left(\int_0^{d_{k_0}} \sigma(y)\,dy \right)^{q/p} \\
&\le C\left(\frac{1}{\lambda} \int_0^{d_{k_0}} h(y)\sigma(y)\,dy \right)^{q/p} \\
&\le C\left(\frac{1}{\lambda} \int_0^{\infty} h(y)\sigma(y)\,dy \right)^{q/p},
\end{aligned}$$

where the constant C does not depend on K. This ends the proof. □

3 The proofs of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4 We first prove (5). By the Hölder inequality and condition (4), we have

$$\begin{aligned}
&\int_0^\infty \big|P_\alpha f(x)\big|^q u(x)^q\,dx \\
&\quad= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{x^{1-\alpha}} \int_0^x f(y)\,dy \right|^q u(x)^q\,dx \\
&\quad\le \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^{j(1-\alpha)}} \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} f(y)\,dy \right|^q u(x)^q\,dx \\
&\quad\le \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} u(x)^q\,dx \\
&\qquad\times \left(\sum_{k=-\infty}^{j} \left(\int_{2^k}^{2^{k+1}} \big|f(y)\big|^p v(y)^p\,dy \right)^{1/p} \left(\int_{2^k}^{2^{k+1}} v(y)^{-p'}\,dy \right)^{1/p'} \right)^q \\
&\quad\le \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} u(x)^q\,dx \\
&\qquad\times \left(\sum_{k=-\infty}^{j} \left(\int_{2^k}^{2^{k+1}} \big|f(y)\big|^p v(y)^p\,dy \right)^{1/p} \left(\int_{2^k}^{2^{k+1}} v(y)^{-rp'}\,dy \right)^{1/rp'} \left(\int_{2^k}^{2^{k+1}} 1\,dy \right)^{1/r'p'} \right)^q
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j}2^{\frac{(k-j)}{r'p'}}\left(\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\right)^q\\
&\le C\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j}2^{\frac{(k-j)}{2r'}}\right)^{q/p'}\left(\sum_{k=-\infty}^{j}2^{\frac{(k-j)p}{2r'p'}}\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}\\
&\le C\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{j}2^{\frac{(k-j)p}{2r'p'}}\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}\\
&\le C\left(\int_0^{\infty}\left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}.
\end{aligned}
$$

Next we will prove the following inequalities (7). By the Hölder inequality and condition (6), we have

$$
\begin{aligned}
&\int_0^{\infty}\left|Q_\alpha f(x)\right|^q u(x)^q\,dx\\
&\quad=\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\left|\int_x^{\infty}\frac{f(y)}{y^{1-\alpha}}\,dy\right|^q u(x)^q\,dx\\
&\quad\le\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\left|\sum_{k=j}^{\infty}\frac{1}{2^{k(1-\alpha)}}\int_{2^k}^{2^{k+1}}\left|f(y)\right|dy\right|^q u(x)^q\,dx\\
&\quad\le\sum_{j=-\infty}^{\infty}\left(\int_{2^j}^{2^{j+1}}u(x)^{rq}\,dx\right)^{1/r}\left(\int_{2^j}^{2^{j+1}}1\,dx\right)^{1/r'}\\
&\qquad\times\left(\sum_{k=j}^{\infty}\frac{1}{2^{k(1-\alpha)}}\left(\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\left(\int_{2^k}^{2^{k+1}}v(y)^{-p'}\,dy\right)^{1/p'}\right)^q\\
&\quad\le C\sum_{j=-\infty}^{\infty}\left(\sum_{k=j}^{\infty}2^{\frac{(j-k)}{r'q}}\left(\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\right)^q\\
&\quad\le C\sum_{j=-\infty}^{\infty}\left(\sum_{k=j}^{\infty}2^{\frac{(j-k)p'}{2r'q}}\right)^{q/p'}\left(\sum_{k=j}^{\infty}2^{\frac{(j-k)p}{2r'q}}\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}\\
&\quad\le C\left(\int_0^{\infty}\left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}.
\end{aligned}
$$

□

Lemma 3.1 ([10]) *Let* $b\in \mathrm{CMO}^1$, $j,k\in\mathbb{Z}$, *then*

$$
\left|b(t)-b_{(0,2^{j+1}]}\right|\le\left|b(t)-b_{(0,2^{k+1}]}\right|+2|j-k|\|b\|_{\mathrm{CMO}^1}.
$$

Proof of Theorem 1.5 We first prove (9). We have

$$
\begin{aligned}
&\int_0^{\infty}\left|P_\alpha^b f(x)\right|^q u(x)^q\,dx\\
&\quad=\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\left|\frac{1}{x^{1-\alpha}}\int_0^x\big(b(x)-b(y)\big)f(y)\,dy\right|^q u(x)^q\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^{j(1-\alpha)}} \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} \left| \big(b(x)-b(y)\big) f(y) \right| dy \right|^q u(x)^q \, dx \\
&\leq 2^{q/q'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^{j(1-\alpha)}} \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} \left| \big(b(x)-b_{(0,2^{j+1}]}\big) f(y) \right| dy \right|^q u(x)^q \, dx \\
&\quad + 2^{q/q'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \frac{1}{2^{j(1-\alpha)}} \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} \left| b(y)-b_{(0,2^{j+1}]}\big) f(y) \right| dy \right|^q u(x)^q \, dx \\
&= 2^{q/q'} (\mathrm{I} + \mathrm{II}).
\end{aligned}
$$

For I, by the Hölder inequality and condition (8), we have

$$
\begin{aligned}
\mathrm{I} &= \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} \left| b(x) - b_{(0,2^{j+1}]} \right|^q u(x)^q \, dx \left(\sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} \left| f(y) \right| dy \right)^q \\
&\leq \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \left(\int_{2^j}^{2^{j+1}} \left| b(x) - b_{(0,2^{j+1}]} \right|^{r'q} dx \right)^{1/r'} \left(\int_{2^j}^{2^{j+1}} u(x)^{rq} \, dx \right)^{1/r} \\
&\quad \times \left(\sum_{k=-\infty}^{j} \left(\int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p \, dy \right)^{1/p} \left(\int_{2^k}^{2^{k+1}} v(y)^{-rp'} \, dy \right)^{1/rp'} \left(\int_{2^k}^{2^{k+1}} dy \right)^{1/r'p'} \right)^q \\
&\leq C \|b\|_{\mathrm{CMO}^{r'q}}^q \sum_{j=-\infty}^{\infty} \frac{2^{j+1}}{2^{jq(1-\alpha)}} \left(\frac{1}{2^{j+1}} \int_0^{2^{j+1}} u(x)^{rq} \, dx \right)^{1/r} \\
&\quad \times \left(\sum_{k=-\infty}^{j} 2^{\frac{j+1}{rp'} + \frac{k+1}{r'p'}} \left(\int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p \, dy \right)^{1/p} \left(\frac{1}{2^{j+1}} \int_0^{2^{j+1}} v(y)^{-rp'} \, dy \right)^{1/rp'} \right)^q \\
&\leq C \|b\|_{\mathrm{CMO}^{r'q}}^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j} 2^{\frac{(k-j)}{r'p'}} \left(\int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p \, dy \right)^{1/p} \right)^q \\
&\leq C \|b\|_{\mathrm{CMO}^{r'q}}^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j} 2^{\frac{(k-j)}{2r'}} \right)^{q/p'} \left(\sum_{k=-\infty}^{j} 2^{\frac{(k-j)p}{2r'p'}} \int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p \, dy \right)^{q/p} \\
&\leq C \|b\|_{\mathrm{CMO}^{r'q}}^q \left(\int_0^{\infty} \left| f(y) \right|^p v(y)^p \, dy \right)^{q/p}.
\end{aligned}
$$

For II, by Lemma 3.1, we have

$$
\begin{aligned}
\mathrm{II} &\leq \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} \left| \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} \left| \big(b(y) - b_{(0,2^{k+1}]}\big) f(y) \right| dy \right|^q u(x)^q \, dx \\
&\quad + \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} \left| \sum_{k=-\infty}^{j} \int_{2^k}^{2^{k+1}} 2(j-k) \|b\|_{\mathrm{CMO}^1} \left| f(y) \right| dy \right|^q u(x)^q \, dx \\
&= \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}
$$

For II_1, by the Hölder inequality and condition (8), we have

$$\begin{aligned}\mathrm{II}_1 &\le \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} u(x)^q\,dx \Bigg(\sum_{k=-\infty}^{j} \left(\int_{2^k}^{2^{k+1}} \left| b(y) - b_{(0,2^{k+1}]} \right|^{r'p'} dy \right)^{1/r'p'} \\ &\quad \times \left(\int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p\,dy \right)^{1/p} \left(\int_{2^k}^{2^{k+1}} v(y)^{-rp'}\,dy \right)^{1/rp'} \Bigg)^q \\ &\le C\|b\|_{\mathrm{CMO}^{r'p'}}^q \sum_{j=-\infty}^{\infty} \frac{2^{j+1}}{2^{jq(1-\alpha)}} \left(\sum_{k=-\infty}^{j} 2^{\frac{(k+1)}{r'p'} + \frac{(j+1)}{rp'}} \left(\int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p\,dy \right)^{1/p} \right)^q \\ &\le C\|b\|_{\mathrm{CMO}^{r'p'}}^q \left(\int_0^\infty \left| f(y) \right|^p v(y)^p\,dy \right)^{q/p}.\end{aligned}$$

For II_2, we have

$$\begin{aligned}\mathrm{II}_2 &= C\|b\|_{\mathrm{CMO}^1}^q \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} u(x)^q\,dx \left(\sum_{k=-\infty}^{j} (j-k) \int_{2^k}^{2^{k+1}} \left| f(y) \right| dy \right)^q \\ &\le C\|b\|_{\mathrm{CMO}^1}^q \sum_{j=-\infty}^{\infty} \frac{1}{2^{jq(1-\alpha)}} \int_{2^j}^{2^{j+1}} u(x)^q\,dx \Bigg(\sum_{k=-\infty}^{j} (j-k) \\ &\quad \times \left(\int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p\,dy \right)^{1/p} \left(\int_{2^k}^{2^{k+1}} v(y)^{-rp'}\,dy \right)^{1/rp'} \left(\int_{2^k}^{2^{k+1}} dy \right)^{1/r'p'} \Bigg)^q \\ &\le C\|b\|_{\mathrm{CMO}^{r'p'}}^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{j} (j-k) 2^{\frac{(k-j)}{r'p'}} \left(\int_{2^k}^{2^{k+1}} \left| f(y) \right|^p v(y)^p\,dy \right)^{1/p} \right)^q \\ &\le C\|b\|_{\mathrm{CMO}^{r'p'}}^q \left(\int_0^\infty \left| f(y) \right|^p v(y)^p\,dy \right)^{q/p}.\end{aligned}$$

Now we prove (10). We have

$$\begin{aligned}&\int_0^\infty \left| Q_\alpha^b f(x) \right|^q u(x)^q\,dx \\ &\quad = \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \int_x^\infty \frac{(b(x)-b(y))f(y)}{y^{1-\alpha}}\,dy \right|^q u(x)^q\,dx \\ &\quad \le \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{2^k}^{2^{k+1}} \left| \big(b(x)-b(y)\big) f(y) \right| dy \right|^q u(x)^q\,dx \\ &\quad \le 2^{q/q'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{2^k}^{2^{k+1}} \left| \big(b(x)-b_{(0,2^{j+1}]}\big) f(y) \right| dy \right|^q u(x)^q\,dx \\ &\qquad + 2^{q/q'} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left| \sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{2^k}^{2^{k+1}} \left| \big(b(y)-b_{(0,2^{j+1}]}\big) f(y) \right| dy \right|^q u(x)^q\,dx \\ &\quad = 2^{q/q'} (\mathrm{J} + \mathrm{JJ}).\end{aligned}$$

For J, by the Hölder inequality and condition (8), we have

$$\begin{aligned}
\mathrm{J} &= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left|b(x) - b_{(0,2^{j+1}]}\right|^q u(x)^q\,dx \left(\sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{2^k}^{2^{k+1}} \left|f(y)\right| dy\right)^q \\
&\le \sum_{j=-\infty}^{\infty} \left(\int_{2^j}^{2^{j+1}} \left|b(x) - b_{(0,2^{j+1}]}\right|^{r'q} dx\right)^{1/r'} \left(\int_{2^j}^{2^{j+1}} u(x)^{rq}\,dx\right)^{1/r} \left(\sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}}\right. \\
&\quad \times \left.\left(\int_{2^k}^{2^{k+1}} \left|f(y)\right|^p v(y)^p\,dy\right)^{1/p} \left(\int_{2^k}^{2^{k+1}} v(y)^{-rp'}\,dy\right)^{1/rp'} \left(\int_{2^k}^{2^{k+1}} dy\right)^{1/r'p'}\right)^q \\
&\le C\|b\|_{\mathrm{CMO}^{r'q}}^q \sum_{j=-\infty}^{\infty} 2^{\frac{j+1}{r'}} \left(\int_0^{2^{j+1}} u(x)^{rq}\,dx\right)^{1/r} \\
&\quad \times \left(\sum_{k=j}^{\infty} 2^{-k(1-\alpha)+\frac{k+1}{rp'}+\frac{k+1}{r'p'}} \left(\int_{2^k}^{2^{k+1}} \left|f(y)\right|^p v(y)^p\,dy\right)^{1/p} \left(\frac{1}{2^{k+1}} \int_0^{2^{k+1}} v(y)^{-rp'}\,dy\right)^{1/rp'}\right)^q \\
&\le C\|b\|_{\mathrm{CMO}^{r'q}}^q \sum_{j=-\infty}^{\infty} 2^{\frac{j+1}{r'}} \left(\sum_{k=j}^{\infty} 2^{\frac{-(k+1)}{r'q}} \cdot \left(2^{k+1}\right)^{(\frac{1}{q}+\alpha-\frac{1}{p})} \left(\int_{2^k}^{2^{k+1}} \left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\right. \\
&\quad \times \left.\left(\frac{1}{2^{k+1}} \int_0^{2^{k+1}} u(x)^{rq}\,dx\right)^{1/rq} \left(\frac{1}{2^{k+1}} \int_0^{2^{k+1}} v(y)^{-rp'}\,dy\right)^{1/rp'}\right)^q \\
&\le C\|b\|_{\mathrm{CMO}^{r'q}}^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=j}^{\infty} 2^{\frac{(j-k)}{r'q}} \left(\int_{2^k}^{2^{k+1}} \left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\right)^q \\
&\le C\|b\|_{\mathrm{CMO}^{r'q}}^q \sum_{j=-\infty}^{\infty} \left(\sum_{k=j}^{\infty} 2^{\frac{(j-k)p'}{2r'q}}\right)^{q/p'} \left(\sum_{k=j}^{\infty} 2^{\frac{(j-k)p}{2r'q}} \int_{2^k}^{2^{k+1}} \left|f(y)\right|^p v(y)^p\,dy\right)^{q/p} \\
&\le C\|b\|_{\mathrm{CMO}^{r'q}}^q \left(\int_0^{\infty} \left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}.
\end{aligned}$$

For JJ, by Lemma 3.1, we have

$$\begin{aligned}
\mathrm{JJ} &\le \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left|\sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{2^k}^{2^{k+1}} \left|\left(b(y) - b_{(0,2^{k+1}]}\right) f(y)\right| dy\right|^q u(x)^q\,dx \\
&\quad + \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left|\sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{2^k}^{2^{k+1}} 2(k-j)\|b\|_{\mathrm{CMO}^1} \left|f(y)\right| dy\right|^q u(x)^q\,dx \\
&= \mathrm{JJ}_1 + \mathrm{JJ}_2.
\end{aligned}$$

For JJ_1, we have

$$\begin{aligned}
\mathrm{JJ}_1 &\le \sum_{j=-\infty}^{\infty} \left(\int_{2^j}^{2^{j+1}} u(x)^{rq}\,dx\right)^{1/r} 2^{j/r'} \left(\sum_{k=j}^{\infty} \frac{1}{2^{k(1-\alpha)}} \left(\int_{2^k}^{2^{k+1}} \left|b(y) - b_{(0,2^{k+1}]}\right|^{r'p'} dy\right)^{1/r'p'}\right. \\
&\quad \times \left.\left(\int_{2^k}^{2^{k+1}} \left|f(y)\right|^p v(y)^p\,dy\right)^{1/p} \left(\int_{2^k}^{2^{k+1}} v(y)^{-rp'}\,dy\right)^{1/rp'}\right)^q
\end{aligned}$$

$$\leq C\|b\|^q_{\mathrm{CMO}^{r'p'}}\sum_{j=-\infty}^{\infty}\left(\sum_{k=j}^{\infty}2^{\frac{(j-k)}{r'q}}\left(\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\right)^q$$

$$\leq C\|b\|^q_{\mathrm{CMO}^{r'p'}}\left(\int_0^{\infty}\left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}.$$

For JJ_2, we have

$$\mathrm{JJ}_2\leq C\|b\|^q_{\mathrm{CMO}^1}\sum_{j=-\infty}^{\infty}\left(\int_{2^j}^{2^{j+1}}u(x)^{rq}\,dx\right)^{1/r}\left(\int_{2^j}^{2^{j+1}}1\,dx\right)^{1/r'}\left(\sum_{k=j}^{\infty}\frac{(k-j)}{2^{k(1-\alpha)}}\right.$$

$$\left.\times\left(\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\left(\int_{2^k}^{2^{k+1}}v(y)^{-rp'}\,dy\right)^{1/rp'}\left(\int_{2^k}^{2^{k+1}}1\,dy\right)^{1/r'p'}\right)^q$$

$$\leq C\|b\|^q_{\mathrm{CMO}^{r'p'}}\sum_{j=-\infty}^{\infty}\left(\sum_{k=j}^{\infty}(k-j)2^{\frac{(j-k)}{r'q}}\left(\int_{2^k}^{2^{k+1}}\left|f(y)\right|^p v(y)^p\,dy\right)^{1/p}\right)^q$$

$$\leq C\|b\|^q_{\mathrm{CMO}^{r'p'}}\left(\int_0^{\infty}\left|f(y)\right|^p v(y)^p\,dy\right)^{q/p}.$$

This ends the proof. □

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