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A logarithmic estimate for harmonic sums and the digamma function, with an application to the Dirichlet divisor problem

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Abstract

Let $H_n = \sum_{r=1}^{n} 1/r$ and $H_n(x) = \sum_{r=1}^{n} 1/(r+x)$. Let $\psi(x)$ denote the digamma function. It is shown that $H_n(x) + \psi(x+1)$ is approximated by $\frac{1}{2}\log f(n+x)$, where $f(x) = x^2 + x + \frac{1}{3}$, with error term of order $(n+x)^{-5}$. The cases $x = 0$ and $n = 0$ equate to estimates for $H_n - \gamma$ and $\psi(x+1)$ itself. The result is applied to determine exact bounds for a remainder term occurring in the Dirichlet divisor problem.

MSC: Primary 26D15; 33B15; secondary 11N37

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1 Introduction and summary of results

Write H_n for the harmonic sum $\sum_{r=1}^{n} \frac{1}{r}$. The following well-known estimation can be established by an Euler–Maclaurin summation, or by the logarithmic and binomial series:

$$H_n - \gamma = \log n + \frac{1}{2n} - \frac{1}{12n^2} + r_n, \tag{1}$$

where γ is Euler's constant and

$$0 < r_n \le \frac{1}{120n^4}.$$

Since $\log(n+\frac{1}{2}) = \log n + \frac{1}{2n} + O(\frac{1}{n^2})$, it is a natural idea to absorb the term $\frac{1}{2n}$ into the logarithmic term and compare $H_n - \gamma$ directly with $\log(n+\frac{1}{2})$. This was done by De Temple [6]: he showed that

$$H_n - \gamma = \log\left(n + \frac{1}{2}\right) + \frac{1}{24(n+\frac{1}{2})^2} - r_n, \tag{2}$$

where

$$\frac{7}{960(n+1)^4} \le r_n \le \frac{7}{960n^4}.$$

(We will repeatedly re-use the notation r_n for the remainder term in estimations like this, with a new meaning each time.)

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Negoi [9] demonstrated that the n^{-2} term can be absorbed into the log term by considering $\log h(n)$, where $h(n) = n + \frac{1}{2} + \frac{1}{24n}$. His result is

$$H_n - \gamma = \log h(n) - r_n, \tag{3}$$

where $\frac{1}{48}(n+1)^{-3} \leq r_n \leq \frac{1}{48}n^{-3}$. Numerous more recent articles have developed this process further. For example, Chen and Mortici [3] show that

$$H_n - \gamma = \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) + O\left(n^{-5}\right). \tag{4}$$

Further variations and extensions are given, for example, in [7] and [4], and in other references listed in these papers. One natural extension is the replacement of $H_n - \gamma$ by $\psi(x+1)$, where $\psi(x)$ is the digamma function $\Gamma'(x)/\Gamma(x)$, since $H_n - \gamma = \psi(n+1)$.

A slightly different approach, which has proved quite effective, is to compare $H_n - \gamma$ or $\psi(x+1)$ with expressions of the form $\frac{1}{2}\log f(n)$. Using no more than the elementary inequalities $H_n - \gamma \leq \log n + \frac{1}{2n}$ and $e^x \leq 1 + x + x^2$, it is easily shown that $H_n - \gamma \leq \frac{1}{2}\log(n^2 + n + 1)$. With $h(n)$ as in (3), we have $h(n)^2 = n^2 + n + \frac{1}{3} + O(\frac{1}{n})$, suggesting that the right comparison is with $f(n) = n^2 + n + \frac{1}{3}$. Indeed, with $f(n)$ defined in this way, Batir [2] and Lu [5] have shown, by different methods, that

$$H_n - \gamma = \frac{1}{2}\log f(n) - r_n, \tag{5}$$

where $r_n \sim 1/(180n^4)$. Lu obtained (5) as one case of a rather complicated analysis extending to further terms and parameters.

Our objective here is to refine and extend (5) by considering the sum

$$H_n(x) = \sum_{r=1}^{n} \frac{1}{r+x}, \tag{6}$$

To identify the limit of $H_n(x) - \log n$, recall Euler's limit formula for the Gamma function: this can be written as $\Gamma(x+1) = \lim_{n\to\infty} G_n(x+1)$, where

$$G_n(x+1) = \frac{n^x(n+1)!}{(x+1)\ldots(x+n)},$$

from which it follows that $\lim_{n\to\infty}[H_n(x) - \log n] = -\psi(x+1)$. (No further facts about $\psi(x)$ are needed for our purposes.) We compare the difference $H_n(x) + \psi(x+1)$ with $\frac{1}{2}\log f(n+x)$. The case $x = 0$ reproduces (5), while the case $n = 0$ (with $H_0(x) = 0$) gives a similar estimate for $\psi(x+1)$. Unlike [2] and [5], we give explicit upper and lower bounds for the error term, which we will require in the subsequent application. The exact statement is as follows.

Theorem 1 *Let $n \geq 1$ and $x > -1$, or $n = 0$ and $x > 0$, and let $H_n(x)$ be defined by* (6). *Let $f(x) = x^2 + x + \frac{1}{3}$. Then*

$$H_n(x) + \psi(x+1) = \frac{1}{2}\log f(n+x) - r(n,x), \tag{7}$$

where

$$\frac{1}{180(n+x+1)^4} \le r(n,x) \le \frac{1}{180(n+x)^4}. \tag{8}$$

In particular,

$$H_n - \gamma = \frac{1}{2}\log f(n) - r_n, \tag{9}$$

where

$$\frac{1}{180(n+1)^4} \le r_n \le \frac{1}{180n^4}. \tag{10}$$

Also, for $x > 0$,

$$\psi(x+1) = \frac{1}{2}\log f(x) - r(x), \tag{11}$$

where

$$\frac{1}{180(x+1)^4} \le r(x) \le \frac{1}{180x^4}. \tag{12}$$

Our proof, given in Sect. 2, is a development of De Temple's original method.

Note that the difference between the upper and lower bounds in (10) is less than $1/45n^5$. Of course, (9) is actually a special case of (11).

The case $x = -\frac{1}{2}$ in (7) leads to the following estimation of sums of odd reciprocals (which could not be derived from (9)). The proof is very short, so we include it here.

Corollary 1 *Let* $U_n = \sum_{r=1}^{n} \frac{1}{2r-1}$. *Then*

$$U_n - \frac{1}{2}\gamma - \log 2 = \frac{1}{4}\log\left(n^2 + \frac{1}{12}\right) - r_n,$$

where

$$\frac{1}{360(n+\frac{1}{2})^4} \le r_n \le \frac{1}{360(n-\frac{1}{2})^4}.$$

Proof Note that $2U_n = H_n(-\frac{1}{2})$. Also, $2U_n = 2H_{2n} - H_n$, from which it follows easily that $-\psi(\frac{1}{2}) = \lim_{n\to\infty}[2U_n - \log n] = \gamma + 2\log 2$. Finally, $f(n-\frac{1}{2}) = n^2 + \frac{1}{12}$. □

Theorem 1 has a rather surprising application to an expression that arises in the Dirichlet divisor problem. Denote the divisor function by $\tau(n)$ and its summation function $\sum_{n\le x}\tau(n)$ by $T(x)$. Write

$$F(x) = x\log x + (2\gamma - 1)x$$

and

$$T(x) = F(x) + \Delta(x).$$

The most basic form of Dirichlet's theorem (e.g. [1, p. 59]) states that $\Delta(x) = O(x^{1/2})$. The problem of determining the true order of magnitude of $\Delta(x)$ is the "Dirichlet divisor problem". Denote by θ_0 the infimum of numbers θ such that $\Delta(x) = O(x^{\theta})$. It was already shown by Voronoi in 1903 that $\theta_0 \le \frac{1}{3}$ (e.g. see [10, Sect. 1.6.4]). The estimate has been gradually reduced in a long series of studies: the current best value [8] is $\theta_0 \le \frac{131}{416}$.

Write $[x]$ for the integer part of x and let

$$B(x) = x - [x] - \frac{1}{2},$$

the 1-periodic extension of the function $x - \frac{1}{2}$ on $[0, 1)$. Further, let

$$S(x) = 2 \sum_{j \le x^{1/2}} B\left(\frac{x}{j}\right).$$

Note that $|B(x)| \le \frac{1}{2}$, and hence $|S(x)| \le [x^{1/2}]$, for all $x > 0$. A key, if small, step in the proof of Voronoi's theorem and later refinements is the statement

$$\Delta(x) = -S(x) + q(x), \tag{13}$$

where $q(x)$ is bounded. A version of the usual proof can be found in [10, Sect. 1.6.4]; if scrutinised carefully, it gives the bound 3 for $|q(x)|$. This is quite good enough for the purpose of proving Voronoi's theorem: the serious work is the estimation of $S(x)$ by exponential sums. However, it is still of some interest to determine the true bounds for $q(x)$, along with some other facts about its nature. We will see that $q(x)$ is continuous at integers, and that (9), together with (1), is exactly what is needed to establish:

Theorem 2 *With $q(x)$ defined as above, we have, for all $x \ge 1$,*

$$-\frac{1}{6} < q(x) < \frac{1}{3}. \tag{14}$$

Both bounds are optimal.

2 The proof of Theorem 1

The key step is the following lemma.

Lemma 1 *Let $f(x) = x^2 + x + \frac{1}{3}$. Then, for all $x \ge 1$,*

$$\log f(x) - \log f(x-1) = \frac{2}{x} - \delta(x), \tag{15}$$

where

$$\frac{2}{45x^5} < \delta(x) < \frac{2}{45(x - \frac{1}{2})^5}. \tag{16}$$

Proof We start with $f(x) = x^2 + x + c$ and allow the choice of c to emerge from the reasoning. Let $\delta(x)$ be defined by (15). Now $f(x)/f(x-1) \to 1$, and hence $\delta(x) \to 0$, as $x \to \infty$. So

$\delta(x) = -\int_x^\infty \delta'(t)\,dt$ for all $x > 0$. Now

$$\begin{aligned}-\delta'(t) &= \frac{2t+1}{f(t)} - \frac{2t-1}{f(t-1)} + \frac{2}{t^2}\\ &= \frac{G(t)}{t^2 f(t)f(t-1)},\end{aligned}$$

where

$$\begin{aligned}G(t) &= t^2(2t+1)\bigl(t^2-t+c\bigr) - t^2(2t-1)\bigl(t^2+t+c\bigr) + 2\bigl(t^2+t+c\bigr)\bigl(t^2-t+c\bigr)\\ &= -t^2\bigl(2t^2-2c\bigr) + 2\bigl[t^4 + (2c-1)t^2 + c^2\bigr]\\ &= 2(3c-1)t^2 + 2c^2.\end{aligned}$$

To eliminate the t^2 term, we now choose $c = \frac{1}{3}$, so that $G(t) = \frac{2}{9}$ and

$$-\delta'(t) = \frac{2}{9t^2 f(t)f(t-1)}.$$

Now $f(t)f(t-1) = t^4 - \frac{1}{3}t^2 + \frac{1}{9} < t^4$ for $t \geq 1$, so

$$\delta(x) > \int_x^\infty \frac{2}{9t^6}\,dt = \frac{2}{45x^5}$$

for $x \geq 1$. On the other hand, $f(t) > f(t-1) > (t-\frac{1}{2})^2$, hence

$$\delta(x) < \int_x^\infty \frac{2}{9(t-\frac{1}{2})^6}\,dt = \frac{2}{45(x-\frac{1}{2})^5}.$$ □

Proof of Theorem 1 Apply the identity (15) to $r + x$ for $1 \leq r \leq n$ and add: we find

$$\log f(n+x) - \log f(x) = 2H_n(x) - \sum_{r=1}^{n} \delta(r+x),$$

equivalently

$$2H_n(x) - \log f(n+x) = -\log f(x) + \sum_{r=1}^{n} \delta(r+x). \tag{17}$$

Now $f(n+x)/n^2 \to 1$, so $\log f(n+x) - 2\log n \to 0$, as $n \to \infty$. Taking the limit in (17), we see that

$$-2\psi(x+1) = -\log f(x) + \sum_{r=1}^{\infty} \delta(r+x). \tag{18}$$

Now taking the difference, we have

$$2H_n(x) - \log f(n+x) + 2\psi(x+1) = -2r(n,x),$$

where

$$2r(n,x) = \sum_{r=n+1}^{\infty} \delta(r+x).$$

The condition $x > -1$ ensures that the inequality (16) applies to $\delta(r+x)$ for $r \geq 2$. By integral estimation, we now have, for $n \geq 1$,

$$r(n,x) \geq \sum_{r=n+1}^{\infty} \frac{1}{45(r+x)^5} > \int_{n+1}^{\infty} \frac{1}{45(t+x)^5}\,dt = \frac{1}{180(n+x+1)^4}.$$

At the same time,

$$r(n,x) \leq \sum_{r=n+1}^{\infty} \frac{1}{45(r-\frac{1}{2}+x)^5}.$$

The function $1/(t+x)^5$ is convex, and convex functions $h(t)$ satisfy $h(y-\frac{1}{2}) \leq \int_{y-1}^{y} h(t)\,dt$, hence

$$r(n,x) \leq \int_{n}^{\infty} \frac{1}{45(t+x)^5}\,dt = \frac{1}{180(n+x)^4}.$$

For $x > 0$, the case $n = 0$ follows similarly from (18) (note that (16) now applies also to $\delta(1+x)$). □

Note 1 The upper bounds for $\delta(x)$ in Lemma 1 and $r(n,x)$ in Theorem 1 can be slightly improved. One can verify that $t^2 f(t) f(t-1) \geq (t-\frac{1}{12})^6$ where we previously used $(t-\frac{1}{2})^6$. This leads to $\delta(x) < 2/[45(x-\frac{1}{12})^5]$ and $r(n,x) \leq 1/[180(n+x+\frac{5}{12})^4]$.

Note 2 In principle, one could derive Stirling-type approximations to $\log \Gamma(x)$ and $\Gamma(x)$ from (11), but only in terms of the rather unpleasant antiderivative of $\log f(x)$.

3 The remainder in the divisor problem

We return to the divisor problem. The starting point is Dirichlet's hyperbola identity [1, p. 59]:

$$T(x) = 2\sum_{j \leq x^{1/2}} \left[\frac{x}{j}\right] - \left[x^{1/2}\right]^2. \tag{19}$$

With our previous notation, note that

$$T(x) + S(x) = F(x) + q(x).$$

Write $I_n = [n^2, (n+1)^2]$.

Lemma 2 *For* $x \in I_n$,

$$q(x) = xH_n - F(x) - n(n+1). \tag{20}$$

The function $q(x)$ *is continuous for all* $x \geq 1$, *and concave on each interval* I_n.

Proof We work with $T(x) + S(x)$. For $n^2 \le x < (n+1)^2$, we have by (19)

$$\begin{aligned} T(x) &= 2\sum_{j=1}^{n}\left(\frac{x}{j} - B\left(\frac{x}{j}\right) - \frac{1}{2}\right) - n^2 \\ &= 2xH_n - S(x) - n - n^2, \end{aligned}$$

which equates to (20). We check that this remains valid at $x = (n+1)^2$. By what we have just shown, with n replaced by $n+1$,

$$\begin{aligned} T\left[(n+1)^2\right] + S\left[(n+1)^2\right] &= 2H_{n+1}(n+1)^2 - (n+1)(n+2) \\ &= 2H_n(n+1)^2 + 2(n+1) - (n+1)(n+2) \\ &= 2H_n(n+1)^2 - n(n+1), \end{aligned}$$

agreeing with (20). Hence $F(x) + q(x)$, and consequently $q(x)$ itself, is continuous for all $x \ge 1$. Also, $q'(x) = 2H_n - F'(x) = 2H_n - \log x - 2\gamma$, which is decreasing, so $q(x)$ is concave on I_n. □

So in fact $F(x) + q(x)$ is linear on I_n. For example,

$$F(x) + q(x) = \begin{cases} 2x - 2 & \text{for } 1 \le x \le 4, \\ 3x - 6 & \text{for } 4 \le x \le 9. \end{cases}$$

The reason for continuity of $T(x) + S(x)$ is easily seen directly. At non-square integers k, $T(x)$ increases by $\tau(k)$. Meanwhile, for each divisor j of k with $j < k^{1/2}$, $[x/j]$ increases by 1, so $B(x/j)$ decreases by 1. There are $\frac{1}{2}\tau(k)$ such divisors j, so $S(x)$ decreases by $\tau(k)$. At square integers $k = n^2$, the new term $2B(k/n) = -1$ enters the sum, so again the decrease in $S(x)$ is $\tau(k)$.

To determine the lower bound of $q(x)$, we consider $q(n^2)$ and apply (1).

Lemma 3 *We have $q(x) > -\frac{1}{6}$ for all $x \ge 1$. Further, $q(n^2) \le -\frac{1}{6} + \frac{1}{60n^2}$, so $\inf_{x\ge 1} q(x) = -\frac{1}{6}$.*

Proof By (20) and (1) (with r_n as in (1)),

$$\begin{aligned} q\left(n^2\right) &= 2n^2H_n - 2n^2\log n - (2\gamma - 1)n^2 - n - n^2 \\ &= 2n^2(H_n - \log n - \gamma) - n \\ &= 2n^2\left(\frac{1}{2n} - \frac{1}{12n^2} + r_n\right) - n \\ &= -\frac{1}{6} + 2n^2 r_n. \end{aligned}$$

So

$$-\frac{1}{6} < q\left(n^2\right) \le -\frac{1}{6} + \frac{1}{60n^2}.$$

This applies also at $(n+1)^2$. Since $q(x)$ is concave on I_n, it follows that $q(x) > -\frac{1}{6}$ throughout this interval. □

Finally, we apply (9) and (10) to identify the upper bound.

Proof of the upper bound in Theorem 2 Let x_n be the point where $q(x)$ attains its maximum in I_n. Since $q'(x) = 2H_n - \log x - 2\gamma$, we have $\log x_n = 2H_n - 2\gamma$, hence the maximum value is

$$\begin{aligned} q(x_n) &= 2H_n x_n - F(x_n) - n(n+1) \\ &= (\log x_n + 2\gamma)x_n - F(x_n) - n(n+1) \\ &= x_n - n(n+1). \end{aligned} \tag{21}$$

By (9) and (10) (using only $r_n > 0$), we have $\log x_n < \log f(n)$, so $x_n < f(n) = n(n+1) + \frac{1}{3}$, hence $q(x_n) < \frac{1}{3}$.

We show that, conversely,

$$q(x_n) > \frac{1}{3} - \frac{1}{45n^2}$$

for $n \geq 2$, so that $\sup_{x\geq 1} q(x) = \frac{1}{3}$. By (21), this is equivalent to $x_n > f(n) - \frac{1}{45n^2}$. By (10),

$$\log x_n = 2H_n - 2\gamma \geq \log f(n) - \frac{1}{90n^4}.$$

Now using the inequality $e^{-x} \geq 1 - x$, together with $f(n) \leq 2n^2$, we have

$$x_n \geq f(n)e^{-1/90n^4} \geq f(n)\left(1 - \frac{1}{90n^4}\right) \geq f(n) - \frac{1}{45n^2}.$$ □

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References

1. Apostol, T.M.: Introduction to Analytic Number Theory. Springer, Berlin (1976)
2. Batir, N.: Sharp bounds for the psi function and harmonic numbers. Math. Inequal. Appl. **14**, 917–925 (2011)
3. Chen, C., Mortici, C.: New sequence converging towards the Euler–Mascheroni constant. Comput. Math. Appl. **64**, 391–398 (2012)

4. Dawei, L.: Some new convergent sequences and inequalities of Euler's constant. J. Math. Anal. Appl. **419**, 541–552 (2014)
5. Dawei, L.: Some quicker classes of sequences convergent to Euler's constant. Appl. Math. Comput. **232**, 172–177 (2014)
6. De Temple, D.W.: A quicker convergence to Euler's constant. Am. Math. Mon. **100**, 468–470 (1993)
7. Elezovic, N.: Estimations of psi function and harmonic numbers. Appl. Math. Comput. **258**, 192–205 (2015)
8. Huxley, M.N.: Exponential sums and lattice points III. Proc. Lond. Math. Soc. (3) **87**, 591–609 (2003)
9. Negoi, T.: A faster convergence to Euler's constant. Math. Gaz. **83**, 487–489 (1999)
10. Tenenbaum, G.: Introduction to Analytic and Probabilistic Number Theory. Cambridge Univ. Press, Cambridge (1995)