Luo and Yang *Journal of Inequalities and Applications* (2019) 2019:142
https://doi.org/10.1186/s13660-019-2095-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Parameterized discrete Hilbert-type inequalities with intermediate variables

Ricai Luo[1*] and Bicheng Yang[2]

*Correspondence: hcxylor@126.com
[1] School of Mathematics and Statistics, Hechi University, Yizhou, P.R. China
Full list of author information is available at the end of the article

Abstract

By means of the weight coefficients and the idea of introducing parameters, a discrete Hilbert-type inequality with the general homogeneous kernel and the intermediate variables is given. The equivalent form is obtained. The equivalent statements of the best possible constant factor related to some parameters, the operator expressions, and a few particular cases are considered.

MSC: 26D15

Keywords: Weight coefficient; Hilbert-type inequality; Equivalent statement; Parameter; Operator expression

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the following Hardy–Hilbert inequality with the best possible constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

For $p = q = 2$, inequality (1) reduces to the well-known Hilbert inequality.

If $f(x), g(y) \geq 0$, $0 < \int_0^{\infty} f^p(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^q(y)\,dy < \infty$, then we have the following Hardy–Hilbert integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)\,dy\right)^{\frac{1}{q}} \tag{2}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 316).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang [2, 3] gave an extension of (2) (for $p = q = 2$) with the best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ as follows:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}\,dx\,dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(\int_0^{\infty} x^{1-\lambda}f^2(x)\,dx\int_0^{\infty} y^{1-\lambda}g^2(y)\,dy\right)^{\frac{1}{2}}. \tag{3}$$

Inequalities (1), (2), (3) and their extensions are important in analysis and its applications (cf. [4–15]).

The following half-discrete Hilbert-type inequality was provided (cf. [1], Theorem 351): If $K(x)$ $(x > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^\infty K(x)x^{s-1}\,dx < \infty$, then

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions of (4) with their applications were provided by [16–21].

In 2016, by the use of the technique of real analysis, Hong [22] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. The other similar works about the extensions of (2) and (3) were given by [23–27].

In this paper, following the way of [22], by means of the weight functions and the idea of introducing parameters, a discrete Hilbert-type inequality with the general homogeneous kernel and the intermediate variables is given, which is an extension of (1). The equivalent form is obtained. The equivalent statements of the best possible constant factor related to parameters, the operator expressions, and a few particular cases are considered.

2 Some lemmas

In what follows, we suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta > 0$, $\lambda \in \mathrm{R}$, $\lambda_2, \lambda - \lambda_1 \le \frac{1}{\beta}$, $\lambda_1, \lambda - \lambda_2 \le \frac{1}{\alpha}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ satisfying, for any $u, x, y > 0$,

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x,y).$$

Also, $k_\lambda(x,y)$ is strictly decreasing with respect to $x, y > 0$ such that, for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$k_\lambda(\gamma) := \int_0^\infty k_\lambda(u,1)u^{\gamma-1}\,du \in \mathrm{R}_+ = (0,\infty). \tag{5}$$

We still assume that $a_m, b_n \ge 0$ $(m, n \in \mathrm{N} = \{1, 2, \ldots\})$ such that

$$0 < \sum_{m=1}^\infty m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q < \infty.$$

Definition 1 We define the following weight coefficients:

$$\omega_\lambda(\lambda_2, m) := m^{\alpha(\lambda-\lambda_2)}\sum_{n=1}^\infty k_\lambda\left(m^\alpha, n^\beta\right)n^{\beta\lambda_2-1} \quad (m \in \mathrm{N}), \tag{6}$$

$$\varpi_\lambda(\lambda_1, n) := n^{\beta(\lambda-\lambda_1)}\sum_{m=1}^\infty k_\lambda\left(m^\alpha, n^\beta\right)m^{\alpha\lambda_1-1} \quad (n \in \mathrm{N}). \tag{7}$$

Lemma 1 *We have the following inequalities*:

$$\omega_\lambda(\lambda_2, m) < \frac{1}{\beta}k_\lambda(\lambda - \lambda_2) \quad (m \in \mathrm{N}), \tag{8}$$

$$\varpi_\lambda(\lambda_1, n) < \frac{1}{\alpha}k_\lambda(\lambda_1) \quad (n \in \mathrm{N}). \tag{9}$$

Proof For $\beta\lambda_2 - 1 \leq 0$, it is evident that $k_\lambda(m^\alpha, y^\beta)y^{\beta\lambda_2-1}$ is strictly decreasing with respect to $y > 0$. By the decreasingness property, setting $u = \frac{m^\alpha}{y^\beta}$, we find that

$$\begin{aligned}\omega_\lambda(\lambda_2, m) &< m^{\alpha(\lambda-\lambda_2)} \int_0^\infty k_\lambda\left(m^\alpha, y^\beta\right) y^{\beta\lambda_2-1}\,dy \\ &= \frac{1}{\beta}\int_0^\infty k_\lambda(u,1)u^{(\lambda-\lambda_2)-1}\,du = \frac{1}{\beta}k_\lambda(\lambda-\lambda_2).\end{aligned}$$

Hence, we have (8).

For $\alpha\lambda_1 - 1 \leq 0$, it is evident that $k_\lambda(x^\alpha, n^\beta)x^{\alpha\lambda_1-1}$ is strictly decreasing with respect to $x > 0$. By the decreasingness property, setting $u = \frac{x^\alpha}{n^\beta}$, we find that

$$\begin{aligned}\varpi_\lambda(\lambda_1, n) &< n^{\beta(\lambda-\lambda_1)} \int_0^\infty k_\lambda\left(x^\alpha, n^\beta\right) x^{\alpha\lambda_1-1}\,dx \\ &= \frac{1}{\alpha}\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}\,du = \frac{1}{\alpha}k_\lambda(\lambda_1).\end{aligned}$$

Hence, we have (9). □

Lemma 2 *We have the following inequality*:

$$\begin{aligned} I &:= \sum_{n=1}^\infty \sum_{m=1}^\infty k_\lambda\left(m^\alpha, n^\beta\right) a_m b_n \\ &< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{\sum_{m=1}^\infty m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=1}^\infty n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{10}$$

Proof By Hölder's inequality with weight (cf. [28]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^\infty \sum_{m=1}^\infty k_\lambda\left(m^\alpha, n^\beta\right) \left[\frac{n^{(\beta\lambda_2-1)/p}}{m^{(\alpha\lambda_1-1)/q}} a_m\right]\left[\frac{m^{(\alpha\lambda_1-1)/q}}{n^{(\beta\lambda_2-1)/p}} b_n\right] \\ &\leq \left[\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda\left(m^\alpha, n^\beta\right) \frac{n^{\beta\lambda_2-1}}{m^{(\alpha\lambda_1-1)(p-1)}} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty \sum_{m=1}^\infty k_\lambda\left(m^\alpha, n^\beta\right) \frac{m^{\alpha\lambda_1-1}}{n^{(\beta\lambda_2-1)(q-1)}} b_n^q\right]^{\frac{1}{q}} \\ &= \left\{\sum_{m=1}^\infty \omega_\lambda(\lambda_2, m) m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty \varpi_\lambda(\lambda_1, n) n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then, by (8) and (9), we have (10). □

Remark 1 (i) By (10), for $\lambda_1 + \lambda_2 = \lambda$, we find

$$0 < \sum_{m=1}^\infty m^{p(1-\alpha\lambda_1)-1} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{q(1-\beta\lambda_2)-1} b_n^q < \infty,$$

and the following inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda\left(m^\alpha,n^\beta\right)a_m b_n < \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1)\left[\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{11}$$

In particular, for $\alpha = \beta = 1$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda(m,n)a_m b_n < k_\lambda(\lambda_1)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{12}$$

(ii) For $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (12) reduces to (1). Hence, (11) is an extension of (12) and (1).

Lemma 3 *The constant factor* $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1)$ *in* (11) *is the best possible.*

Proof For any $\varepsilon > 0$, we set

$$\tilde{a}_m := m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}, \qquad \tilde{b}_n := n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1} \quad (m,n \in \mathrm{N}).$$

If there exists a constant M $(\le \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1))$ such that (11) is valid when replacing $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1)$ by M, then, in particular, we have

$$\tilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda\left(m^\alpha,n^\beta\right)\tilde{a}_m\tilde{b}_n < M\left[\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}\tilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}\tilde{b}_n^q\right]^{\frac{1}{q}}.$$

By the decreasingness property, we obtain

$$\begin{aligned}
\tilde{I} &< M\left[\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}m^{p\alpha\lambda_1-\alpha\varepsilon-p}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}n^{q\beta\lambda_2-\beta\varepsilon-q}\right]^{\frac{1}{q}} \\
&= M\left(1+\sum_{m=2}^{\infty}m^{-\alpha\varepsilon-1}\right)^{\frac{1}{p}}\left(1+\sum_{n=2}^{\infty}n^{-\beta\varepsilon-1}\right)^{\frac{1}{q}} \\
&< M\left(1+\int_1^{\infty}t^{-\alpha\varepsilon-1}\,dt\right)^{\frac{1}{p}}\left(1+\int_1^{\infty}t^{-\beta\varepsilon-1}\,dt\right)^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon}\left(\varepsilon+\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\varepsilon+\frac{1}{\beta}\right)^{\frac{1}{q}}.
\end{aligned}$$

By the decreasingness property and Fubini theorem (cf. [29]), we find

$$\begin{aligned}
\tilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda\left(m^\alpha,n^\beta\right)\frac{m^{\alpha\lambda_1-1}}{m^{\varepsilon\alpha/p}}\cdot\frac{n^{\beta\lambda_2-1}}{n^{\varepsilon\beta/q}} \\
&\ge \int_1^{\infty}\left[\int_1^{\infty}k_\lambda\left(x^\alpha,y^\beta\right)\frac{x^{\alpha\lambda_1-1}}{x^{\varepsilon\alpha/p}}\cdot\frac{y^{\beta\lambda_2-1}}{y^{\varepsilon\beta/q}}\,dx\right]dy \quad \left(u = \frac{x^\alpha}{y^\beta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha}\int_1^\infty y^{-\beta\varepsilon-1}\left(\int_{\frac{1}{y^\beta}}^\infty k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du\right)dy \\
&= \frac{1}{\alpha}\int_1^\infty y^{-\beta\varepsilon-1}\left(\int_{\frac{1}{y^\beta}}^1 k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du\right)dy \\
&\quad + \frac{1}{\alpha}\int_1^\infty y^{-\beta\varepsilon-1}\,dy\int_1^\infty k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du \\
&= \frac{1}{\alpha}\int_0^1\left(\int_{u^{-1/\beta}}^\infty y^{-\beta\varepsilon-1}\,dy\right)k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du \\
&\quad + \frac{1}{\alpha\beta\varepsilon}\int_1^\infty k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du \\
&= \frac{1}{\alpha\beta\varepsilon}\left(\int_0^1 k_\lambda(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}\,du + \int_1^\infty k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\frac{1}{\alpha\beta}\left(\int_0^1 k_\lambda(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}\,du + \int_1^\infty k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du\right) \\
&\quad < M\left(\varepsilon+\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\varepsilon+\frac{1}{\beta}\right)^{\frac{1}{q}}.
\end{aligned}
$$

For $\varepsilon \to 0^+$, by Fatou's lemma (cf. [29]), we find

$$
\begin{aligned}
\frac{1}{\alpha\beta}k_\lambda(\lambda_1) &= \frac{1}{\alpha\beta}\left(\int_0^1 \lim_{\varepsilon\to 0^+} k_\lambda(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}\,du + \int_1^\infty \lim_{\varepsilon\to 0^+} k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du\right) \\
&\le \frac{1}{\alpha\beta}\varliminf_{\varepsilon\to 0^+}\left(\int_0^1 k_\lambda(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}\,du + \int_1^\infty k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du\right) \\
&\le M\varliminf_{\varepsilon\to 0^+}\left(\varepsilon+\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\varepsilon+\frac{1}{\beta}\right)^{\frac{1}{q}} = M\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\beta}\right)^{\frac{1}{q}},
\end{aligned}
$$

namely $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1) \le M$. Hence, $M = \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1)$ is the best possible constant factor of (11). □

Remark 2 Setting $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$
\begin{aligned}
&\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \\
&\hat{\lambda}_1 \le \frac{1}{p\alpha} + \frac{1}{q\alpha} = \frac{1}{\alpha}, \qquad \hat{\lambda}_2 \le \frac{1}{q\beta} + \frac{1}{p\beta} = \frac{1}{\beta},
\end{aligned}
$$

and by Hölder's inequality (cf. [28]), we obtain

$$
\begin{aligned}
0 &< k_\lambda(\lambda-\hat{\lambda}_2) = k_\lambda(\hat{\lambda}_1) = k_\lambda\left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}\right) \\
&= \int_0^\infty k_\lambda(u,1)u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}\,du = \int_0^\infty k_\lambda(u,1)\left(u^{\frac{\lambda-\lambda_2-1}{p}}\right)\left(u^{\frac{\lambda_1-1}{q}}\right)du
\end{aligned}
$$

$$\leq \left(\int_0^\infty k_\lambda(u,1)u^{\lambda-\lambda_2-1}\,du\right)^{\frac{1}{p}}\left(\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}\,du\right)^{\frac{1}{q}}$$
$$= k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) < \infty. \tag{13}$$

We can reduce (10) as follows:

$$I < \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^\infty m^{p(1-\alpha\hat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{14}$$

Lemma 4 *If the constant factor* $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in* (10) *is the best possible, then* $\lambda_1+\lambda_2=\lambda$.

Proof If the constant factor $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (10) is the best possible, then by (14) and (11) the unique best possible constant factor must be $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\hat{\lambda}_1)$ ($\in \mathrm{R}_+$), namely

$$k_\lambda(\hat{\lambda}_1) = k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1).$$

We observe that (13) keeps the form of equality if and only if there exist constants A and B such that they are not all zero and (cf. [28])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \quad \text{a.e. in } \mathrm{R}_+.$$

Assuming that $A \neq 0$ (otherwise, $B = A = 0$), it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in R_+, and then $\lambda-\lambda_2-\lambda_1 = 0$, namely $\lambda_1+\lambda_2=\lambda$. □

3 Main results

Theorem 1 *Inequality* (10) *is equivalent to*

$$J := \left[\sum_{n=1}^\infty n^{p\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}\left(\sum_{m=1}^\infty k_\lambda\left(m^\alpha,n^\beta\right)a_m\right)^p\right]^{\frac{1}{p}}$$
$$< \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=1}^\infty m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}. \tag{15}$$

If the constant factor in (10) *is the best possible, then so is the constant factor in* (15).

Proof Suppose that (15) is valid. By Hölder's inequality (cf. [28]), we find

$$I = \sum_{n=1}^\infty\left[n^{\frac{-1}{p}+\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}\sum_{m=1}^\infty k_\lambda\left(m^\alpha,n^\beta\right)a_m\right]\left[n^{\frac{1}{p}-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}b_n\right]$$
$$\leq J\left\{\sum_{n=1}^\infty n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}. \tag{16}$$

Then by (15) we obtain (10).

On the other hand, assuming that (10) is valid, we set

$$b_n := n^{p\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}\left(\sum_{m=1}^{\infty} k_\lambda\left(m^\alpha, n^\beta\right)a_m\right)^{p-1}, \quad n \in \mathbf{N}.$$

If $J = 0$, then (15) is naturally valid; if $J = \infty$, then it is impossible to make (15) valid, namely $J < \infty$. Suppose that $0 < J < \infty$. By (10), it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q \\
&\quad = J^p = I < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \\
&\qquad \times \left\{\sum_{m=1}^{\infty} m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}, \\
&J = \left\{\sum_{n=1}^{\infty} n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{p}} \\
&\quad < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{\sum_{m=1}^{\infty} m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}},
\end{aligned}$$

namely (15) follows, which is equivalent to (10).

If the constant factor in (10) is the best possible, then so is constant factor in (15). Otherwise, by (16), we would reach a contradiction that the constant factor in (10) is not the best possible. □

Theorem 2 *The following statements* (i), (ii), (iii), *and* (iv) *are equivalent*:

(i) $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is independent of* p, q;

(ii) $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is expressible as a single integral*;

(iii) $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is the best possible constant factor of* (10);

(iv) $\lambda_1 + \lambda_2 = \lambda$.

If the statement (iv) *follows, namely* $\lambda_1 + \lambda_2 = \lambda$, *then we have* (11) *and the following equivalent inequality with the best possible constant factor* $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1)$:

$$\left[\sum_{n=1}^{\infty} n^{p\beta\lambda_2-1}\left(\sum_{m=1}^{\infty} k_\lambda\left(m^\alpha, n^\beta\right)a_m\right)^p\right]^{\frac{1}{p}} < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1)\left[\sum_{m=1}^{\infty} m^{p(1-\alpha\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}. \tag{17}$$

Proof (i) ⇒ (ii). Since $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is independent of p, q, we find

$$k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty}\lim_{q\to 1^+} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1),$$

namely $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral

$$k_\lambda(\lambda_1) = \int_0^\infty k_\lambda(u,1) u^{\lambda_1-1}\,du.$$

(ii) ⇒ (iv). In (13), if $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral $k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (13) keeps the form of equality, from which it follows that $\lambda_1+\lambda_2=\lambda$.

(iv) ⇒ (i). If $\lambda_1+\lambda_2=\lambda$, then $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=k_\lambda(\lambda_1)$, which is independent of p, q. Hence, we have (i) ⇔ (ii) ⇔ (iv).

(iii) ⇒ (iv). By Lemma 4, we have $\lambda_1+\lambda_2=\lambda$.

(iv) ⇒ (iii). By Lemma 3, for $\lambda_1+\lambda_2=\lambda$,

$$\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\quad\left(=\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1)\right)$$

is the best possible constant factor of (10). Therefore, we find (iii) ⇔ (iv).

Hence, statements (i), (ii), (iii), and (iv) are equivalent. □

Remark 3 (i) For $\lambda=\alpha=\beta=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$ in (11) and (17), we have the following equivalent inequalities with the best possible constant factor $k_1(\frac{1}{q})$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_1(m,n)a_mb_n<k_1\left(\frac{1}{q}\right)\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}},\tag{18}$$

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}k_1(m,n)a_m\right)^p\right]^{\frac{1}{p}}<k_1\left(\frac{1}{q}\right)\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}.\tag{19}$$

(ii) For $\lambda=\alpha=\beta=1$, $\lambda_1=\frac{1}{p}$, $\lambda_2=\frac{1}{q}$ in (11) and (17), we have the following equivalent inequalities with the best possible constant factor $k_1(\frac{1}{p})$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_1(m,n)a_mb_n<k_1\left(\frac{1}{p}\right)\left(\sum_{m=1}^{\infty}m^{p-2}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{q-2}b_n^q\right)^{\frac{1}{q}},\tag{20}$$

$$\left[\sum_{n=1}^{\infty}n^{p-2}\left(\sum_{m=1}^{\infty}k_1(m,n)a_m\right)^p\right]^{\frac{1}{p}}<k_1\left(\frac{1}{p}\right)\left(\sum_{m=1}^{\infty}m^{p-2}a_m^p\right)^{\frac{1}{p}}.\tag{21}$$

(iii) For $p=q=2$, both (18) and (20) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_1(m,n)a_mb_n<k_1\left(\frac{1}{2}\right)\left(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}},\tag{22}$$

and both (19) and (21) reduce to the equivalent form of (22) as follows:

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}k_1(m,n)a_m\right)^2\right]^{\frac{1}{2}}<k_1\left(\frac{1}{2}\right)\left(\sum_{m=1}^{\infty}a_m^2\right)^{\frac{1}{2}}.\tag{23}$$

4 Operator expressions and some particular cases

We set functions

$$\phi(m):=m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1},\qquad\psi(n):=n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1},$$

from which

$$\psi^{1-p}(n) = n^{p\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1} \quad (m, n \in \mathrm{N}).$$

Define the following real normed spaces:

$$l_{p,\phi} := \left\{a = \{a_m\}_{m=1}^{\infty}; \|a\|_{p,\phi} := \left(\sum_{m=1}^{\infty} \phi(m)|a_m|^p\right)^{\frac{1}{p}} < \infty\right\},$$

$$l_{q,\psi} := \left\{b = \{b_n\}_{n=1}^{\infty}; \|b\|_{q,\psi} := \left(\sum_{n=1}^{\infty} \psi(n)|b_n|^q\right)^{\frac{1}{q}} < \infty\right\},$$

$$l_{p,\psi^{1-p}} := \left\{c = \{c_n\}_{n=1}^{\infty}; \|c\|_{p,\psi^{1-p}} := \left(\sum_{n=1}^{\infty} \psi^{1-p}(n)|b_n|^p\right)^{\frac{1}{p}} < \infty\right\}.$$

Assuming that $a \in l_{p,\phi}$, setting

$$c = \{c_n\}_{n=1}^{\infty}, \qquad c_n := \sum_{m=1}^{\infty} k_\lambda\left(m^\alpha, n^\beta\right) a_m, \quad n \in \mathrm{N},$$

we can rewrite (15) as follows:

$$\|c\|_{p,\psi^{1-p}} < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\phi} < \infty,$$

namely $c \in l_{p,\psi^{1-p}}$.

Definition 2 Define a Hilbert-type operator $T : l_{p,\phi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\phi}$,there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta, b) := \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} k_\lambda\left(m^\alpha, n^\beta\right) a_m\right) b_n,$$

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\phi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\phi}}.$$

By Theorem 1 and Theorem 2, we have the following.

Theorem 3 *If* $a \in l_{p,\phi}$, $b \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, *then we have the following equivalent inequalities*:

$$(Ta, b) < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \tag{24}$$

$$\|Ta\|_{p,\psi^{1-p}} < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\phi}. \tag{25}$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$ *if and only if the constant factor*

$$\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1)$$

in (24) *and* (25) *is the best possible, namely*

$$\|T\| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1). \tag{26}$$

Example 1 We set $k_\lambda(x,y) := \frac{1}{(cx+y)^\lambda}$ $(c, \lambda > 0; x, y > 0)$. Then we find

$$k_\lambda\left(m^\alpha, n^\beta\right) = \frac{1}{(cm^\alpha + n^\beta)^\lambda}.$$

For $0 < \lambda_1, \lambda - \lambda_2 \le \frac{1}{\alpha}$, $0 < \lambda_2, \lambda - \lambda_1 \le \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is strictly decreasing with respect to $x, y > 0$, and for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}}{(cu+1)^\lambda}\,du = \frac{1}{c^\gamma}\int_0^\infty \frac{v^{\gamma-1}}{(v+1)^\lambda}\,dv = \frac{1}{c^\gamma}B(\gamma, \lambda-\gamma) \in \mathrm{R}_+.$$

In view of Theorem 3, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}} \cdot \frac{1}{c^{\lambda_1}} B(\lambda_1, \lambda_2).$$

Example 2 We set $k_\lambda(x,y) := \frac{\ln(cx/y)}{(cx)^\lambda - y^\lambda}$ $(c, \lambda > 0; x, y > 0)$. Then we find

$$k_\lambda\left(m^\alpha, n^\beta\right) = \frac{\ln(cm^\alpha/n^\beta)}{c^\lambda m^{\lambda\alpha} - n^{\lambda\beta}}.$$

For $0 < \lambda_1, \lambda - \lambda_2 \le \frac{1}{\alpha}$, $0 < \lambda_2, \lambda - \lambda_1 \le \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is strictly decreasing with respect to $x, y > 0$ (cf. [4], Example 2.2.1), and for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}\ln(cu)}{(cu)^\lambda - 1}\,du = \frac{1}{c^\gamma\lambda^2}\int_0^\infty \frac{v^{(\gamma/\lambda)-1}\ln v}{v-1}\,dv = \frac{1}{c^\gamma}\left[\frac{\pi}{\lambda\sin(\pi\gamma/\lambda)}\right]^2 \in \mathrm{R}_+.$$

In view of Theorem 3, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}} \cdot \frac{1}{c^{\lambda_1}}\left[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\right]^2.$$

Example 3 For $s \in \mathrm{N}$, we set $k_\lambda(x,y) := \frac{1}{\prod_{k=1}^s (x^{\lambda/s} + c_k y^{\lambda/s})}$ $(0 < c_1 \le \cdots \le c_s, \lambda > 0; x, y > 0)$. Then we find

$$k_\lambda\left(m^\alpha, n^\beta\right) = \frac{1}{\prod_{k=1}^s (m^{\alpha\lambda/s} + c_k n^{\beta\lambda/s})}.$$

For $0 < \lambda_1, \lambda - \lambda_2 \le \frac{1}{\alpha}$, $0 < \lambda_2, \lambda - \lambda_1 \le \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is strictly decreasing with respect to $x, y > 0$, and for $\gamma = \lambda_1, \lambda - \lambda_2$, by Example 1 of [30], we have

$$k_\lambda^{(s)}(\gamma) = \int_0^\infty \frac{t^{\gamma-1}}{\prod_{k=1}^s (t^{\lambda/s} + c_k)}\,dt = \frac{\pi s}{\lambda\sin(\frac{\pi s\gamma}{\lambda})}\sum_{k=1}^s c_k^{\frac{s\gamma}{\lambda}-1}\prod_{j=1 (j\ne k)}^s \frac{1}{c_j - c_k} \in \mathrm{R}_+.$$

In view of Theorem 3, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{(s)}(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}} \cdot \frac{\pi s}{\lambda \sin(\frac{\pi s\lambda_1}{\lambda})} \sum_{k=1}^{s} c_k^{\frac{s\lambda_1}{\lambda}-1} \prod_{j=1 (j\neq k)}^{s} \frac{1}{c_j - c_k}.$$

In particular, for $c_1 = \cdots = c_s = c$, we have $k_\lambda(x,y) = \frac{1}{(x^{\lambda/s}+cy^{\lambda/s})^s}$ and

$$\begin{aligned}\tilde{k}_\lambda^{(s)}(\lambda_1) &:= \int_0^\infty \frac{t^{\lambda_1-1}}{(t^{\lambda/s}+c)^s}\,dt \\ &= \frac{s}{\lambda c^{(1-\frac{\lambda_1}{\lambda})s}} \int_0^\infty \frac{v^{\frac{s\lambda_1}{\lambda}-1}}{(v+1)^s}\,dv = \frac{s}{\lambda c^{(1-\frac{\lambda_1}{\lambda})s}} B\left(\frac{s\lambda_1}{\lambda}, \frac{s\lambda_2}{\lambda}\right).\end{aligned}$$

If $s = 1$, then we have $k_\lambda(x,y) = \frac{1}{x^\lambda + cy^\lambda}$ and

$$\|T\| = \frac{1}{\beta^{1/p}\alpha^{1/q}} \tilde{k}_\lambda^{(1)}(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}} \cdot \frac{1}{\lambda c^{1-\frac{\lambda_1}{\lambda}}} \frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}.$$

5 Conclusions

In this paper, by means of the weight coefficients and the idea of introducing parameters, a discrete Hilbert-type inequality with the general homogeneous kernel and the intermediate variables is obtained which is an extension of (1). The equivalent forms are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to some parameters are considered in Theorem 2. The operator expressions, some particular cases, and examples are given in Theorem 3, Remark 1, Remark 3, and Examples 1–3. The lemmas and theorems provide an extensive account of this type of inequalities.

Funding
This work is supported by the National Natural Science Foundation (No. 61772140) and Guangxi Natural Science Foundation (No. 2016GXNSFAA380125).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. RL participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Hechi University, Yizhou, P.R. China. [2]Department of Mathematics, Guangdong University of Education, Guangzhou, China.

Publisher's Note

Received: 17 February 2019 Accepted: 10 May 2019 Published online: 21 May 2019

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