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On extreme points and product properties of a new subclass of p-harmonic functions

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Abstract

In this paper, we introduce a new subclass of p-harmonic functions and investigate the univalence and sense-preserving, extreme points, distortion bounds, convex combination, neighborhoods of mappings belonging to the subclass. Relevant connections of the results presented here with the results of previous research are briefly indicated. Finally, we also prove new properties of the Hadamard product of these classes.

MSC: Primary 30C45; secondary 30C80

Keywords: Analytic function; p-harmonic functions; Extreme points; Neighborhood; Hadamard product

1 Introduction

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions $f = h + \bar{g}$ in $\mathbb{U} = \{z : |z| < 1\}$, where h and g are analytic in $\mathbb{U}$ and normalized such that

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{\infty} a_j z^j + \overline{\sum_{j=1}^{\infty} b_j z^j}. \tag{1.1}$$

A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{U}$ is that $J_f = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ in $\mathbb{U}$ (see [1, 2]). Let S_H denote the subclass of $\mathcal{H}$ consisting of sense-preserving univalent functions in $\mathbb{U}$. Then the function $f \in S_H$ of the form (1.1) satisfies the condition $|b_1| < 1$.

A $2p$-times continuously differentiable complex-valued function $F = u + iv$ in a domain $\mathbb{U}$ is p-harmonic if F satisfies the p-harmonic equation $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$ ($p = 1, 2, \ldots$), where Δ represents the complex Laplacian operator:

$$\Delta = 4\frac{\partial^2}{\partial z \, \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Obviously, if we take $p = 1$ and $p = 2$, then F is harmonic and biharmonic, respectively.

A function F is p-harmonic in a simply connected domain $\mathbb{U}$ if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} f_{p-k+1}(z) \quad \bigl(k \in \{1, 2, \ldots, p\}\bigr), \tag{1.2}$$

where each $f_{p-k+1}(z)$ is harmonic (or $\triangle f_{p-k+1} = 0$) (see [3]) and $f_{p-k+1}(z)$ has the form

$$f_{p-k+1} = h_{p-k+1} + \bar{g}_{p-k+1}, \tag{1.3}$$

where

$$h_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^{j} \quad (a_{1,p} = 1, k \geq 1), \tag{1.4}$$

$$g_{p-k+1}(z) = \sum_{j=1}^{\infty} b_{j,p-k+1} z^{j} \quad (k \geq 1). \tag{1.5}$$

Denote by SH_p the class of functions F of the form (1.2) that are p-harmonic, univalent, and sense-preserving in the unit disk. Recently, there has been significant interest in results about the class SH_p (see, for details, [4–9]).

Denote by $HL_p(\alpha, \lambda)$ $(0 \leq \alpha < 1, \lambda \geq 0)$ the class of all mappings of the form (1.2) which satisfy the condition

$$\begin{aligned} &\sum_{k=1}^{p} \sum_{j=2}^{\infty} \biggl(2(k-1) + \frac{j^{\lambda}(j-\alpha)}{1-\alpha}\biggr) \bigl(|a_{j,p-k+1}| + |b_{j,p-k+1}|\bigr) \\ &\quad \leq 1 - |b_{1,p}| - \sum_{k=2}^{p} (2k-1) \bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr) \end{aligned} \tag{1.6}$$

with

$$0 \leq |b_{1,p}| + \sum_{k=2}^{p} (2k-1) \bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr) < 1. \tag{1.7}$$

Clearly, inequality (1.6) implies that

$$\sum_{k=1}^{p} \sum_{j=1}^{\infty} \biggl((k-1) + \frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\biggr) \bigl(|a_{j,p-k+1}| + |b_{j,p-k+1}|\bigr) \leq 1, \tag{1.8}$$

where $a_{1,p} = 1, k \in \{1, \ldots, p\}$.

It is easy to see that various subclasses of SH_p consisting of mappings $F(z)$ of the form (1.2) and (1.3) can be represented as $HL_p(\alpha, \lambda)(b_{1,p} = a_{1,p-k+1} = b_{1,p-k+1} = 0, k = 2, \ldots, p)$ for suitable choices of p, α, and λ in the earlier studies by various authors.

(i) $HL_p(0, 0) = HS_p$ and $HL_p(0, 1) = HC_p$ (see Qiao and Wang [4]);

(ii) $HL_p(\alpha, 0) = HS_p(\alpha)$ and $HL_p(\alpha, 1) = HC_p(\alpha)$ (see Saurabh Porwal and Dixit [5]);

(iii) $HL_1(\alpha, 0) = HS(\alpha)$ and $HL_1(\alpha, 1) = HC(\alpha)$ (see Öztürk and Yalcin [10]);

(vi) $HL_1(0,0) = HS$ and $HL_1(0,1) = HC$ (see Avci and Zlotkiewicz [11]).

For $\lambda \in \mathbb{N} = \{1,2,\ldots\} \cup \{0\}$, we have the following inclusion relation:

$$HL_p(\alpha,\lambda) \subset HL_p(\alpha,\lambda-1) \subset \cdots \subset HL_p(\alpha,2) \subset HC_p(\alpha) \subset HS_p(\alpha).$$

Suppose that F is a p-harmonic mapping with expression (1.2). Following Ruscheweyh [12], we use $N^{\delta}_{\lambda,\alpha}(F)$ to denote the δ-neighborhood of F in p-harmonic mappings:

$$\begin{aligned} N^{\delta}_{\lambda,\alpha}(F) = \Bigg\{ \tilde{F} : |b_{1,p} - B_{1,p}| &+ \sum_{k=2}^{p} (2k-1)\big(|a_{1,p-k+1} - A_{1,p-k+1}| + |b_{1,p-k+1} - B_{1,p-k+1}|\big) \\ &+ \sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1) + \frac{j^{\lambda}(j-\alpha)}{1-\alpha}\right)\big(|a_{j,p-k+1} - A_{j,p-k+1}| \\ &+ |b_{j,p-k+1} - B_{j,p-k+1}|\big) \le \delta \Bigg\}, \end{aligned}$$

where

$$\tilde{F} = z + \sum_{j=2}^{\infty} A_{j,p} z^j + \sum_{j=1}^{\infty} \bar{B}_{j,p}\bar{z}^j + \sum_{k=2}^{p} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} A_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{B}_{j,p-k+1}\bar{z}^j\right).$$

If $F, G \in SH_p$ satisfy

$$F = \sum_{k=1}^{p} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,p-k+1}\bar{z}^j\right)$$

and

$$G = \sum_{k=1}^{p} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} A_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{B}_{j,p-k+1}\bar{z}^j\right),$$

then the convolution $F * G$ of F and G is defined to be the mapping

$$F * G = \sum_{k=1}^{p} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{j,p-k+1}A_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,p-k+1}\bar{B}_{j,p-k+1}\bar{z}^j\right).$$

Let

$$TH_p = \big\{F(z) : F \in SH_p \text{ with } a_{1,p} = 1, a_{j,p-k+1} \ge 0, b_{j,p-k+1} \ge 0 \text{ for } j \ge 1, k = 1,\ldots,p\big\}$$

and denote $\overline{HL}_p(\alpha,\lambda) = HL_p(\alpha,\lambda) \cap TH_p$.

The main objective of the paper is to introduce a new subclass of p-harmonic mappings and investigate the univalence and sense-preserving, extreme points, neighborhoods and Hadamard product of mappings for the above subclass. Relevant connections of the results presented here with the results of Qiao et al. [4] and Porwal et al. [5] are briefly indicated. Finally, we also prove new properties of the Hadamard product of these classes.

2 Main results

Firstly, we discuss the inclusion relation of $HL_p(\alpha,\lambda)$.

Theorem 2.1 *Let* $\lambda_2 \geq \lambda_1 \geq 0, 1 > \alpha_2 \geq \alpha_1 \geq 0$, *then* $HL_p(\alpha_2,\lambda_2) \subseteq HL_p(\alpha_1,\lambda_1)$.

Proof Let $F \in HL_p(\alpha_2,\lambda_2)$, then using (1.6), we have

$$\begin{aligned}
&\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j^{\lambda_1}(j-\alpha_1)}{1-\alpha_1}\right)\left(|a_{j,p-k+1}|+|b_{j,p-k+1}|\right)\\
&\quad\leq \sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j^{\lambda_2}(j-\alpha_2)}{1-\alpha_2}\right)\left(|a_{j,p-k+1}|+|b_{j,p-k+1}|\right)\\
&\quad\leq 1-|b_{1,p}|-\sum_{k=2}^{p}(2k-1)\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right),
\end{aligned}$$

therefore $F \in HL_p(\alpha_1,\lambda_1)$, and so $HL_p(\alpha_2,\lambda_2) \subseteq HL_p(\alpha_1,\lambda_1)$. □

Next, we prove that the mapping in $HL_p(\alpha,\lambda)$ is univalent and sense-preserving.

Theorem 2.2 *Each mapping in* $HL_p(\alpha,\lambda)$ *is univalent and sense-preserving.*

Proof Let $F \in HL_p(\alpha,\lambda)$ and $z_1, z_2 \in \mathbb{U}$ with $z_1 \neq z_2$, so that $|z_1| \leq |z_2|$:

$$\begin{aligned}
&\left|F(z_1)-F(z_2)\right|\\
&\quad=\left|\sum_{k=1}^{p}\left(|z_1|^{2(k-1)}f_{p-k+1}(z_1)-|z_2|^{2(k-1)}f_{p-k+1}(z_2)\right)\right|\\
&\quad\geq |z_1-z_2|\left\{1-\left|\sum_{j=2}^{\infty}a_{j,p}\frac{z_1^j-z_2^j}{z_1-z_2}+\sum_{j=1}^{\infty}\bar{b}_{j,p}\frac{\bar{z}_1^j-\bar{z}_2^j}{z_1-z_2}\right|\right\}\\
&\qquad-\left|\sum_{k=2}^{p}\left(\sum_{j=1}^{\infty}a_{j,p-k+1}\frac{|z_1|^{2(k-1)}z_1^j-|z_2|^{2(k-1)}z_2^j}{z_1-z_2}\right.\right.\\
&\qquad\left.\left.+\sum_{j=1}^{\infty}\bar{b}_{j,p-k+1}\frac{|z_1|^{2(k-1)}\bar{z}_1^j-|z_2|^{2(k-1)}\bar{z}_2^j}{z_1-z_2}\right)\right|\\
&\quad\geq |z_1-z_2|\left(1-|b_{1,p}|-|z_2|\sum_{j=2}^{\infty}j\left(|a_{j,p}|+|b_{j,p}|\right)\right.\\
&\qquad\left.-|z_2|\sum_{k=2}^{p}\sum_{j=1}^{\infty}\left(2(k-1)+j\right)\left(|a_{j,p-k+1}|+|b_{j,p-k+1}|\right)\right)\\
&\quad\geq |z_1-z_2|\left(1-|b_{1,p}|-|z_2|\sum_{j=2}^{\infty}\frac{j^{\lambda}(j-\alpha)}{1-\alpha}\left(|a_{j,p}|+|b_{j,p}|\right)\right.\\
&\qquad\left.-|z_2|\sum_{k=2}^{p}\sum_{j=1}^{\infty}\left(2(k-1)+\frac{j^{\lambda}(j-\alpha)}{1-\alpha}\right)\left(|a_{j,p-k+1}|+|b_{j,p-k+1}|\right)\right)
\end{aligned}$$

$$\begin{aligned}&\geq |z_1 - z_2|\bigl(1-|b_{1,p}|\bigr)\bigl(1-|z_2|\bigr)\\&> 0,\end{aligned}$$

which proves univalence.

In order to prove that F is sense-preserving, we need to show that $J_F = |F_z|^2 - |F_{\bar{z}}|^2 > 0$:

$$\begin{aligned}J_F(z) &= |F_z|^2 - |F_{\bar{z}}|^2 = \bigl(|F_z| + |F_{\bar{z}}|\bigr)\bigl(|F_z| - |F_{\bar{z}}|\bigr)\\
&= \bigl(|F_z| + |F_{\bar{z}}|\bigr)\Biggl\{\Biggl|1 + \sum_{j=2}^{\infty} j a_{j,p} z^{j-1} + \sum_{k=2}^{p}\sum_{j=2}^{\infty} |z|^{2(k-1)} j a_{j,p-k+1} z^{j-1}\\
&\quad + \sum_{k=2}^{p} |z|^{2(k-1)} j a_{1,p-k+1} + \sum_{k=2}^{p} (k-1)|z|^{2(k-1)}\\
&\quad \times \Biggl(\sum_{j=1}^{\infty} a_{j,p-k+1} z^{j-1} + \frac{\bar{z}}{z}\sum_{j=1}^{\infty} \bar{b}_{j,p-k+1} \bar{z}^{j-1}\Biggr)\Biggr|\\
&\quad - \Biggl|\sum_{j=1}^{\infty} j\bar{b}_{j,p} \bar{z}^{j-1} + \sum_{k=2}^{p}\sum_{j=2}^{\infty} |z|^{2(k-1)} j\bar{b}_{j,p-k+1} \bar{z}^{j-1}\\
&\quad + \sum_{k=2}^{p} |z|^{2(k-1)} \bar{b}_{1,p-k+1} + \sum_{k=2}^{p} (k-1)|z|^{2(k-1)}\\
&\quad \times \Biggl(\frac{z}{\bar{z}}\sum_{j=1}^{\infty} a_{j,p-k+1} z^{j-1} + \frac{\bar{z}}{z}\sum_{j=1}^{\infty} \bar{b}_{j,p-k+1} \bar{z}^{j-1}\Biggr)\Biggr|\Biggr\}\\
&\geq \bigl(|F_z| + |F_{\bar{z}}|\bigr)\Biggl[1 - |b_{1,p}| - \sum_{k=2}^{p} (2k-1)\bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr)\\
&\quad - |z|\sum_{k=1}^{p}\sum_{j=2}^{\infty} \bigl(2(k-1) + j\bigr)\bigl(|a_{j,p-k+1}| + |b_{j,p-k+1}|\bigr)\Biggr]\\
&\geq \bigl(|F_z| + |F_{\bar{z}}|\bigr)\Biggl[1 - |b_{1,p}| - \sum_{k=2}^{p} (2k-1)\bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr)\\
&\quad - |z|\sum_{k=1}^{p}\sum_{j=2}^{\infty} \Biggl(2(k-1) + \frac{j^{\lambda}(j-\alpha)}{1-\alpha}\Biggr)\bigl(|a_{j,p-k+1}| + |b_{j,p-k+1}|\bigr)\Biggr]\\
&\geq \bigl(|F_z| + |F_{\bar{z}}|\bigr)\bigl(1 - |b_{1,p}|\bigr)\bigl(1 - |z|\bigr)\\
&> 0.\end{aligned}$$

From $z \neq 0$ and the obvious fact $J_F(0) > 0$, we thus complete the proof. □

Example 2.1 Let $F(z) = z + \frac{1}{(2p-1)}|z|^{2(p-1)}\bar{z}$. Then $F(z)$ is a p-harmonic function and

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\Biggl((k-1) + \frac{j^{\lambda+1}}{2}\Biggr)\bigl(|a_{j,p-k+1}| + |b_{j,p-k+1}|\bigr) < 1,$$

using (1.8), we get $F \in HL_p(0,\lambda)$.

Also, we determine the extreme points of $\overline{HL}_p(\alpha,\lambda)$.

Theorem 2.3 *Let F be given by (1.2). Then $F\in\overline{HL}_p(\alpha,\lambda)$ if and only if*

$$F(z)=\sum_{k=1}^{p}\sum_{j=1}^{\infty}\bigl(X_{j,p-k+1}h_{j,p-k+1}(z)+Y_{j,p-k+1}g_{j,p-k+1}(z)\bigr), \tag{2.1}$$

where

$$h_{j,p-k+1}(z)=z+|z|^{2(k-1)}\frac{z^j}{(k-1)+\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\quad(2\le k\le p; j\ge 1),$$

$$g_{j,p-k+1}(z)=z+|z|^{2(k-1)}\frac{\overline{z}^j}{(k-1)+\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\quad(2\le k\le p; j\ge 1),$$

$$h_{1,p}(z)=z,\qquad h_{1,j}(z)=z+\frac{z^j}{\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\quad(j\ge 2),$$

$$g_{1,j}(z)=z+\frac{\overline{z}^j}{\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\quad(j\ge 1)$$

and

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(X_{j,p-k+1}+Y_{j,p-k+1})=1\quad(X_{j,p-k+1}\ge 0, Y_{j,p-k+1}\ge 0).$$

In particular, the extreme points of $\overline{HL}_p(\alpha,\lambda)$ are $\{h_{j,p-k+1}(z)\}$ and $\{g_{j,p-k+1}(z)\}$, where $j\ge 1$ and $1\le k\le p$.

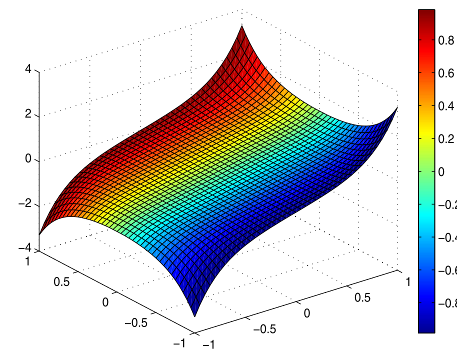
Proof Since

$$\begin{aligned}F(z)&=\sum_{k=1}^{p}\sum_{j=1}^{\infty}\bigl(X_{j,p-k+1}h_{j,p-k+1}(z)+Y_{j,p-k+1}g_{j,p-k+1}(z)\bigr)\\&=z+\sum_{k=2}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\left(\frac{X_{j,p-k+1}}{(k-1)+\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}z^j+\frac{Y_{j,p-k+1}}{(k-1)+\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\overline{z}^j\right)\\&\quad+\sum_{j=1}^{\infty}\frac{X_{j,p}}{\frac{j^\lambda(j-\alpha)}{1-\alpha}}z^j+\sum_{j=1}^{\infty}\frac{Y_{j,p}}{\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\overline{z}^j\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left((k-1)+\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}\right)\left(\left|\frac{X_{j,p-k+1}}{(k-1)+\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\right|+\left|\frac{Y_{j,p-k+1}}{(k-1)+\frac{j^\lambda(j-\alpha)}{2(1-\alpha)}}\right|\right)\\&\quad+|Y_{1,p}|+\sum_{k=2}^{p}\frac{(2k-1)}{2}\left(\left|\frac{X_{1,p-k+1}}{\frac{2k-1}{2}}\right|+\left|\frac{Y_{1,p-k+1}}{\frac{2k-1}{2}}\right|\right)\\&\le\sum_{k=1}^{p}\sum_{j=2}^{\infty}(X_{j,p-k+1}+Y_{j,p-k+1})+\sum_{k=2}^{p}(X_{1,p-k+1}+Y_{1,p-k+1})+Y_{1,p}\end{aligned}$$

Figure 1 When $F(z)$ is a 4-harmonic function



$$\leq 1 - X_{1,p}$$
$$\leq 1,$$

we see that $F \in \overline{HL}_p(\alpha,\lambda)$.

Conversely, assuming that $F \in \overline{HL}_p(\alpha,\lambda)$ and setting

$$X_{j,p-k+1} = \left((k-1) + \frac{j^\lambda(j-\alpha)}{2(1-\alpha)}\right)a_{j,p-k+1} \quad (2 \leq k \leq p, j \geq 1),$$
$$X_{j,p} = \frac{j^\lambda(j-\alpha)}{2(1-\alpha)}a_{j,p} \quad (j \geq 2),$$
$$Y_{j,p-k+1} = \left((k-1) + \frac{j^\lambda(j-\alpha)}{2(1-\alpha)}\right)b_{j,p-k+1} \quad (1 \leq k \leq p, j \geq 1)$$

and

$$X_{1,p} = 1 - \sum_{k=1}^{p}\sum_{j=2}^{\infty}(X_{j,p-k+1} + Y_{j,p-k+1}) - \sum_{k=2}^{p}(X_{1,p-k+1} + Y_{1,p-k+1}) - Y_{1,p},$$

where $X_{1,p} \geq 0$. Then, as required, we obtain

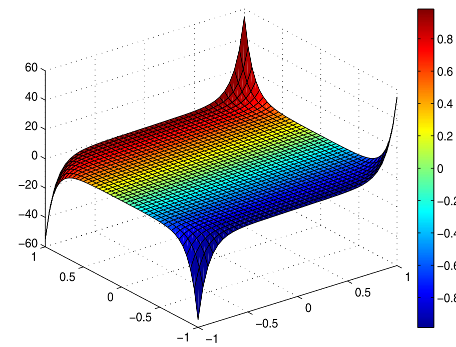
$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty}\bigl(X_{j,p-k+1}h_{j,p-k+1}(z) + Y_{j,p-k+1}g_{j,p-k+1}(z)\bigr). \qquad \square$$

Example 2.2 Let $F(z) = z + \frac{1}{(2p-1)}|z|^{2(p-1)}z + \frac{1}{(2p-1)}|z|^{2(p-1)}\bar{z}$. Then $F(z)$ is a p-harmonic function, and using Theorem 2.3, we have $F \in \overline{HL}_p(0,\lambda)$. Here, we give the figures for $p = 4$ and $p = 10$, respectively (see Fig. 1 and Fig. 2).

Theorem 2.4 *Let F be given by* (1.2) *and $F \in \overline{HL}_p(\alpha,\lambda)$. Then, for $|z| = r < 1$, we have*

$$\begin{aligned}\bigl|F(z)\bigr| \leq{}& \left(\sum_{k=1}^{p}\bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr)\right)r \\ &+ \frac{1}{\psi_{2,1}(\lambda,\alpha)}\left(1 - |b_{1,p}| - \sum_{k=2}^{p}\bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr)\right)r^2 \end{aligned} \tag{2.2}$$

Figure 2 When $F(z)$ is a 10-harmonic function



and

$$\begin{aligned}\left|F(z)\right| &\geq \left(1-|b_{1,p}|-\sum_{k=2}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r \\ &\quad -\frac{1}{\psi_{2,1}(\lambda,\alpha)}\left(1-|b_{1,p}|-\sum_{k=2}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r^2, \end{aligned} \tag{2.3}$$

where

$$\psi_{j,k}(\lambda,\alpha)=(k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}. \tag{2.4}$$

Proof Let $F\in\overline{HL}_p(\alpha,\lambda)$. Taking the absolute value of $F(z)$, we have

$$\begin{aligned}\left|F(z)\right| &\leq \left(\sum_{k=1}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r+\left(\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r^2 \\ &\leq \left(\sum_{k=1}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r \\ &\quad +\left(\frac{1}{\psi_{2,1}(\lambda,\alpha)}\sum_{k=1}^{p}\sum_{j=2}^{\infty}\psi_{j,k}(\lambda,\alpha)\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r^2 \\ &\leq \left(\sum_{k=1}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r \\ &\quad +\frac{1}{\psi_{2,1}(\lambda,\alpha)}\left(1-|b_{1,p}|-\sum_{k=2}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r^2\end{aligned}$$

and

$$\begin{aligned}\left|F(z)\right| &\geq \left(\sum_{k=1}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r-\left(\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r^2 \\ &\geq \left(\sum_{k=1}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{1}{\psi_{2,1}(\lambda,\alpha)} \sum_{k=1}^{p} \sum_{j=2}^{\infty} \psi_{j,k}(\lambda,\alpha) \bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr) \right) r^2 \\
\geq &\left(1 - |b_{1,p}| - \sum_{k=2}^{p} \bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr) \right) r \\
&- \frac{1}{\psi_{2,1}(\lambda,\alpha)} \left(1 - |b_{1,p}| - \sum_{k=2}^{p} \bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr) \right) r^2. \qquad \square
\end{aligned}
$$

Corollary 2.5 *Let F be given by (1.2) and $F \in \overline{HL}_p(\alpha,\lambda)$. Then*

$$
\bigl\{\omega : |\omega| < \rho\bigr\} \subset F(\mathbb{U}),
$$

where

$$
\rho = \frac{1 + \psi_{2,1}(\lambda,\alpha)}{\psi_{2,1}(\lambda,\alpha)} - \frac{1 - \psi_{2,1}(\lambda,\alpha)}{\psi_{2,1}(\lambda,\alpha)} \left(|b_{1,p}| + \sum_{k=2}^{p} \bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}|\bigr) \right)
$$

and $\psi_{j,k}(\lambda,\alpha)$ is given by (2.4).

Theorem 2.6 *The class $F \in \overline{HL}_p(\alpha,\lambda)$ is closed under combination.*

Proof For $i = 1, 2, \ldots$, let $F_i \in \overline{HL}_p(\alpha,\lambda)$, where

$$
F_i(z) = z + \sum_{j=2}^{\infty} a_{ij,p} z^j + \sum_{j=1}^{\infty} b_{ij,p} \bar{z}^j + \sum_{k=2}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \bigl(|a_{j,p-k+1}| z^j + |\bar{b}_{j,p-k+1}| \bar{z}^j \bigr).
$$

Then, by (1.6) and (2.4), we get

$$
\sum_{k=1}^{p} \sum_{j=2}^{\infty} \psi_{j,k}(\lambda,\alpha) \bigl(|a_{j,p-k+1}| + |b_{j,p-k+1}| \bigr) \leq 1 - |b_{1,p}| - \sum_{k=2}^{p} \bigl(|a_{1,p-k+1}| + |b_{1,p-k+1}| \bigr). \tag{2.5}
$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of F_i may be written as

$$
\begin{aligned}
\sum_{i=1}^{\infty} t_i F_i = z &- \sum_{j=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i \bigl[|a_{ij,p}| z^j + |b_{ij,p}| \bar{z}^j \bigr] \right) \\
&- \sum_{k=2}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \bigl[|a_{ij,p-k+1}| z^j + |b_{ij,p-k+1}| \bar{z}^j \bigr] \right).
\end{aligned}
$$

Then, by (2.5), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{p} \sum_{j=2}^{\infty} \psi_{j,k}(\lambda,\alpha) \left(\sum_{i=1}^{\infty} t_i \bigl[|a_{ij,p-k+1} + |b_{ij,p-k+1}| \bigr] \right) \\
&= \sum_{i=1}^{\infty} t_i \left[\sum_{k=1}^{p} \sum_{j=2}^{\infty} \psi_{j,k}(\lambda,\alpha) \cdot \bigl(|a_{ij,p-k+1}| + |b_{ij,p-k+1}| \bigr) \right]
\end{aligned}
$$

$$\leq \left[(1-|b_{1,p}| - \sum_{k=2}^{p} \big(|a_{1,p-k+1}| + |b_{1,p-k+1}|\big) \right] \sum_{i=1}^{\infty} t_i$$

$$= 1-|b_{1,p}| - \sum_{k=2}^{p} \big(|a_{1,p-k+1}| + |b_{1,p-k+1}|\big).$$

Therefore, using (1.6), we obtain $\sum_{i=1}^{\infty} t_i F_i \in \overline{HL}_p(\alpha,\lambda)$. □

Theorem 2.7 *Let*

$$F_1(z) = z + \sum_{j=2}^{\infty} a_{j,p} z^j + \sum_{j=2}^{p} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,p-k+1} \bar{z}^j \right)$$

belong to $\overline{HL}_p(\alpha,\lambda_2)$. *If* $\lambda_2 > \lambda_1 \geq 0$ *and*

$$\delta \leq (1-c_0)\big(1-|b_{1,p}|\big) - \sum_{k=2}^{p} (2k-1)\big(|a_{1,p-k+1}| + |b_{1,p-k+1}|\big), \tag{2.6}$$

then $N^{\delta}_{\lambda_1,\alpha}(F_1) \subset HL_p(\alpha,\lambda_1)$, *where*

$$c_0 = \frac{2(p-1)(1-\alpha) + 2^{\lambda_1}(2-\alpha)}{2(p-1)(1-\alpha) + 2^{\lambda_2}(2-\alpha)}. \tag{2.7}$$

Proof The δ-neighborhood of F_1 is the set

$$\begin{aligned} N^{\delta}_{\lambda_1,\alpha}(F_1) = \Bigg\{ F_2 : \sum_{k=1}^{p} \sum_{j=2}^{\infty} \left(2(k-1) + \frac{j^{\lambda_1}(j-\alpha)}{1-\alpha} \right) \big(|a_{j,p-k+1} - A_{j,p-k+1}| \\ + |b_{j,p-k+1} - B_{j,p-k+1}| \big) + |b_{1,p} + B_{1,p}| + \sum_{k=2}^{p} (2k-1)\big(|a_{1,p-k+1} - A_{1,p-k+1}| \\ + |b_{1,p-k+1} - B_{1,p-k+1}| \big) \leq \delta \Bigg\}, \end{aligned}$$

where

$$F_2(z) = z + \sum_{j=2}^{\infty} A_{j,p} z^j + \sum_{j=1}^{\infty} \bar{B}_{j,p} \bar{z}^j + \sum_{k=2}^{\infty} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} A_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{B}_{j,p-k+1} \bar{z}^j \right).$$

If

$$\delta \leq (1-c_0)\big(1-|b_{1,p}|\big) - \sum_{k=2}^{p} (2k-1)\big(|a_{1,p-k+1}| + |b_{1,p-k+1}|\big),$$

then we have

$$\sum_{j=2}^{\infty} \frac{j^{\lambda_1}(j-\alpha)}{1-\alpha} |A_{j,p}| + \sum_{j=1}^{\infty} \frac{j^{\lambda_1}(j-\alpha)}{1-\alpha} |B_{j,p}|$$

$$
\begin{aligned}
&+\sum_{k=2}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j^{\lambda_1}(j-\alpha)}{1-\alpha}\right)\bigl(|A_{j,p-k+1}|+|B_{j,p-k+1}|\bigr)\\
&\leq\sum_{k=2}^{p}(2k-1)\bigl(|a_{1,p-k+1}-A_{1,p-k+1}|+|b_{1,p-k+1}-B_{1,p-k+1}|+|b_{1,p}-B_{1,p}|\bigr)\\
&+\sum_{k=2}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j^{\lambda_1}(j-\alpha)}{1-\alpha}\right)\bigl((|a_{j,p-k+1}-A_{1,p-k+1}|+|b_{j,p-k+1}-B_{j,p-k+1}|\bigr)\\
&+\sum_{k=2}^{p}(2k-1)\bigl(|a_{1,p-k+1}|+|b_{1,p-k+1}|\bigr)+|b_{1,p}|\\
&+\sum_{k=2}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j^{\lambda_1}(j-\alpha)}{1-\alpha}\right)\bigl((|a_{j,p-k+1}|+|b_{j,p-k+1}|\bigr)\\
&\leq\delta+\sum_{k=2}^{p}(2k-1)\bigl(|a_{1,p-k+1}|+|b_{1,p-k+1}|\bigr)+|b_{1,p}|\\
&+c_0\sum_{k=2}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j^{\lambda_2}(j-\alpha)}{1-\alpha}\right)\bigl(|a_{j,p-k+1}|+|b_{j,p-k+1}|\bigr)\\
&\leq\delta+c_0+(1-c_0)\left(\sum_{k=2}^{p}(2k-1)\bigl(|a_{1,p-k+1}|+|b_{1,p-k+1}|\bigr)+|b_{1,p}|\right)\\
&\leq 1.
\end{aligned}
$$

Hence $F_2\in\overline{HL}_p(\alpha,\lambda_1)$. □

Remark 2.8

1. If $\alpha=0,\lambda=0$ and $\alpha=0,\lambda=1$, then Theorem 2.2, Theorem 2.4, and Theorem 2.7, respectively, coincide with Theorem 3.1, Theorem 4.3, Theorem 4.4, Lemma 4.1, and Theorem 5.1 in [4].
2. If $\lambda=0$ and $\lambda=1$, then Theorem 2.2, Theorem 2.3, and Theorem 2.7, respectively, coincide with Theorem 3.1, Theorem 3.6, Theorem 3.7, and Theorem 4.1 in [5].

At last, we discuss the Hadamard product of $\overline{HL}_p(\alpha,\lambda)$.

Theorem 2.9 *Let $\lambda\geq 0$, $0\leq\alpha<1$, $p\in\{1,2,\ldots\}$. If F, $G\in\overline{HL}_p(\alpha,\lambda)$, then $F*G\in\overline{HL}_p(\alpha,\lambda)$, where*

$$2^{\lambda-1}(2-\alpha)\geq(1-\alpha)p^2\,. \tag{2.8}$$

Proof Let $F,G\in\overline{HL}_p(\alpha,\lambda)$, then, from (1.8), we know that, in order to prove $F*G\in\overline{HL}_p(\alpha,\lambda)$, we need to show that

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\bigl(|A_{j,p-k+1}||a_{j,p-k+1}|+|B_{j,p-k+1}||b_{j,p-k+1}|\bigr)\leq 1. \tag{2.9}$$

Since $F, G \in \overline{HL}_p(\alpha,\lambda)$, using (1.8), we have

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\bigl(|a_{j,p-k+1}|+|b_{j,p-k+1}|\bigr)\leq 1 \tag{2.10}$$

and

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\bigl(|A_{j,p-k+1}|+|B_{j,p-k+1}|\bigr)\leq 1. \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\bigl(|a_{j,p-k+1}|+|b_{j,p-k+1}|\bigr)\leq 1 \tag{2.12}$$

and

$$\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\bigl(|A_{j,p-k+1}|+|B_{j,p-k+1}|\bigr)\leq 1. \tag{2.13}$$

Using the Cauchy–Schwarz inequations, from (2.12) and (2.13), we get

$$\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\sqrt{\bigl(|A_{j,p-k+1}|+|B_{j,p-k+1}|\bigr)\bigl(|a_{j,p-k+1}|+|b_{j,p-k+1}|\bigr)}\leq 1, \tag{2.14}$$

because

$$\begin{aligned}&\bigl(|A_{j,p-k+1}||a_{j,p-k+1}|+|B_{j,p-k+1}||b_{j,p-k+1}|\bigr)\\&\quad\leq\bigl(|A_{j,p-k+1}|+|B_{j,p-k+1}|\bigr)\bigl(|a_{j,p-k+1}|+|b_{j,p-k+1}|\bigr)\quad(1\leq k\leq 1, j\in\mathbb{N}).\end{aligned} \tag{2.15}$$

So from (2.14) and (2.15), we have

$$\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\sqrt{\bigl(|A_{j,p-k+1}||a_{j,p-k+1}|+|B_{j,p-k+1}||b_{j,p-k+1}|\bigr)}\leq 1,$$

and hence

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\left((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)}\right)\sqrt{\bigl(|A_{j,p-k+1}||a_{j,p-k+1}|+|B_{j,p-k+1}||b_{j,p-k+1}|\bigr)}\leq p, \tag{2.16}$$

which implies that

$$\sqrt{\bigl(|A_{j,p-k+1}||a_{j,p-k+1}|+|B_{j,p-k+1}||b_{j,p-k+1}|\bigr)}\leq\frac{p}{((k-1)+\frac{j^{\lambda}(j-\alpha)}{2(1-\alpha)})}. \tag{2.17}$$

In addition, if

$$\bigl(|A_{j,p-k+1}||a_{j,p-k+1}|+|B_{j,p-k+1}||b_{j,p-k+1}|\bigr)\leq\frac{1}{p}\sqrt{\bigl(|A_{j,p-k+1}||a_{j,p-k+1}|+|B_{j,p-k+1}||b_{j,p-k+1}|\bigr)},$$

that is,

$$\sqrt{\left(|A_{j,p-k+1}||a_{j,p-k+1}| + |B_{j,p-k+1}||b_{j,p-k+1}|\right)} \leq \frac{1}{p}, \tag{2.18}$$

then we obtain the conditions of satisfaction (2.9). Again, combining (2.17) and (2.18) with $k = 1$ and $j = 2$, we can get

$$\frac{p}{(\frac{2^{\lambda}(2-\alpha)}{2(1-\alpha)})} \leq \frac{1}{p},$$

which deduces condition (2.8). The proof is completed. □

Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 2.9, respectively, we obtain the following corollaries.

Corollary 2.10 *Let* $0 \leq \alpha < 1, 2 - \alpha \geq 2(1-\alpha)p^2 (p \geq 1)$. *If* $F, G \in HS_p(\alpha)$, *then* $F * G \in HS_p(\alpha)$.

Corollary 2.11 *Let* $0 \leq \alpha < 1, 2 - \alpha \geq (1-\alpha)p^2 (p \geq 1)$. *If* $F, G \in HC_p(\alpha)$, *then* $F * G \in HC_p(\alpha)$.

3 Conclusions

In this paper, we mainly introduce a new subclass of p-harmonic mappings and investigate the univalence and sense-preserving, extreme points, distortion bounds, convex combination, neighborhoods of mappings belonging to the subclass. Relevant connections of the results presented here with the results of Qiao et al. [4] and Porwal et al. [5] are briefly indicated. Finally, we also prove new properties of the Hadamard product of these classes.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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