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# The convergence of $(p,q)$-Bernstein operators for the Cauchy kernel with a pole via divided difference

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**Abstract**

In this paper, some qualitative approximation results for the $(p,q)$-Bernstein operators $B^n_{p,q}(f;x)$ are obtained for the Cauchy kernel $\frac{1}{x-\alpha}$ with a pole $\alpha \in [0,1]$ for $q > p > 1$. The main focus lies in the study of behavior of operators $B^n_{p,q}(f;x)$ for the function $f_m(x) = \frac{1}{x-p^m q^{-m}}$, $x \neq p^m q^{-m}$ and $f_m(p^m q^{-m}) = a$, $a \in \mathbb{R}$ and the extra parameter $p$ provides flexibility for the approximation.

**MSC:** 41A10; 41A25; 41A36

**Keywords:** $(p,q)$-integer; $(p,q)$-Bernstein operator; Convergence; Approximation of unbounded function; Cauchy kernel

## 1 Introduction and preliminaries

The uniform convergence of a sequence of operators to a continuous function was introduced by Bohman [9] and Korovkin [16]. Through $q$-calculus various modifications of Bernstein operators [7] have been studied so far [10, 18, 31]. The $(p,q)$-integers are the generalization of the $q$-integers, which has an important role in the representation theory of quantum calculus in the physics literature. Recently, the approximation by the $(p,q)$-analog of a positive linear operator has become an active area of research. For the theory and numerical implementations of the $(p,q)$-analog of Bernstein operators introduced by Mursaleen et al. [22] and other $(p,q)$-analogs, the reader may refer to [1–5, 11–15, 19–21] and [32]. For most recent work on the $(p,q)$-approximation we refer to [8, 24, 26].

The $(p,q)$-integer, $(p,q)$-binomial expansion and the $(p,q)$-binomial coefficients are defined by

$$[m]_{p,q} := \frac{p^m - q^m}{p-q}, \quad m = 0,1,2,\dots, p > q \geq 1,$$

$$(a+b)^m_{p,q} := (a+b)(pa+qb)\left(p^2a+q^2b\right)\cdots\left(p^{m-1}a+q^{m-1}b\right)$$

$$= \sum_{r=0}^{k} p^{\frac{(m-r)(m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} m \\ r \end{bmatrix}_{p,q} a^r,$$

$$\begin{bmatrix} m \\ r \end{bmatrix}_{p,q} := \frac{[m]_{p,q}!}{[r]_{p,q}![m-r]_{p,q}!}.$$

It can easily be verified by induction that

$$(1+a)(p+qa)\left(p^2+q^2a\right)\cdots\left(p^{n-1}+q^{n-1}a\right)=\sum_{r=0}^{k}p^{\frac{(m-r)(m-r-1)}{2}}q^{\frac{r(r-1)}{2}}\begin{bmatrix} m \\ r \end{bmatrix}_{p,q}a^r.$$

The $(p,q)$-analog of Euler's identity is defined by

$$\prod_{s=0}^{m-1}\left(p^s-q^sa\right):=\sum_{k=0}^{m}p^{\frac{(m-k)(m-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix} m \\ k \end{bmatrix}_{p,q}a^k.$$

Let $f:[0,1]\longrightarrow\mathbb{R}$ and $q>p>1$. The $(p,q)$-Bernstein operators [22] of $f$ is defined as

$$B_{p,q}^{n}(f;x):=\sum_{k=0}^{n}f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)p_{n,k}(p,q;x),\quad n\in\mathbb{N},\tag{1.1}$$

where the polynomial $p_{n,k}(p,q;x)$ is given by

$$p_{n,k}(p,q;x)=\frac{1}{p^{\frac{n(n-1)}{2}}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}\left(p^s-q^sx\right),\quad x\in[0,1],0<q<p<1.\tag{1.2}$$

For $p=1$, $B_{p,q}^{n}(f;x)$ turns into the $q$-Bernstein operator. We have

$$B_{p,q}^{n}(f;0)=f(0),\qquad B_{p,q}^{n}(f;1)=f(1),\quad n\in\mathbb{N}.\tag{1.3}$$

The following $(p,q)$-difference form of Bernstein operators [25] is given by

$$B_{p,q}^{n}(f;x):=\sum_{r=0}^{n}\lambda_{p,q}^{n}f\left[0,\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}},\ldots,\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\right]x^r,\tag{1.4}$$

where $f[x_0,x_1,\ldots,x_n]$ indicates the $n$th order divided difference of $f$ with pairwise distinct node, that is,

$$f[x_0]=f(x_0),\qquad f[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0},$$
$$f[x_0,x_1,\ldots,x_n]=\frac{f[x_1,\ldots,x_n]-f[x_0,\ldots,x_{n-1}]}{[x_n-x_0]}$$

and $\lambda_{p,q}^{n}$ is given by

$$\begin{aligned}\lambda_{p,q}^{n}&=\begin{bmatrix} n \\ r \end{bmatrix}_{p,q}\frac{[r]_{p,q}!}{[n]_{p,q}^{r}}p^{\frac{(n-r)(n-r-1)}{2}}q^{\frac{r(r-1)}{2}}\\&=\left(1-\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\left(1-\frac{p^{n-2}[2]_{p,q}}{[n]_{p,q}}\right)\cdots\left(1-\frac{p^{n-r+1}[r-1]_{p,q}}{[n]_{p,q}}\right),\end{aligned}\tag{1.5}$$

and $\lambda_{p,q}^{0}=\lambda_{p,q}^{1}=1$, $0\leq\lambda_{p,q}^{n}\leq1$, $r=0,1,\ldots,n$.

In this paper, some qualitative approximation results for the $(p,q)$-Bernstein operators $B_{p,q}^{n}(f;x)$ have been obtained for the Cauchy kernel $\frac{1}{x-\alpha}$ with a pole $\alpha\in[0,1]$ for $q>p>1$.

The main focus lies in the study of behavior of operators $B^n_{p,q}(f;x)$ for the function $f_m(x) = \frac{1}{x-p^m q^{-m}}$, $x \neq p^m q^{-m}$ and $f_m(p^m q^{-m}) = a$, $a \in \mathbb{R}$ and the extra parameter $p$ provides flexibility for the approximation.

The time scale $\mathbb{J}_{p,q}$ for $q > p > 1$ is denoted and defined as

$$\mathbb{J}_{p,q} = \{0\} \cup \{p^k q^{-k}\}_{k=0}^{\infty}. \tag{1.6}$$

Here, we consider the $(p,q)$-Bernstein operators with the Cauchy kernel $\frac{1}{x-\alpha}$, $\alpha \in [0,1]$. The previously obtained results [27–30] lead to the following conclusions.

- If $\alpha = 0$, that is, $f(x) = \frac{1}{x}$, $x \neq 0$ and $f(0) = a$, then, for $q \geq 2$,

$$\lim_{n\to\infty} B^n_{p,q}(f;x) = \begin{cases} f(x), & x \in \mathbb{J}_{p,q}, \\ \infty, & x \notin \mathbb{J}_{p,q}. \end{cases} \tag{1.7}$$

- If $\alpha \in \mathbb{J}_{p,q} \setminus [0,1]$ that is $f(x) = \frac{1}{(x-\alpha)}$ if $x \neq \alpha$ and $f(\alpha) = a$, then

$$\lim_{n\to\infty} B^n_{p,q}(f;x) = f(x), \quad x \in \mathbb{J}_{p,q}.$$

Furthermore, as $n \to \infty$, $B^n_{p,q}(f;x) \to f(x)$ uniformly on any compact subset of $(-\alpha,\alpha)$ and $B^n_{p,q}(f;x) \to \infty$ for $|x| > \alpha$, $x \notin \mathbb{J}_{p,q}$. Therefore, it is left to examine the case $\alpha \in \mathbb{J}_{p,q} \setminus \{0\}$ which is exactly the subject of the present paper. Let the function $f_m : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_m(x) = \begin{cases} \frac{1}{(x-p^m q^{-m})^j}, & x \in \mathbb{R} \setminus \{p^m q^{-m}\}, \\ a, & x = p^m q^{-m}, \end{cases} \quad m \in \mathbb{N}_0, a \in \mathbb{R}. \tag{1.8}$$

## 2 Some auxiliary results

In this section, we prove some important lemmas.

**Lemma 2.1** *For the function $f_m$ defined by* (1.8), *we have*

(a) *for $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} B^n_{p,q}(f_m; p^j q^{-j}) = f_m(p^j q^{-j}), \quad j \in \mathbb{N}_0 \setminus \{m, m+1\}.$$

*Besides,*

$$\lim_{n\to\infty} B^n_{p,q}(f_m; p^m q^{-m}) = -\infty, \quad \text{and}$$

$$\lim_{n\to\infty} B^n_{p,q}(f_m; p^{-(m+1)} q^{-(m+1)}) = f_m(p^{m+1} q^{-(m+1)}) - \frac{p^{-m} q^m [m+1]_{p,q}}{p^{-1}(q-1)[m]_{p,q}}.$$

(b) *For $m = 0$,*

$$\lim_{n\to\infty} B^n_{p,q}(f_0; p^j q^{-j}) = f_0(p^j q^{-j}), \quad j \in \mathbb{N}_0$$

*i.e., $B^n_{p,q}(f_0;\cdot)$ approximates $f_0$ on $\mathbb{J}_{p,q}$.*

This describes the behavior of $B^n_{p,q}(f_m;\cdot)$ on the time scale $\mathbb{J}_{p,q}$.

*Proof* (a) From (1.2), we can easily see that $p_{n,n-k}(p,q;p^jq^{-j}) = 0$ for $k > j$, whence

$$B^n_{p,q}\bigl(f;p^{-j}q^j\bigr) = \sum_{k=0}^{\min\{k,j\}} f\biggl(\frac{[n-k]_{p,q}}{[n]_{p,q}}\biggr)p_{n,n-k}\bigl(p,q;p^jq^{-j}\bigr). \tag{2.1}$$

Besides

$$\lim_{n\to\infty} p_{n,n-k}\bigl(p,q;p^jq^{-j}\bigr) = \delta_{k,j} \quad \text{and} \quad \lim_{n\to\infty}\frac{[n-k]_{p,q}}{[n]_{p,q}} = p^kq^{-k}. \tag{2.2}$$

Thus, $\lim_{n\to\infty} f_m(\frac{[n-k]_{p,q}}{[n]_{p,q}})p_{n,n-k}(p,q;p^jq^{-j}) = f_m(p^kq^{-k})\delta_{j,k}$ for all $k \neq m$.

Now by easy calculation, we have

$$\lim_{n\to\infty} f\biggl(\frac{[n-k]_{p,q}}{[n]_{p,q}}\biggr)p_{n,n-k}\bigl(p,q;p^jq^{-j}\bigr) = \begin{cases} -\infty, & j = m, \\ -\frac{p^{-m}q^m[m+1]_{p,q}}{p^{-1}(q-1)[m]_{p,q}}, & j = m+1, \\ 0, & \geq m+2, \end{cases}$$

and combining with (2.1) and (2.2) yields the result.

(b) It can be obtained easily from (1.3) and (2.2) as $f_0$ is continuous at all points $p^jq^{-j}$, $j \in \mathbb{N}$. □

The next lemma is related to the coefficient of $B^n_{p,q}(f_0;\cdot)$.

**Lemma 2.2** *Let $f_m$ be a function as in (1.8). If $B^n_{p,q}(f_m;x) = \sum_{k=0}^n C^{p,q}_{k,n}x^k$ and $\frac{[k]_{p,q}}{[n]_{p,q}} \neq p^mq^{-m}$ for $k = 0,1,2,\ldots,n$, then*

$$C^{p,q}_{k,n} = -\frac{\lambda^{p,q}_{k,n}p^{-m(k+1)}q^{m(k+1)}}{(1-\frac{p^{n-m-1}q^m[1]_{p,q}}{[n]_{p,q}})(1-\frac{p^{n-m-2}q^m[2]_{p,q}}{[n]_{p,q}})\cdots(1-\frac{p^{n-m-k}q^m[k]_{p,q}}{[n]_{p,q}})}, \tag{2.3}$$

*where $\lambda^{p,q}_{k,n}$ are given by (1.5).*

*Proof* Consider $f_m(z) = \frac{1}{z-p^mq^{-m}}$, which is analytic function in $\mathbb{C}\setminus\{p^mq^{-m}\}$. It is well known that [17] the $k$th order divided difference of $f$ can be expressed as

$$f[x_0,x_1,\ldots,x_k] = \frac{1}{2\pi i}\oint_{\mathcal{L}}\frac{f(\zeta)\,d\zeta}{(\zeta-x_0)(\zeta-x_1)\cdots(\zeta-x_k)},$$

where $\mathcal{L}$ is contour encircling $x_0,\ldots,x_k$ and $f$ is assumed to be analytic on and within $\mathcal{L}$. Hence, when the nodes $0, \frac{[1]_{p,q}}{[n]_{p,q}}, \frac{[2]_{p,q}}{[n]_{p,q}},\ldots,\frac{[k]_{p,q}}{[n]_{p,q}}$ are inside $\mathcal{L}$ and the pole $\alpha = p^mq^{-m}$ is outside, one has

$$f\biggl[0,\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}},\ldots,\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\biggr] = \frac{1}{2\pi i}\oint_{\mathcal{L}}\frac{f_m(\zeta)\,d\zeta}{\zeta(\zeta-\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}})\cdots(\zeta-\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}})}. \tag{2.4}$$

By the residue theorem

$$
\begin{aligned}
f\left[0,\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}},\ldots,\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\right] &= \sum_{j=0}^{k}\operatorname{Res}_{z=p^{n-j}\frac{[j]_{p,q}}{[n]_{p,q}}}\frac{f_m(z)}{\prod_{j=0}^{k}(z-p^{n-j}\frac{[j]_{p,q}}{[n]_{p,q}})}\\
&= -\operatorname{Res}_{z=p^m q^{-m}}\frac{f_m(z)}{\prod_{j=0}^{k}(z-p^{n-j}\frac{[j]_{p,q}}{[n]_{p,q}})}\\
&= -\frac{p^{-m(k+1)}q^{m(k+1)}}{\prod_{j=1}^{k}(1-p^{n-m-j}\frac{[j]_{p,q}}{[n]_{p,q}})}.
\end{aligned}
$$

Since $f_m(z)=f_m(x)$ for $z=x\in[0,1]$, the statement follows from the divided difference representation (1.4). □

Now, we find the asymptotic estimates for the coefficient $C_{k,n}^{p,q}$ in the next lemma.

**Lemma 2.3** *We have*

$$
\lim_{n\to\infty}\prod_{k=1}^{n-j}\left(1-p^{n-m-j}q^m\frac{[k]_{p,q}}{[n]_{p,q}}\right)=\left(\frac{p^{2j-m}}{q^{j-m}};\frac{p}{q}\right)_\infty
$$

*for* $j>m$, $q>p>1$.

*Proof* It is clear that

$$
\log\prod_{k=1}^{n-j}\left(1-p^{n-m-j}q^m\frac{[k]_{p,q}}{[n]_{p,q}}\right)=\sum_{k=j}^{n-1}\log\left(1-p^{n-m-j}q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)=\sum_{k=j}^{\infty}a_{k,n}^{p,q},
$$

where

$$
a_{k,n}^{p,q}=\begin{cases}\log(1-p^{n-m-j}q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}), & k<n,\\ 0, & k\geq n.\end{cases}
$$

Since

$$
\begin{aligned}
\left|a_{k,n}^{p,q}\right| &\leq \left|\log\left(1-p^{n-m-j}q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)\right|\\
&\leq \frac{q}{q-p}\frac{p^{n-m-k}q^m[n-k]_{p,q}}{[n]_{p,q}}\leq\frac{q}{q-p}p^{n-k-m}q^m,
\end{aligned}
$$

which gives $\sum_{k=j}^{\infty}|a_{k,n}^{p,q}|<\infty$, and by the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{k=j}^{n-1}\log\left(1-p^{n-m-j}q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right) &= \sum_{k=j}^{\infty}\left(\lim_{n\to\infty}\log\left(1-p^{n-m-j}q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)\right)\\
&= \sum_{k=j}^{\infty}\lim_{n\to\infty}\left(1-p^{n-k-m}\frac{q^m}{q^k}\right),
\end{aligned}
$$

as a result

$$\lim_{n\to\infty} \log \prod_{k=1}^{n-j} \left(1 - p^{n-m-j} q^m \frac{[n-k]_{p,q}}{[n]_{p,q}}\right) = \log \prod_{k=j}^{\infty} \left(1 - p^{n-k-m} \frac{q^m}{p^k}\right),$$

which completes the proof. □

The following lemma gives an upper bound for $n - m - 1$.

**Lemma 2.4** *If* $m \in \mathbb{N}$, $k = 0, 1, 2, \ldots, n - m - 1$, *then*

$$\left|C_{k,n}^{p,q}\right| \leq \mathcal{C}_{m,p,q} p^{-mn} q^{mn},$$

*where* $\mathcal{C}$ *in RHS is a positive constant, whose value need not to be addressed.*

*Proof* For $n > m + 1$ and from (2.3), we have

$$\begin{aligned}
\left|C_{k,n}^{p,q}\right| &\leq \frac{p^{-m(k+1)} q^{m(k+1)}}{(1 - \frac{p^{n-m-1} q^m [1]_{p,q}}{[n]_{p,q}})(1 - \frac{p^{n-m-2} q^m [2]_{p,q}}{[n]_{p,q}}) \cdots (1 - \frac{p^{n-m-k} q^m [k]_{p,q}}{[n]_{p,q}})} \\
&\leq \frac{p^{-m(n-m)} q^{m(n-m)}}{(1 - \frac{p^{2(n-m-1)} q^m}{q^{n-1}})(1 - \frac{p^{2(n-m-2)} q^m [2]_{p,q}}{[n]_{p,q}}) \cdots (1 - \frac{p^{n-m-k} q^m [k]_{p,q}}{[n]_{p,q}})}, \\
\left|C_{k,n}^{p,q}\right| &\leq \frac{p^{-mn} q^{mn}}{p^{2m} q^{2m} (\frac{p}{q}; \frac{p}{q})_\infty}.
\end{aligned}$$

Further, we discuss the nature of $C_{n-m+1,n}, \ldots, C_{n,n}$ as follows. □

**Lemma 2.5** *For* $m \in \mathbb{N}$, $q > p > 1$,

$$\left|C_{n-m,n}^{p,q}\right| \sim \mathcal{C}_{p,q,m} p^{-(m+1)n} q^{(m+1)n}, \quad n \to \infty.$$

*Proof* Using (2.3), we obtain the following

$$\left|C_{n-m}^{p,q}\right| = \lambda_{n-m,n}^{p,q} \frac{p^{-m(n-m+1)} q^{m(n-m+1)}}{(1 - \frac{p^{n-m-1} q^m [1]_{p,q}}{[n]_{p,q}})(1 - \frac{p^{n-m-2} q^m [2]_{p,q}}{[n]_{p,q}}) \cdots (1 - \frac{p^{n-m-(n-m)} q^m [n-m]_{p,q}}{[n]_{p,q}})}.$$

From Lemma 2.3, we have

$$\begin{aligned}
\left|C_{n-m}^{p,q}\right| &\sim \frac{(\frac{p^{m+1}}{q^{m+1}}; \frac{p}{q})_\infty q^{m(n-m+1)} p^{-m(n-m+1)} p^m (p^n - q^n)}{(\frac{p}{q}; \frac{p}{q})_\infty p^n (p^m - q^m)} \\
&\sim \frac{(\frac{p^{m+1}}{q^{m+1}}; \frac{p}{q})_\infty q^{mn} p^{-mn} p^m (p^n - q^n)}{(\frac{p}{q}; \frac{p}{q})_\infty q^{m(m-1)} p^{m(m-1)} (p^m - q^m)}, \qquad (2.5) \\
C_{n-m,n}^{p,q} &= \mathcal{C}_m^{p,q} p^{-n(m+1)} q^{n(m+1)}.
\end{aligned}$$

The nature of the remaining coefficients $C_{n-m+1,n}, \ldots, C_{n,n}$ is given as follows. □

**Lemma 2.6** *For $j = 1, 2, \dots, m$, we have*

$$\lim_{n\to\infty} \frac{C^{p,q}_{n-m+j,n}}{C^{p,q}_{n-m,n}} = (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}}.$$

*Proof* Using (2.3) and (1.5), we get

$$C^{p,q}_{n-m+j,n} = C^{p,q}_{n-m,n} \frac{(1 - \frac{p^m[n-m]_{p,q}}{[n]_{p,q}}) \cdots (1 - p^{m-j+1}\frac{[n-m+j-1]_{p,q}}{[n]_{p,q}})}{(1 - p^{-1}q^m \frac{[n-m+1]}{[n]_{p,q}}) \cdots (1 - p^{-j}q^m \frac{[n-m+j]_{p,q}}{[n]_{p,q}})},$$

$$\lim_{n\to\infty} \frac{C^{p,q}_{n-m+j,n}}{C^{p,q}_{n-m,n}} = \frac{(1 - \frac{p^m}{q^m}) \cdots (1 - \frac{p^{m-j+1}}{q^{m-j+1}})}{(1 - \frac{q}{p}) \cdots (1 - \frac{q^j}{p^j})},$$

$$\lim_{n\to\infty} \frac{C^{p,q}_{n-m+j,n}}{C^{p,q}_{n-m,n}} = (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}}.$$ □

**Corollary 2.7** *The following estimate holds*:

$$\left|C^{p,q}_{k,n}\right| \le \mathcal{C}_{p,q,m} p^{-(m+1)n} q^{(m+1)n}, \quad k = 0, 1, 2, \dots, n, \tag{2.6}$$

*and $\mathcal{C}_{p,q,m}$ is independent of both $k$ and $n$.*

**Corollary 2.8** *We have the following*:

$$\lim_{n\to\infty} \frac{C_{n-m,n} + \cdots + C_{n-m+j,n}x^j + \cdots + C_{n,n}x^n}{C_{n-m,n}} = (x; p, q)_m. \tag{2.7}$$

*Proof* The statement follows from Rothe's identity [6],

$$(x; p, q)_m = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}}.$$ □

## 3 Main results

- First we consider the case when pole $\alpha \in \mathbb{J}_{p,q} \setminus \{0, 1\}$.

Now, we obtain the results that concern with the uniform approximation of $f_m(x)$, $m \in \mathbb{N}$ by its $(p, q)$-Bernstein operators. It may be noted that, while the case when $\alpha \in [0, 1] \setminus \mathbb{J}_{p,q}$ can easily be examined by using the result and method of [27], the condition $\alpha \in \mathbb{J}_{p,q}$ requires a different approach.

**Theorem 3.1** *If $m \in \mathbb{N}$, then $B^n_{p,q}(f_m; x) \to f_m(x)$ as $n \to \infty$ uniformly on any compact subset of $(-p^{(m+1)}q^{(m+1)}, p^{(m+1)}q^{(m+1)})$.*

*Proof* We consider the complex $(p, q)$-Bernstein operators given by

$$B^n_{p,q}(f; x) = \sum_{k=0}^{n} f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) p_{n,k}(p, q; x), \quad n \in \mathbb{N}, z \in \mathbb{C}, \tag{3.1}$$

and the function $f_m(z) = \frac{1}{(z-p^m q^{-m})}$, $z \in \mathbb{C}$. Let $n$ be large enough to satisfy the condition $\frac{[k]_{p,q}}{[n]_{p,q}} \neq p^m q^{-m}$. Then

$$B^n_{p,q}(f_m;z) = \sum_{k=0}^{n} C^{p,q}_{k,n} z^k,$$

where $C^{p,q}_{k,n}$ is given by (2.3). Let $\rho \in (0, p^{(m+1)} q^{-(m+1)})$. Therefore for $|z| \leq \rho$ the following estimate is valid by Corollary 2.7:

$$\left|B^n_{p,q}(f_m;z)\right| \leq \sum_{k=0}^{n} \left|C^{p,q}_{k,n}\rho^k\right| \leq \mathcal{C}_{p,q,m} \sum_{k=0}^{n} \left(p^{-(m+1)} q^{(m+1)} \rho\right)^k \leq \mathcal{C}_{p,q,m} \frac{1}{(1-p^{-(m+1)} q^{(m+1)} \rho)}.$$

Hence it follows that the operators $\{B^n_{p,q}(f_m,z)\}$ are uniformly bounded in the disk $\{z : |z| \leq \rho\}$ and convergent on the sequence $\{p^j q^{-j}\}_{j=m+2}^{\infty}$ having an accumulation point at 0 to the function $f_m(z)$ analytic in this disc. Using Vitali's convergence theorem, we have $B^n_{p,q}(f_m;z) \to f_m(z)$ $(n \to \infty)$ uniformly on any compact set in $\{z : |z| \leq \rho\}$ as $\rho \in (0, p^{(m+1)} q^{-(m+1)})$ was arbitrary. This completes the proof. □

Next we demonstrate that, outside of the interval, operators diverge everywhere except a finite number of points.

**Theorem 3.2** *If* $m \in \mathbb{N}$, *then* $\lim_{n\to\infty} B^n_{p,q}(f_m;x) = \infty$ *for* $|x| > p^{(m+1)} q^{-(m+1)}$, $x \neq p^{(m+1)} \times q^{-(m+1)}$, $x \neq p^{(m-1)} q^{-(m-1)}$, $x \neq p^{(m-2)} q^{-(m-2)}, \ldots, 1$.

*Proof* For exceptional points $p^{(m-1)} q^{-(m-1)}, p^{(m-2)} q^{-(m-2)}, \ldots, 1$, the situation has been analyzed in Lemma 2.1(a). We take $x$ satisfying $|x| > p^{(m-1)} q^{-(m-1)}$ different from these values. Let $n > m$ be sufficiently large such that (2.3) holds. By Lemma 2.4, we obtain

$$\begin{aligned}\left|\sum_{k=0}^{\infty} C^{p,q}_{k,n} x^k\right| &\leq \mathcal{C}_{m,p,q} \sum_{k=0}^{n-m-1} p^{-mk} q^{mk} x^k = \mathcal{C}_{m,p,q} \frac{(p^{-m} q^m x)^{n-m} - 1}{p^{-m} q^m x - 1} \\ &= o\left(\left(p^{-(m+1)} q^{(m+1)} x\right)^n\right), \quad n \to \infty.\end{aligned}$$

Hence

$$\begin{aligned}B^n_{p,q}(f_m;x) &= \sum_{k=n-m}^{n} C^{p,q}_{k,n} x^k + o\left(\left(p^{-(m+1)} q^{(m+1)} x\right)^n\right) \\ &= C^{p,q}_{n-m} x^{n-m} g_n(x) + o\left(\left(p^{-(m+1)} q^{(m+1)} x\right)^n\right), \quad n \to \infty.\end{aligned}$$

By Lemma 2.5, $|C^{p,q}_{n-m}| \sim \mathcal{C}_{p,q,m}(x)(p^{-(m+1)} q^{(m+1)} x)^n$ as $n \to \infty$ whenever $|x| > p^{(m+1)} q^{-(m+1)}$, since $\lim_{n\to\infty} g_n(x) = (x;p,q)_m \neq 0$, when $x \notin \{p^{(m+1)} q^{-(m+1)}, \ldots, 1\}$. □

**Lemma 3.1** *Let* $f_0$ *be given by putting* $m = 0$ *in* (1.8). *If* $B^n_{p,q}(f_0;x) = \sum_{k=0}^{n} C^{p,q}_{k,n} x^k$ *then*

$$C^{p,q}_{k,n} = \frac{-1}{(1-p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}})}, \quad k = 0, 1, 2, \ldots, n-1, \qquad C^{p,q}_{n,n} = a + \sum_{k=0}^{n-1} \frac{1}{(1-p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}})}.$$

*Proof* For $k = 0, 1, \dots, n-1$, on a specific choice of the contour $\mathcal{L}$, such that the nodes $0, \frac{[1]_{p,q}}{[n]_{p,q}}, \dots, \frac{[k]_{p,q}}{[n]_{p,q}}$ are inside $\mathcal{L}$ while the pole $\alpha = 1$ is outside, formula (2.4) implies

$$C_{k,n}^{p,q} = \frac{-\lambda_{k,n}^{p,q}}{\prod_{j=0}^{k}(1 - p^{n-j}\frac{[j]_{p,q}}{[n]})} = \frac{-1}{(1 - p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}})},$$

since by (1.3), $B_{p,q}^{n}(f_0; 1) = f_0(1) = a$ and the statement is proved. □

**Corollary 3.2** *For $k = 0, 1, 2, \dots, n-1$ with $q > p > 1$ we have the following result*:

$$\left|C_{k,n}^{p,q}\right| \leq \frac{q}{q-p}.$$

- Now, we consider the case when pole $\alpha = 1$.

Here the point of singularity $x = 1$ is one of the nodes $\frac{[k]_{p,q}}{[n]_{p,q}}$. Consider the function $f_0$

$$f_0(x) = \begin{cases} \frac{1}{x-1}, & x \in \mathbb{R} \setminus \{1\}, \\ a, & x = 1. \end{cases} \tag{3.2}$$

**Theorem 3.3** *If $f_0$ is given by* (3.2), *then the following holds*:

(1) *For all $x \in (-1, 1]$*,

$$\lim_{n\to\infty} B_{p,q}^{n}(f_0; x) = f_0(x)$$

*and the convergence is uniform on any compact subset of* $(-1, 1)$.

(2) *For all $x \in \mathbb{R} \setminus (-1, 1]$*,

$$\lim_{n\to\infty} B_{p,q}^{n}(f_0; x) = \infty.$$

*Proof* (1) Since $B_{p,q}^{n}(f_0; 1) = f_0(1)$, we need to prove only the uniform convergence of the compact subset of $(-1, 1)$. For any $\rho \in (0, 1)$ and $|z| \leq \rho$. From Corollary 3.2, we have

$$\left|\sum_{k=0}^{n-1} C_{k,n}^{p,q} z^k\right| \leq \frac{\mathcal{C}_{p,q}}{1-\rho}.$$

Apart from that,

$$\left|C_{n,n}^{p,q} z^k\right| \leq |a| + \sum_{k=0}^{n-1} \frac{1}{1 - p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}} \leq |a| + n\left|C_{k,n}^{p,q}\right| \leq |a| + \frac{nq}{q-p},$$

whence

$$\left|C_{n,n}^{p,q} z^n\right| \leq |a|\rho^n + \rho^n \frac{nq}{q-p} \leq \mathcal{C}_{p,q,\rho}.$$

Therefore, we conclude that the operators $B_{p,q}^{n}(f; z)$ are uniformly bounded in any disk $\{z : |z| \leq \rho\}$ where $\rho \in (0, 1)$. From Lemma 2.1(b) and Vitali's convergence theorem we arrive at our result.

(2) Given that $x$ satisfies $|x| > 1$, by Able's inequality, we have

$$\left|\sum_{k=0}^{n-1} C_{k,n}^{p,q} x^k\right| \leq \frac{|x|^n - 1}{|x| - 1}\left(\left|C_{0,n}^{p,q}\right| + 2\left|C_{n-1,n}^{p,q}\right|\right) \leq \frac{|x|^n}{|x|-1}\left(1 + \frac{2p}{p-q}\right) = \mathcal{C}_{p,q,x}|x|^n.$$

Meanwhile,

$$\left|C_{n,n}^{p,q} x^n\right| \geq \left(\sum_{k=0}^{n-1} \frac{1}{1 - p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}}\right)|x|^n - |a| \cdot |x|^n \geq \left(n - |a|\right)|x|^n.$$

Thus, $|B_{p,q}^n(f_0;x)| \geq n|x|^n - (\mathcal{C}_{p,q,x} + |a|)|x|^n \to \infty$ as $n \to \infty$.

At $x = -1$, we have

$$B_{p,q}^n(f_0;-1) = \sum_{k=0}^{n-1} C_{k,n}^{p,q}(-1)^k + \left(a + \sum_{k=0}^{n-1} \frac{1}{1 - p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}}\right)(-1)^n,$$

and again applying Able's inequality,

$$\left|\sum_{k=0}^{n-1} C_{k,n,}^{p,q}(-1)^k\right| \leq |C_{0,n}| + 2|C_{n-1,n}| \leq 1 + \frac{2p}{p-q}.$$

On the other hand

$$\left|\left(a + \sum_{k=0}^{n-1} \frac{1}{1 - p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}}\right)(-1)^n\right| \geq n - |a|,$$

which implies that

$$\left|B_{p,q}^n(f_0;-1)\right| \geq n - |a| - \left(1 + \frac{2p}{p-q}\right) \to \infty, \quad n \to \infty. \qquad \square$$

*Remark* For justification of the statement that the extra parameter $p$ provides flexibility for approximation, one can see Remark 1 of [23].

Moreover, since for $q > p = 1$ we recapture the $q$-Bernstein operators studied in [30], it is clear that the interval of uniform convergence for $B_{p,q}^n$ in Theorem 3.1, i.e. $(-p^{m+1}q^{m+1}, p^{m+1}q^{m+1})$, is larger than the interval of uniform convergence $(-q^{m+1}, q^{m+1})$, obtained by Theorem 2.1 in [30].

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The authors declare that they have no competing interests.

**Authors' contributions**
All the authors contributed equally and significantly in writing this paper. All of them read and approved the final manuscript.

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