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The method of lower and upper solutions for the cantilever beam equations with fully nonlinear terms

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Abstract

In this paper we discuss the existence of solutions of the fully fourth-order boundary value problem

$$\begin{cases} u^{(4)} = f(t,u,u',u'',u'''), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

which models the deformations of an elastic cantilever beam in equilibrium state, where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous. Using the method of lower and upper solutions and the monotone iterative technique, we obtain some existence results under monotonicity assumptions on nonlinearity.

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1 Introduction

In this paper, we are concerned with the existence of the fully fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t)), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous. This equation models the deformations of an elastic beam in equilibrium state, whose one end-point is fixed and the other is free, and in mechanics it is called cantilever beam equation. In the equation, the physical meaning of the derivatives of the deformation function $u(t)$ is as follows: $u^{(4)}$ is the load density stiffness, u''' is the shear force stiffness, u'' is the bending moment stiffness, and u' is the slope [1–4].

For the special case of BVP (1.1) that f does not contain any derivative terms, namely the simply fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.2}$$

and f only contains first-order derivative term u', namely the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.3}$$

the existence of positive solutions has been discussed by some authors, see [5–9]. The methods applied in these works are not applicable to BVP (1.1) since they cannot deal with the derivative terms u'' and u'''.

For the cantilever beam equation with a nonlinear boundary condition of third-order derivative

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ u(0) = u'(0) = u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases} \tag{1.4}$$

the existence of solution has also been discussed by some authors, see [10–13]. The boundary condition in (1.4) means that the left end of the beam is fixed and the right end of the beam is attached to an elastic bearing device, see [10].

The purpose of this paper is to obtain existence results of solutions to the fully fourth-order nonlinear boundary value problem (1.1). For fully fourth-order nonlinear BVPs with the boundary condition in BVP (1.1) or other boundary conditions, the existence of solution has discussed by several authors, see [14–20]. In [14], Kaufmann and Kosmatov considered a symmetric fully fourth-order nonlinear boundary value problem. They used a triple fixed point theorem of cone mapping to obtain existence results of triple positive symmetric solutions when f satisfies some range conditions dependent upon three positive parameters a, b and d. Since they did not give the method to determine these parameters, the range conditions are difficult to verify. The authors of [15] used the method of lower and upper solutions to discuss the existence of solution of the fully fourth-order nonlinear boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & 0 \le t \le 1, \\ u(0) = u'(1) = u''(0) = u'''(1) = 0, \end{cases} \tag{1.5}$$

where the discussed problem has a pair of ordered lower and upper solutions. But they did not discuss how they found a pair of ordered lower and upper solutions. Under the case that $f(t, x_0, x_1, x_2, x_3)$ is sublinear growth on x_0, x_1, x_2, x_3, the existence of the following fully fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.6}$$

is discussed in [16]. In this case, using the method in [16], we can obtain existence results for BVP (1.1). Usually the superlinear problems are more difficult to treat than the sublinear problems. In [17], the present author discussed the case that $f(t,x_0,x_1,x_2,x_3)$ may be superlinear growth on x_0, x_1, x_2, x_3 when nonlinearity f is nonnegative by using the fixed point index theory in cones. In recent paper [18], Dang and Ngo dealt with the solvability of BVP (1.1) by using the contraction mapping principle. They showed that if there exists a region

$$\mathcal{D}_M = \left\{(t,x_0,x_1,x_2,x_3) \,\middle|\, t \in [0,1], |x_0| \le \frac{M}{8}, |x_1| \le \frac{M}{6}, |x_2| \le \frac{M}{2}, |x_3| \le M\right\} \tag{1.7}$$

determined by a positive number M such that nonlinearity f satisfies

$$\bigl|f(t,x_0,x_1,x_2,x_3)\bigr| \le M, \tag{1.8}$$

$$\bigl|f(t,x_0,x_1,x_2,x_3) - f(t,y_0,y_1,y_2,y_3)\bigr| \le \sum_{i=0}^{3} c_i|x_i - y_i| \tag{1.9}$$

for any (t,x_0,x_1,x_2,x_3), $(t,y_0,y_1,y_2,y_3) \in \mathcal{D}_M$, where c_0, c_1, c_2, c_3 are positive constants and satisfy

$$q := \frac{c_0}{8} + \frac{c_1}{6} + \frac{c_0}{2} + c_3 < 1, \tag{1.10}$$

then BVP (1.1) has a unique solution u satisfying

$$\bigl(t, u(t), u'(t), u''(t), u'''(t)\bigr) \in \mathcal{D}_M, \quad t \in [0,1]. \tag{1.11}$$

See [18, Theorem 2.2]. A similar result is built for BVP (1.6) in [19] and for a fourth-order BVP of Kirchhoff type equation in [20]. Dang and Ngo's result can be applied to the superlinear equations, and it ensures the uniqueness of solution on $\mathcal{D}_M$. However, the key to the application of this result is how to determine the constant M. For the general nonlinearity f, M is not easy to determine and the Lipschitz coefficients condition (1.10) is not easy to satisfy. In this paper we shall discuss the general case that f may be superlinear growth and have negative value.

We will use the method of lower and upper solutions to discuss BVP (1.1). For BVP (1.1), since the boundary conditions are different from BVP (1.5), the definitions of lower and upper solutions are different from those in [16] and the argument methods in [16] are not applicable to BVP (1.1). In Sect. 2, under $f(t,x_0,x_1,x_2,x_3)$ increasing on x_0, x_1, x_2 and decreasing on x_3 in the domain surrounded by lower and upper solutions, we use a monotone iterative technique to obtain the existence of a solution between lower and upper solutions. In Sect. 3, under $f(t,x_0,x_1,x_2,x_3)$ without monotonicity on x_3, we use a truncating technique to prove the existence of a solution between lower and upper solutions. In Sect. 4, we use the lower and upper theorem built in Sect. 3 to obtain a new existence result of positive solution.

2 Monotone iterative method

The monotone iterative method is an important method for solving nonlinear BVPs. For the special BVP (1.3), a monotone iterative method has been built, see [8]. In this section, we will develop the monotone iterative method of lower and upper solutions for BVP (1.1).

Let $I = [0,1]$ and $C(I)$ denote the Banach space of all continuous functions $u(t)$ on I with norm $\|u\|_C = \max_{t\in I}|u(t)|$. Generally, for $n \in \mathbb{N}$, we use $C^n(I)$ to denote the Banach space of all nth-order continuous differentiable functions on I with the norm $\|u\|_{C^n} = \max\{\|u\|_C, \|u'\|_C, \dots, \|u^{(n)}\|_C\}$. Let $C^+(I)$ denote the cone of all nonnegative functions in $C(I)$.

Let $f : I \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and consider BVP (1.1). If a function $v \in C^4(I)$ satisfies

$$\begin{cases} v^{(4)}(t) \le f(t, v(t), v'(t), v''(t), v'''(t)), & t \in I, \\ v(0) \le 0, \quad v'(0) \le 0, \quad v''(1) \le 0, \quad v'''(1) \ge 0, \end{cases} \tag{2.1}$$

we call it a lower solution of BVP (1.1), and if a function $w \in C^4(I)$ satisfies

$$\begin{cases} w^{(4)}(t) \ge f(t, w(t), w'(t), w''(t), w'''(t)), & t \in I, \\ w(0) \ge 0, \quad w'(0) \ge 0, \quad w''(1) \ge 0, \quad w'''(1) \le 0, \end{cases} \tag{2.2}$$

we call it an upper solution of BVP (1.1).

Lemma 2.1 *Let $v_0 \in C^4(I)$ be a lower solution of BVP (1.1) and w_0 be an upper solution, and ${v_0}''' \ge {w_0}'''$. Then*

$$v_0 \le w_0, \qquad {v_0}' \le {w_0}', \qquad {v_0}'' \le {w_0}''. \tag{2.3}$$

Proof Let $u = w_0 - v_0$, then $u'''(t) \le 0$ for every $t \in I$. By the definitions of lower and upper solutions, we have

$$u''(t) = u''(1) - \int_t^1 u'''(s)\,ds \ge 0, \quad t \in I,$$

$$u'(t) = u'(0) + \int_0^t u''(s)\,ds \ge 0, \quad t \in I,$$

$$u(t) = u(0) + \int_0^t u'(s)\,ds \ge 0, \quad t \in I.$$

Hence, (2.3) holds. □

Given $h \in C(I)$, consider the linear boundary value problem (LBVP)

$$\begin{cases} u^{(4)}(t) = h(t), & t \in I, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{2.4}$$

Lemma 2.2 *For every $h \in C(I)$, LBVP (2.4) has a unique solution $u := Sh \in C^4(I)$. Moreover, the solution operator $S : C(I) \to C^3(I)$ is a completely continuous linear operator.*

Proof For given any $h \in C(I)$, it is easy to verify that

$$u(t) = \int_0^t (t-\tau) \int_\tau^1 (s-\tau)h(s)\,ds\,d\tau := Sh(t), \quad t \in I, \tag{2.5}$$

is a unique solution of LBVP (2.4). From expression (2.5), we easily see that $S : C(I) \to C^3(I)$ is a completely continuous linear operator. □

Lemma 2.3 *If $u \in C^4(I)$ and satisfies*

$$\begin{cases} u^{(4)}(t) \geq 0, & t \in I, \\ u(0) \geq 0, \quad u'(0) \geq 0, \quad u''(1) \geq 0, \quad u'''(1) \leq 0, \end{cases} \tag{2.6}$$

then $u \geq 0$, $u' \geq 0$, $u'' \geq 0$, and $u''' \leq 0$.

Proof Similar to the proof of Lemma 2.1, we have

$$u'''(t) = u'''(1) - \int_t^1 u^{(4)}(s)\,ds \leq 0, \quad t \in I,$$

$$u''(t) = u''(1) - \int_t^1 u''(s)\,ds \geq 0, \quad t \in I,$$

$$u'(t) = u'(0) + \int_0^t u''(s)\,ds \geq 0, \quad t \in I,$$

$$u(t) = u(0) + \int_0^t u'(s)\,ds \geq 0, \quad t \in I.$$

Hence, the conclusion of Lemma 2.3 holds. □

We introduce a semi-ordering $\preceq$ in $C^3(I)$ by

$$v \preceq w \quad \Longleftrightarrow \quad v \leq w, \quad v' \leq w', \quad v'' \leq w'', \quad \text{and} \quad v''' \geq w'''. \tag{2.7}$$

Then $C^3(I)$ is an ordered Banach space by this semi-ordering. We also use $w \succeq v$ to denote $v \preceq w$. Letting $v, w \in C^3(I)$ and $v \preceq w$, we denote the order-interval in $C^3(I)$ by

$$[v, w]_{C^3} = \{u \in C^3(I) : v \preceq u \preceq w\}. \tag{2.8}$$

Theorem 2.1 *Assume that $f : I \times \mathbb{R}^4 \to \mathbb{R}$ is continuous, BVP (1.1) has a lower solution v_0 and an upper solution w_0 with $v_0''' \geq w_0'''$, and f satisfies the following monotone conditions:*

- (F1) *for every $t \in I$ and $x_3 \in [w_0'''(t), v_0'''(t)]$, $f(t, x_0, x_1, x_2, x_3)$ is increasing on x_0, x_1, and x_2 in $[v_0(t), w_0(t)] \times [v_0'(t), w_0'(t)] \times [v_0''(t), w_0''(t)]$;*
- (F2) *for every $t \in I$ and $(x_0, x_1, x_2) \in [v_0(t), w_0(t)] \times [v_0'(t), w_0'(t)] \times [v_0''(t), w_0''(t)]$, $f(t, x_0, x_1, x_2, x_3)$ is decreasing on x_3 in $[w_0'''(t), v_0'''(t)]$.*

Make iterative sequences $\{v_n\}$ and $\{w_n\}$ starting from v_0 and w_0 respectively by using the iterative equation

$$\begin{cases} u_n^{(4)}(t) = f(t, u_{n-1}(t), u_{n-1}'(t), u_{n-1}''(t), u_{n-1}'''(t)), & t \in I, \\ u_n(0) = u_n'(0) = u_n''(1) = u_n'''(1) = 0, \end{cases} \quad n = 1, 2, \ldots. \tag{2.9}$$

Then $\{v_n\}$ and $\{w_n\}$ satisfy the monotone condition

$$v_0 \preceq v_n \preceq v_{n+1} \preceq w_{n+1} \preceq w_n \preceq w_0, \quad n = 1, 2, \ldots, \tag{2.10}$$

and converge in $C^3(I)$. *Moreover,* $\underline{u} = \lim_{n\to\infty} v_n$ *and* $\overline{u} = \lim_{n\to\infty} w_n$ *are minimal and maximal solutions of BVP* (1.1) *in* $[v_0, w_0]_{C^3}$.

Proof By Lemma 2.1, $v_0 \preceq w_0$. Define a mapping $F : C^3(I) \to C(I)$ by

$$F(u)(t) := f\bigl(t, u(t), u'(t), u''(t), u'''(t)\bigr), \quad t \in I, u \in C^3(I). \tag{2.11}$$

Then $F : C^3(I) \to C(I)$ is continuous and by Assumptions (F1) and (F2) we can verify that

$$v_0 \preceq u_1 \preceq u_2 \preceq w_0 \quad \Longrightarrow \quad F(u_1) \le F(u_2). \tag{2.12}$$

By Lemma 2.2, $A = S \circ F : C^3(I) \to C^3(I)$ is completely continuous and the solution of BVP(1) is equivalent to the fixed point of A. By the definition of S, the iterative sequences $\{v_n\}$ and $\{w_n\}$ satisfy

$$v_n = Av_{n-1}, \qquad w_n = Aw_{n-1}, \quad n = 1, 2, \ldots. \tag{2.13}$$

We show that

$$v_0 \preceq v_1, \qquad w_1 \preceq w_0. \tag{2.14}$$

Let $u = v_1 - v_0$. Then by the definition of the lower solution v_0, u satisfies (2.6). By Lemma 2.3, $u \succeq 0$, and hence $v_0 \preceq v_1$. Similarly, $w_1 \preceq w_0$ can be showed. By Lemma 2.3 and (2.12), we can prove that

$$v_0 \preceq u_1 \preceq u_2 \preceq w_0 \quad \Longrightarrow \quad Au_1 \preceq Au_2. \tag{2.15}$$

By (2.14) and (2.15), we see that (2.10) holds. Note that $\{v_n\} = \{S(F(v_{n-1}))\}$ and $\{w_n\} = \{S(F(w_{n-1}))\}$ are relatively compact in $C^3(I)$ by the complete continuity of S. Combining this fact with (2.10), we conclude that

$$v_n \to \underline{u}, \qquad w_n \to \overline{u} \quad \text{in } C^3(I). \tag{2.16}$$

By (2.13) and (2.15) we can prove that $\underline{u}$ and $\overline{u}$ are minimal and maximal fixed points of A in $[v_0, w_0]_{C^3}$. Hence, they are minimal and maximal solutions of BVP (1.1) in $[v_0, w_0]_{C^3}$. □

Example 2.1 Consider the following fourth-order boundary value problem with superlinear terms:

$$\begin{cases} u^{(4)}(t) = \frac{1}{3}\sqrt[3]{u} + (u')^2 + (u'')^3 - \frac{1}{2}(u''')^3 + t^2(1-t)^2, & t \in I, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{2.17}$$

Corresponding to BVP (1.1), the nonlinearity is

$$f(t, x_0, x_1, x_2, x_3) = \frac{1}{3}\sqrt[3]{x_0} + x_1^2 + x_2^3 - \frac{1}{2}x_3^3 + t^2(1-t)^2, \tag{2.18}$$

which is cubic growth on x_2 and x_3. We use Theorem 2.1 to show that BVP (2.17) has a positive solution. Clearly, $v_0(t) \equiv 0$ is a lower solution of BVP (2.17). We verify that

$$w_0(t) = \frac{1}{24}t^4 + \frac{1}{12}t^3 + \frac{1}{4}t^2(1-t), \quad t \in I$$

is an upper solution of BVP (2.17). Since

$$\begin{aligned} w_0'(t) &= \frac{1}{6}t^3 + \frac{1}{2}t(1-t) \geq 0, \quad t \in I, \\ w_0''(t) &= \frac{1}{2}(1-t)^2 \geq 0, \quad t \in I, \\ w_0'''(t) &= t-1 \leq 0, \quad t \in I, \end{aligned}$$

we obtain that

$$\|w_0\|_C = \frac{1}{8}, \qquad \|w_0'\|_C = \frac{1}{6}, \qquad \|w_0''\|_C = \frac{1}{2}, \qquad \|w_0'''\|_C = 1. \tag{2.19}$$

By (2.18) and (2.19), we have

$$\begin{aligned} f(t, w_0, w_0', w_0'', w_0''') &= \frac{1}{3}\sqrt[3]{w_0(t)} + \big(w_0'(t)\big)^2 + \big(w_0''(t)\big)^3 - \frac{1}{2}\big(w_0'''(t)\big)^3 + t^2(1-t)^2 \\ &\leq \frac{1}{3}\sqrt[3]{\|w_0\|_C} + \|w_0'\|_C^2 + \|w_0''\|_C^3 + \frac{1}{2}\|w_0'''\|_C^3 + t^2(1-t)^2 \\ &\leq \frac{1}{6} + \frac{1}{36} + \frac{1}{8} + \frac{1}{2} + \frac{1}{16} \\ &< 1 = w_0^{(4)}(t), \quad t \in I. \end{aligned}$$

Hence, w_0 is an upper solution of BVP (2.17). By (2.18), $f(t, x_0, x_1, x_2, x_3)$ is increasing on x_0, x_1, x_2 in $[0, +\infty)^3$ and decreasing on x_3 in all $\mathbb{R}$. Hence f satisfies Assumptions (F1) and (F2). By Theorem 2.1, BVP (2.17) has at least one solution $u_0 \in [v_0, w_0]_{C^3}$, which is a positive solution. Since f does not satisfy the Nagumo condition on x_2 and x_3 in [17], this result cannot be obtained from [17]. This result also cannot be obtained from [18]. In fact, for any $M > 0$, the first term $\frac{1}{3}\sqrt[3]{x_0}$ of expression (2.18) of f does not satisfy the Lipschitz condition on $\mathcal{D}_M$. Hence the Lipschitz condition (1.9) does not hold on $\mathcal{D}_M$. So [18, Theorem 2.2] is not applicable for BVP (2.17), and our existence result for BVP (2.17) cannot be obtained from [18].

3 A theorem of lower and upper solutions

In this section, we discuss the existence of a solution between a lower solution and an upper solution for BVP (1.1) under the case of nonlinearity $f(t, x_0, x_1, x_2, x_3)$ without monotonicity on x_3. In [15], an existence result between a lower solution and an upper solution was established for BVP (1.5), in which the authors requested nonlinearity $f(t, x_0, x_1, x_2, x_3)$ to satisfy a Nagumo-type condition on x_3, see [15, Theorem 3.1]. Since the boundary conditions and the definitions of lower and upper solutions of BVP (1.5) are different from those of BVP (1.1), the results presented in [15] are not applicable to BVP (1.1). We will use a directly truncating function technique to establish a similar existence result. A remarkable difference is that our existence result does not need the Nagumo-type condition. Our result is as follows:

Theorem 3.1 *Let $f : I \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and BVP (1.1) have a lower solution v_0 and an upper solution w_0 with $v_0''' \geq w_0'''$. If f satisfies the following condition*:

(F3) *for any $t \in I$ and $(x_0, x_1, x_2) \in [v_0(t), w_0(t)] \times [v_0'(t), w_0'(t)] \times [v_0''(t), w_0''(t)]$,*

$$f\big(t, x_0, x_1, x_2, v_0'''(t)\big) \geq f\big(t, v_0(t), v_0'(t), v_0''(t), v_0'''(t)\big),$$
$$f\big(t, x_0, x_1, x_2, w_0'''(t)\big) \leq f\big(t, w_0(t), w_0'(t), w_0''(t), w_0'''(t)\big);$$

then BVP (1.1) has at least one solution in $[v_0, w_0]_{C^3}$.

In Theorem 3.1, condition (F3) is weaker than condition (F1) of Theorem 2.1, and Theorem 3.1 does not need the monotonicity of $f(t, x_0, x_1, x_2, x_3)$ on x_3. For the existence, Theorem 3.1 is more applicable than Theorem 2.1, but it has no monotone iterative procedure of seeking solutions. The proof of Theorem 3.1 needs the following lemma.

Lemma 3.1 *Let $f : I \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and bounded. Then BVP (1.1) has at least one solution $u \in C^4(I)$.*

Proof Let $F : C^3(I) \to C(I)$ be the mapping defined by (2.10). Then, by Lemma 2.2, $A = S \circ F : C^3(I) \to C^3(I)$ is completely continuous and the solutions of BVP (1.1) are equivalent to the fixed points of A. We show that A has a fixed point in $C^3(I)$. By the boundedness of f, there exists a positive constant $M > 0$ such that

$$\big|f(t, x_0, x_1, x_2, x_3)\big| \leq M, \quad (t, x_0, x_1, x_2, x_3) \in I \times \mathbb{R}^4. \tag{3.1}$$

By (2.10) and (3.1), $F : C^3(I) \to C(I)$ satisfies

$$\big\|F(u)\big\|_C \leq M, \quad u \in C^3(I). \tag{3.2}$$

Choose $R \geq M\|S\|$ and set $\Omega = \{u \in C^3(I) : \|u\|_{C^3} \leq R\}$, where $\|S\|$ denotes the norm of linear bounded operator $S : C(I) \to C^3(I)$. Then Ω is a bounded and convex closed set in $C^3(I)$. For every $u \in \Omega$, by (3.2), we have

$$\|Au\|_{C^3} = \big\|S\big(F(u)\big)\big\|_{C^3} \leq \|S\| \cdot \big\|F(u)\big\|_C \leq M\|S\| \leq R.$$

Hence $Au \in \Omega$. This means that $A(\Omega) \subset \Omega$. By the Schauder fixed point theorem, A has a fixed point in Ω, which is a solution of BVP (1.1). □

Proof of Theorem 3.1 By Lemma 2.3, $v_0 \leq w_0$, $v_0' \leq w_0'$, $v_0'' \leq w_0''$. Define functions $\eta_0, \eta_1, \eta_2, \eta_3 : T \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \eta_0(t, y) &= \min\big\{\max\big\{v_0(t), y\big\}, w_0(t)\big\}, \\ \eta_1(t, y) &= \min\big\{\max\big\{v_0'(t), y\big\}, w_0'(t)\big\}, \\ \eta_2(t, y) &= \min\big\{\max\big\{v_0''(t), y\big\}, w_0''(t)\big\}, \\ \eta_3(t, y) &= \min\big\{\max\big\{w_0'''(t), y\big\}, v_0'''(t)\big\}. \end{aligned} \tag{3.3}$$

Then $\eta_0, \eta_1, \eta_2, \eta_3 : T \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy

$$\begin{aligned} &v_0(t) \le \eta_0(t,y) \le w_0(t), \quad (t,y) \in I \times \mathbb{R}, \\ &v_0'(t) \le \eta_1(t,y) \le w_0'(t), \quad (t,y) \in I \times \mathbb{R}, \\ &v_0''(t) \le \eta_2(t,y) \le w_0''(t), \quad (t,y) \in I \times \mathbb{R}, \\ &w_0'''(t) \le \eta_3(t,y) \le v_0'''(t), \quad (t,y) \in I \times \mathbb{R}. \end{aligned} \tag{3.4}$$

Make a truncating function f^* of f by

$$\begin{aligned} f^*(t,x_0,x_1,x_2,x_3) = {} & f\bigl(t, \eta_0(t,x_0), \eta_1(t,x_1), \eta_2(t,x_2), \eta_3(t,x_2)\bigr) \\ & + \frac{x_3 - \eta_3(t,x_3)}{{x_3}^2 + 1}, \quad (t,x_0,x_1,x_2,x_3) \in I \times \mathbb{R}^4. \end{aligned} \tag{3.5}$$

Then by (3.3) and (3.4), $f^* : I \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and bounded. By Lemma 3.1, the boundary value problem

$$\begin{cases} u^{(4)}(t) = f^*(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in I, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0 \end{cases} \tag{3.6}$$

has a solution $u_0 \in C^4(I)$. We show that

$$w_0''' \le u_0''' \le v_0'''. \tag{3.7}$$

In fact, if $w_0''' \not\le u_0'''$, then for the function

$$\phi(t) = u_0'''(t) - w_0'''(t), \quad t \in I, \tag{3.8}$$

$\min_{t\in I} \phi(t) < 0$. Since $\phi(1) \ge 0$, there exists $t_0 \in [0,1)$ such that

$$\phi(t_0) = \min_{t\in I} \phi(t) < 0, \qquad \phi'(t_0) \ge 0,$$

from which and (3.8) it follows that

$$u_0'''(t_0) < w_0'''(t_0), \qquad u_0^{(4)}(t_0) \ge w_0^{(4)}(t_0). \tag{3.9}$$

Hence from definition (3.3), we see that

$$\eta_3\bigl(t_0, u_0'''(t_0)\bigr) = w_0'''(t_0). \tag{3.10}$$

By Eq. (3.6), (3.10), (3.4), condition (F3) and the definition of the upper solution w_0, we have

$$\begin{aligned} u_0^{(4)}(t_0) &= f^*\bigl(t_0, u_0(t_0), u_0'(t_0), u_0''(t_0), u_0'''(t_0)\bigr) \\ &= f\bigl(t_0, \eta_0\bigl(t_0, u_0(t_0)\bigr), \eta_1\bigl(t_0, u_0'(t_0)\bigr), \eta_2\bigl(t_0, u_0''(t_0)\bigr), \eta_3\bigl(t_0, u_0'''(t_0)\bigr)\bigr) \end{aligned}$$

$$
\begin{aligned}
&+\frac{u_0'''(t_0)-\eta_3(t_0,u_0'''(t_0))}{[u_0'''(t_0)]^2+1}\\
&=f\big(t_0,\eta_0\big(t_0,u_0(t_0)\big),\eta_1\big(t_0,u_0'(t_0)\big),\eta_2\big(t_0,u_0''(t_0)\big),w_0'''(t_0)\big)\\
&\quad+\frac{u_0'''(t_0)-w_0'''(t_0)}{[u_0'''(t_0)]^2+1}\\
&\le f\big(t_0,w_0(t_0),w_0'(t_0),w_0''(t_0),w_0'''(t_0)\big)+\frac{u_0'''(t_0)-w_0'''(t_0))}{[u_0'''(t_0)]^2+1}\\
&< f\big(t_0,w_0(t_0),w_0'(t_0),w_0''(t_0),w_0'''(t_0)\big)\\
&\le w_0^{(4)}(t_0),
\end{aligned}
$$

that is, $u_0^{(4)}(t_0) < w_0^{(4)}(t_0)$, which contradicts (3.9). Hence, $w_0''' \le u_0'''$.

With a similar argument, we can show that $u_0''' \le v_0'''$, so (3.7) holds. Now by Lemma 2.1,

$$
v_0 \le u_0 \le w_0, \qquad {v_0}' \le {u_0}' \le {w_0}', \qquad {v_0}'' \le {u_0}'' \le {w_0}''. \tag{3.11}
$$

From (3.7), (3.11), and the definition (3.3) of η_i ($i = 0, 1, 2, 3$), it follows that

$$
\eta_i\big(t,u^{(i)}(t)\big) = u^{(i)}(t), \quad t \in I, i = 0,1,2,3.
$$

Hence by Eq. (3.6) we have

$$
\begin{aligned}
u_0^{(4)}(t) &= f^*\big(t,u_0(t),u_0'(t),u_0''(t),u_0'''(t)\big)\\
&= f\big(t,\eta_0\big(t,u_0(t)\big),\eta_1\big(t,u_0'(t)\big),\eta_2\big(t,u_0''(t)\big),\eta_3\big(t,u_0'''(t)\big)\big)\\
&\quad+\frac{u_0'''(t)-\eta_3(t,u_0'''(t))}{[u_0'''(t)]^2+1}\\
&= f\big(t,u_0(t),u_0'(t),u_0''(t),u_0'''(t)\big), \quad t \in I.
\end{aligned}
$$

That is, u_0 is a solution of BVP (1.1) in $[v_0, w_0]_{C^3}$. □

Example 3.1 Consider the fourth-order boundary value problem

$$
\begin{cases}
u^{(4)}(t) = \frac{1}{8}\sqrt[3]{u} + (u')^3 + (u'')^3 + \frac{1}{2}(u''')^3 + t(1-t), & t \in I,\\
u(0) = u'(0) = u''(1) = u'''(1) = 0.
\end{cases} \tag{3.12}
$$

Similar to Example 2.1, one can verify that $v_0 = 0$ is a lower solution and w_0 given by

$$
w_0(t) = \frac{1}{24}t^4 + \frac{1}{12}t^3 + \frac{1}{4}t^2(1-t), \quad t \in I, \tag{3.13}
$$

is an upper solution of BVP (3.12). Since $w_0'''(t) = t - 1 \le 0 = v_0'''(t)$ and the corresponding nonlinearity

$$
f(t,x_0,x_1,x_2,x_3) = \frac{1}{8}\sqrt[3]{x_0} + x_1^3 + x_2^3 + \frac{1}{2}x_3^3 + t(1-t) \tag{3.14}
$$

is increasing on x_0, x_1, x_2, all the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, BVP (3.12) has at least one solution $u_0 \in [v_0, w_0]_{C^3}$; clearly this solution is a positive solution. Since f is increasing on x_3 and it does not satisfy condition (F2), this result cannot be obtained by Theorem 2.1. For any $M > 0$, by expression (3.14) of f, the Lipschitz condition (1.9) does not hold on $\mathcal{D}_M$, and hence the result of [18] is not applicable for BVP (3.12).

4 Existence of positive solutions

In [17], the present first author have discussed the existence of positive solution of BVP (1.1) by using fixed point theory in cones. In this section we present a different existence result of positive for BVP (1.1) by Theorem 3.1.

Theorem 4.1 *Let $f : I \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and satisfy the following conditions*:

(F4) *for every $t \in I$ and $x_3 \in (-\infty, 0]$, $f(t, x_0, x_1, x_2, x_3)$ is increasing on x_0, x_1, and x_2 in* $[0, +\infty)$;

(F5) *there exists a positive constant $\delta > 0$ such that*

$$f(t, x_0, x_1, x_2, x_3) \ge 21x_0 \quad \textit{for all } (t, x_0, x_1, x_2, x_3) \in I \times [0, \delta]^3 \times [-\delta, 0];$$

(F6) *there exist nonnegative constants a_0, a_1, a_2, a_3 satisfying $a_0 + a_1 + a_2 + a_3 < 1$ and a positive constant $C_0 > 0$ such that*

$$f(t, x_0, x_1, x_2, x_3) \le a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 |x_3| + C_0$$

for all $(t, x_0, x_1, x_2, x_3) \in I \times [0, +\infty)^3 \times (-\infty, 0]$.

Then BVP (1.1) has at least one positive solution.

The proof of Theorem 4.1 needs the following existence and uniqueness result of a general fourth-order linear boundary value problem.

Lemma 4.1 *Let a_0, a_1, a_2, a_3 be nonnegative constants and satisfy $a_0 + a_1 + a_2 + a_3 < 1$. Then, for every $h \in C(I)$, the fourth-order linear boundary value problem*

$$\begin{cases} u^{(4)}(t) = a_0 u(t) + a_1 u'(t) + a_2 u''(t) - a_3 u'''(t) + h(t), \quad t \in I, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0 \end{cases} \tag{4.1}$$

has a unique solution $u \in C^4(I)$, and when $h \in C^+(I)$, the solution u satisfies

$$u \ge 0, \qquad u' \ge 0, \qquad u'' \ge 0, \qquad u''' \le 0. \tag{4.2}$$

Proof Choose a closed subset space of $C^3(I)$ by

$$E = \{u \in C^3(I) : u(0) = u'(0) = u''(1) = u'''(1) = 0\}. \tag{4.3}$$

For every $u \in E$, we show that

$$\|u\|_C \le \|u'\|_C \le \|u''\|_C \le \|u'''\|_C. \tag{4.4}$$

For every $t \in I$, by the boundary condition of E, we have

$$\begin{aligned}
|u(t)| &= \left|\int_0^t u'(s)\,ds\right| \le \int_0^t |u'(s)|\,ds \le t\|u'\|_C \le \|u'\|_C, \\
|u'(t)| &= \left|\int_0^t u''(s)\,ds\right| \le \int_0^t |u''(s)|\,ds \le t\|u''\|_C \le \|u''\|_C, \\
|u''(t)| &= \left|-\int_t^1 u''(s)\,ds\right| \le \int_t^1 |u'''(s)|\,ds \le (1-t)\|u'''\|_C \le \|u'''\|_C.
\end{aligned}$$

From these inequalities we conclude that

$$\|u\|_C \le \|u'\|_C, \qquad \|u'\|_C \le \|u''\|_C, \qquad \|u''\|_C \le \|u'''\|_C.$$

Hence, (4.4) holds. By (4.4), we have

$$\|u\|_{C^3} = \|u'''\|_C, \quad u \in E. \tag{4.5}$$

By Lemma 2.2, the solution operator of LBVP (2.3) $S : C(I) \to E$ is a completely linear operator. For every $h \in C(I)$ and $t \in I$, setting $u = Sh$, by Eq. (2.4), we have

$$|u'''(t)| = \left|-\int_t^1 u^{(4)}(s)\,ds\right| = \left|\int_t^1 h(s)\,ds\right| \le \|h\|_C.$$

Hence

$$\|Sh\|_{C^3} = \|u\|_{C^3} = \|u'''\|_C \le \|h\|_C.$$

This means that the norm of the linear bounded operator $S : C(I) \to E$ satisfies

$$\|S\|_{\mathcal{B}(C(I),E)} \le 1. \tag{4.6}$$

Define a linear operator $B : E \to C(I)$ by

$$Bu(t) := a_0u(t) + a_1u'(t) + a_2u''(t) - a_3u'''(t), \quad u \in E, t \in I. \tag{4.7}$$

Then, by the definition of the operator $S : C(I) \to E$, LBVP (4.1) is rewritten to the form of the operator equation in Banach space E:

$$(\mathrm{I} - SB)u = Sh, \tag{4.8}$$

where I is the identity operator in E. We prove that the norm of the composite operator SB in $\mathcal{B}(E,E)$ satisfies $\|TB\|_{\mathcal{B}(E,E)} < 1$.

For every $u \in E$, by the definition of B and (4.4), we have

$$\begin{aligned}
\|Bu\|_C &\le a_0\|u\|_C + a_1\|u'\|_C + a_2\|u''\|_C + a_3\|u'''\|_C \\
&\le (a_0 + a_1 + a_2 + a_3)\|u'''\|_C, \\
&= (a_0 + a_1 + a_2 + a_3)\|u\|_{C^3}.
\end{aligned} \tag{4.9}$$

From (4.6) and (4.9) it follows that

$$\begin{aligned}\|SBu\|_{C^3} &= \big\|S(Bu)\big\|_{C^3} \le \|S\|_{\mathcal{B}(C(I),E)} \cdot \|Bu\|_C \\ &\le (a_0 + a_1 + a_2 + a_3)\|u\|_{C^3}.\end{aligned}$$

This means that $\|SB\|_{\mathcal{B}(E,E)} \le a_0 + a_1 + a_2 + a_3 < 1$.

Since $\|SB\|_{\mathcal{B}(E,E)} < 1$, it follows that $\mathrm{I} - SB$ has a bounded inverse operator given by the series

$$(\mathrm{I} - SB)^{-1} = \sum_{n=0}^{\infty} (SB)^n.$$

Hence, Eq. (4.8), equivalently, LBVP (4.1), has the unique solution

$$u = (\mathrm{I} - SB)^{-1} Sh = \sum_{n=0}^{\infty} (SB)^n Sh. \tag{4.10}$$

Set $K_3 = \{u \in C^3 : u \succeq 0\}$. Then K_3 is a closed convex cone in $C^3(I)$. For every $v \in K_3$, by the definition (4.7) of B, $Bv \in C^+(I)$. By Lemma 2.3, $SBv = S(Bv) \in K_3$. Hence, $SB(K_3) \subset K_3$. Let $h \in C^+(I)$. By Lemma 2.3, $v = Sh \in K_3$. Hence, for every $n \in \mathbb{N}$, $(SB)^n Sh = (SB)^n v \in K_3$. By (4.10) and the completeness of K_3, $u \in K_3$, that is, u satisfies (4.2). □

Proof of Theorem 4.1 By [17, Lemma 2.3 and Lemma 2.4], the fourth-order linear eigenvalue problem(EVP)

$$\begin{cases} u^{(4)}(t) = \lambda u(t), & t \in I, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0 \end{cases} \tag{4.11}$$

has a minimum positive real eigenvalue $\lambda_1 \in [8, 21)$, and λ_1 has a positive unit eigenfunction, namely there exists $\phi_1 \in C^4(I) \cap C^+(I)$ with $\|\phi_1\|_C = 1$ which satisfies the equation

$$\begin{cases} \phi_1^{(4)}(t) = \lambda_1 \phi_1(t), & t \in I, \\ \phi_1(0) = \phi_1'(0) = \phi_1''(1) = \phi_1'''(1) = 0. \end{cases} \tag{4.12}$$

By Lemma 2.3, $\phi_1 \in K_3$. Choose a positive constant

$$\varepsilon = \min\bigl\{\delta / \max\bigl\{1, \|\phi_1'\|_C, \|\phi_1''\|_C, \|\phi_1'''\|_C\bigr\}, C_0/21\bigr\}, \tag{4.13}$$

and let $v_0 = \varepsilon \phi_1(t)$. Then, for every $t \in I$,

$$\begin{aligned} &0 \le v_0(t) \le \varepsilon \|\phi_1\|_C \le \delta, \qquad 0 \le v_0'(t) \le \varepsilon \|\phi_1'\|_C \le \delta \\ &0 \le v_0''(t) \le \varepsilon \|\phi_1''\|_C \le \delta, \qquad 0 \ge v_0'''(t) \ge -\varepsilon \|\phi_1'''\|_C \ge -\delta. \end{aligned}$$

By Assumption (F5), we have

$$f\bigl(t, v_0(t), v_0'(t), v_0''(t), v_0'''(t)\bigr) \ge 21 v_0(t) \ge \lambda_1 v_0(t) = {v_0}^{(4)}(t), \quad t \in I.$$

Hence v_0 is a lower solution of BVP (1.1).

By Lemma 4.1, the linear boundary value

$$\begin{cases} u^{(4)}(t) = a_0 u(t) + a_1 u'(t) + a_2 u''(t) - a_3 u'''(t) + C_0, & t \in I, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0 \end{cases} \tag{4.14}$$

has a unique solution $w_0 \in K_3$, where a_0, a_1, a_2, a_3, and C_0 are the constants in Assumption (F6). By Assumption (F6), w_0 is an upper solution of BVP (1.1). We show that $v_0 \preceq w_0$. Set $u_0 = w_0 - v_0$, since $v_0, w_0 \in K_3$, by the definitions of v_0 and w_0 and (4.13), we have

$$\begin{aligned} u_0^{(4)}(t) &= w_0^{(4)}(t) - v_0^{(4)}(t) \\ &= a_0 w_0(t) + a_1 w_0'(t) + a_2 w_0''(t) - a_3 w_0'''(t) + C_0 - \varepsilon\lambda_1\phi_1(t) \\ &\geq a_0 w_0(t) + a_1 w_0'(t) + a_2 w_0''(t) - a_3 w_0'''(t) \geq 0, \quad t \in I. \end{aligned} \tag{4.15}$$

By this inequality and Lemma 2.3, $u_0 \succeq 0$. Hence $v_0 \preceq w_0$. Now by Assumption (F4), condition (F3) of Theorem 3.1 holds. By Theorem 3.1, BVP (1.1) has a solution between v_0 and w_0, which is a positive solution of BVP (1.1). □

Example 4.1 Consider the following fourth-order nonlinear boundary value problem:

$$\begin{cases} u^{(4)}(t) = a\sqrt{|u|} + b\sqrt[3]{(u')^2} + c\sqrt[5]{(u'')^2} - du''' - e(u''')^4, & t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{4.16}$$

where a, b, c, d, e are positive constants. We verify that the nonlinearity term of BVP (4.16)

$$f(t, x_0, x_1, x_2, x_3) = a\sqrt{|x_0|} + b\sqrt[3]{x_1^2} + c\sqrt[5]{x_2^2} - dx_3 - ex_3^4 \tag{4.17}$$

satisfies conditions (F4)–(F6).

By expression (4.17), for every $t \in I$ and $x_3 \in (-\infty, 0]$, $f(t, x_0, x_1, x_2, x_3)$ is increasing on x_0, x_1, and x_2 on $[0, +\infty)$. Hence (F4) holds. Choose $\delta = \min\{\frac{a^2}{441}, \sqrt[3]{\frac{d}{e}}\}$, then for any $(t, x_0, x_1, x_2, x_3) \in I \times [0, \delta]^3 \times [-\delta, 0]$, by (4.17) we have

$$\begin{aligned} f(t, x_0, x_1, x_2, x_3) \geq a\sqrt{x_0} - x_3\bigl(d + ex_3^3\bigr) &= \frac{a}{\sqrt{x_0}} x_0 + |x_3|\bigl(d - e|x_3|^3\bigr) \\ &\geq \frac{a}{\sqrt{\delta}} x_0 + |x_3|\bigl(d - e\delta^3\bigr) \geq \frac{a}{\sqrt{\delta}} x_0 \geq 21 x_0. \end{aligned}$$

Hence (F5) holds. For any $(t, x_0, x_1, x_2, x_3) \in I \times [0, +\infty)^3 \times (-\infty, 0]$, using the Young inequality

$$rs \leq \frac{1}{\alpha} s^\alpha + \frac{1}{\beta} s^\beta, \qquad \alpha, \beta > 0, \qquad \frac{1}{\alpha} + \frac{1}{\beta} = 1; \qquad r, s \geq 0,$$

we have

$$\begin{aligned} a\sqrt{|x_0|} &= (2a)\bigl(x_0^{1/2}/2\bigr) \le 2a^2 + \frac{1}{8}x_0 \quad (\alpha = \beta = 2), \\ b\sqrt[3]{x_1^2} &= \bigl(2^{2/3}b\bigr)\bigl(x_1^{2/3}/2^{2/3}\bigr) \le \frac{4}{3}b^3 + \frac{1}{3}x_1 \quad (\alpha = 3, \beta = 3/2), \\ c\sqrt[5]{x_2^2} &= \bigl(8^{2/5}c\bigr)\bigl(x_1^{2/5}/8^{2/5}\bigr) \le \frac{12}{5}c^{5/3} + \frac{1}{20}x_2 \quad (\alpha = 5/3, \beta = 5/2). \end{aligned} \tag{4.18}$$

By these inequalities and (4.17), we obtain that

$$\begin{aligned} f(t,x_0,x_1,x_2,x_3) &\le \frac{1}{8}x_0 + \frac{1}{3}x_1 + \frac{1}{20}x_2 + C_0 + |x_3|\bigl(d - e|x_3|^3\bigr) \\ &\le \frac{1}{8}x_0 + \frac{1}{3}x_1 + \frac{1}{20}x_2 + C_0 + \max_{x_3\in\mathbb{R}} |x_3|\bigl(d - e|x_3|^3\bigr) \\ &= \frac{1}{8}x_0 + \frac{1}{3}x_1 + \frac{1}{20}x_2 + C_0 + \frac{3}{4}d\left(\frac{d}{4e}\right)^{1/3}, \end{aligned} \tag{4.19}$$

where $C_0 = 2a^2 + \frac{4}{3}b^3 + \frac{12}{5}c^{5/3}$. Choose $a_0 = \frac{1}{8}, a_1 = \frac{1}{3}, a_2 = \frac{1}{20}, a_3 = 0$, and $C = C_0 + \frac{3}{4}d(\frac{d}{4e})^{1/3}$, then $a_0 + a_1 + a_2 + a_3 = \frac{61}{120} < 1$. From (4.19) it follows that (F6) holds.

Consequently, the nonlinearity f of BVP (4.16) satisfies conditions (F4)–(F6). By Theorem 4.1, BVP (4.16) has at least one positive solution.

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Authors' contributions
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