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Equivalent properties of a Mulholland-type inequality with a best possible constant factor and parameters

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Abstract

By means of the weight coefficients, using the idea of introduced parameters and the techniques of real analysis, a Mulholland-type inequality with a homogeneous kernel and an equivalent form are provided. Some equivalent statements of the best possible constant factor related to a few parameters are considered. Some particular inequalities and the operator expressions are obtained.

MSC: 26D15

Keywords: Weight coefficient; Mulholland-type inequality; Parameter; Equivalent form; Operator expression

1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0$, are such that $0 < \sum_{m=1}^{\infty} a_m^p < \infty, 0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following Hardy–Hilbert-type inequality with the best possible constant factor pq (cf. [1], Theorem 341):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}. \tag{1}$$

Also we have the following Mulholland inequality with the best possible constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 343):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty} m a_m^p\right)^{1/p}\left(\sum_{n=2}^{\infty} n b_n^q\right)^{1/q}. \tag{2}$$

Assuming that $f(x), g(y) \geq 0, 0 < \int_0^{\infty} f^p(x)\,dx < \infty$, and $0 < \int_0^{\infty} g^q(y)\,dy < \infty$, we have the following well known Hardy–Hilbert integral inequality (cf. [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)\,dx\right)^{1/p}\left(\int_0^{\infty} g^q(y)\,dy\right)^{1/q}, \tag{3}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Inequalities (1), (2), and (3) are important in analysis and its applications (cf. [2, 3]). Recently, some new extensions with parameters and applications were given in [4–13].

A half-discrete Hilbert-type inequality was provided in 1934 as follows (cf. [1], Theorem 351): If $K(x)$ is decreasing on $(0,\infty), 0 < \varphi(s) < \int_0^\infty K(x)x^{s-1}\,dx < \infty$, then we have the following inequality with the best possible constant factor $\varphi^p(\frac{1}{q})$:

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \varphi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

Some extensions of (4) were provided in [14–19].

In 2016, by using the techniques of real analysis, Hong [20] considered some equivalent statements of the general form of (1) with the homogeneous kernel related to a few parameters and a best possible constant factor. Other similar results about the extended integral inequalities (3) were obtained in [21–24].

In this paper, following the approach of [20], by means of the weight coefficients, the idea of introduced parameters and using techniques of real analysis, a Mulholland-type inequality with the homogeneous kernel and its equivalent form are obtained in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters are considered in Theorem 2. Some particular cases and the operator expressions are provided by Remark 3 and Theorem 3.

2 An example and some lemmas

In what follows, we assume that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, s \in \mathrm{N} = \{1,2,\ldots\}, 0 < c_1 \le c_2 \le \cdots \le c_s < \infty, \lambda_i + \alpha, \lambda - \lambda_i + \alpha \in (0,1]$ $(i = 1,2)$, $a_m, b_n \ge 0$, are such that

$$0 < \sum_{m=2}^\infty \frac{(\ln m)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}}{m^{1-p}} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=2}^\infty \frac{(\ln n)^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}}{n^{1-p}} b_n^q < \infty. \tag{5}$$

Example 1 For $\mathrm{R}_+ = (0,\infty)$, we set

$$k_\lambda(x,y) := \prod_{k=1}^s \frac{(\min\{x, c_k y\})^{\alpha/s}}{(\max\{x, c_k y\})^{(\alpha+\lambda)/s}} \quad \big((x,y) \in \mathrm{R}_+^2 = \mathrm{R}_+ \times \mathrm{R}_+\big). \tag{6}$$

In view of Example 1 in [25], it follows that for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$\begin{aligned} k_s(\gamma) &:= \int_0^\infty k_s(u,1)u^{\gamma-1}\,du \\ &= \int_0^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\alpha+\lambda)/s}} u^{\gamma-1}\,du \\ &= \frac{c_1^{\gamma+\alpha}}{\gamma+\alpha}\prod_{k=1}^s c_k^{-\frac{\lambda+\alpha}{s}} + \frac{c_s^{\gamma-\lambda-\alpha}}{\lambda-\gamma+\alpha}\prod_{k=1}^s c_k^{\frac{\alpha}{s}} \\ &\quad + \sum_{i=1}^{s-1} \frac{c_{i+1}^{\gamma-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha} - c_i^{\gamma-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha}}{\gamma-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha} \frac{\prod_{k=1}^i c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^s c_k^{\frac{\lambda+\alpha}{s}}}. \end{aligned} \tag{7}$$

In particular, for $s = 1$ (or $c_s = \cdots c_1$), we have $k_\lambda(x,y) = \frac{(\min\{x,c_1 y\})^\alpha}{(\max\{x,c_1 y\})^{\alpha+\lambda}}$ ($(x,y) \in \mathrm{R}_+^2$), and

$$k_1(\gamma) = \int_0^\infty \frac{(\min\{u,c_1\})^\alpha}{(\max\{u,c_1\})^{\alpha+\lambda}} u^{\gamma-1}\,du = \frac{(\lambda+2\alpha)c_1^{\gamma-\lambda}}{(\gamma+\alpha)(\lambda-\gamma+\alpha)}. \tag{8}$$

(a) For $\lambda_2 + \alpha \le 1, \lambda - \lambda_2 + \alpha > 0$, we find that $1 - \lambda_2 - \alpha \ge 0, 1 + \lambda - \lambda_2 + \alpha - \frac{i}{s}(\lambda + 2\alpha) > 0$ ($i = 1, \ldots, s-1$), $1 + \lambda - \lambda_2 + \alpha > 1 > 0$, and then, for fixed $x > 0$,

$$\begin{aligned} &k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}} \\ &\quad = \frac{1}{y^{1-\lambda_2}} \prod_{k=1}^s \frac{(\min\{c_k^{-1}x,y\})^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda}{s}}(\max\{c_k^{-1}x,y\})^{\frac{\lambda+\alpha}{s}}} \\ &\quad = \begin{cases} \frac{1}{y^{1-\lambda_2-\alpha}} \prod_{k=1}^s \frac{(\min\{c_k^{-1}x,y\})^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda}{s}}(c_k^{-1}x)^{\frac{\lambda+\alpha}{s}}}, & 0 < y \le c_s^{-1}x, \\ \frac{1}{y^{1+\lambda-\lambda_2+\alpha-\frac{i}{s}(\lambda+2\alpha)}} \frac{\prod_{k=i+1}^s (c_k^{-1}x)^{\frac{\alpha}{s}}}{\prod_{k=1}^s c_k^{\frac{\lambda}{s}} \prod_{k=1}^i (c_k^{-1}x)^{\frac{\lambda+\alpha}{s}}}, & c_{i+1}^{-1}x < y \le c_i^{-1}x \ (i = 1, \ldots, s-1), \\ \frac{1}{y^{1+\lambda-\lambda_2+\alpha}} \prod_{k=1}^s \frac{(c_k^{-1}x)^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda}{s}}}, & c_1^{-1}x < y < \infty \end{cases} \end{aligned}$$

is decreasing for $y > 0$ and strictly decreasing for $y > c_1^{-1}x$.

(b) In the same way, for $\lambda_1 + \alpha \le 1, \lambda - \lambda_1 + \alpha > 0$, we find that for fixed $y > 0$, $k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}$ is decreasing for $x > 0$ and strictly decreasing for $x > c_s y$.

Definition 1 Define the following weight coefficients:

$$\omega_s(\lambda_2,m) := \ln^{\lambda-\lambda_2} m \sum_{n=2}^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} \frac{\ln^{\lambda_2-1} n}{n} \quad \bigl(m \in \mathrm{N}\backslash\{1\}\bigr), \tag{9}$$

$$\varpi_s(\lambda_1,n) := \ln^{\lambda-\lambda_1} n \sum_{m=2}^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} \frac{\ln^{\lambda_1-1} m}{m} \quad \bigl(n \in \mathrm{N}\backslash\{1\}\bigr). \tag{10}$$

Lemma 1 *We have the following inequalities*:

$$\omega_s(\lambda_2,m) < k_s(\lambda - \lambda_2) \quad \bigl(m \in \mathrm{N}\backslash\{1\}\bigr), \tag{11}$$

$$\varpi_s(\lambda_1,n) < k_s(\lambda_1) \quad \bigl(n \in \mathrm{N}\backslash\{1\}\bigr). \tag{12}$$

Proof For fixed $m \ge 2, \lambda_2 + \alpha \le 1, \lambda - \lambda_2 + \alpha > 0$, in view of Example 1(a), we find that the positive function

$$k_\lambda(\ln m, \ln t)\frac{1}{t \ln^{1-\lambda_2} t} = \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln t\})^{\alpha/s}}{(\max\{\ln m, c_k \ln t\})^{\lambda+\alpha}} \frac{1}{t \ln^{1-\lambda_2} t}$$

is strictly decreasing for $t > 1$. By the decreasingness property, setting $u = \frac{\ln m}{\ln t}$, we find that

$$\omega_s(\lambda_2,m) < \ln^{\lambda-\lambda_2} m \int_1^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln t\})^{\alpha/s}}{(\max\{\ln m, c_k \ln t\})^{(\lambda+\alpha)/s}} \frac{1}{t \ln^{1-\lambda_2} t}\,dt$$

$$= \int_0^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{\lambda-\lambda_2-1}\,du = k_s(\lambda - \lambda_2).$$

Hence, we have (11).

In the same way, for fixed $n \geq 2, \lambda_1 + \alpha \leq 1, \lambda - \lambda_1 + \alpha > 0$, in view of Example 1(b), by the decreasingness property, setting $u = \frac{\ln t}{\ln n}$, we find that

$$\begin{aligned} \varpi_s(\lambda_1, n) &< \ln^{\lambda_1} n \int_1^\infty \prod_{k=1}^s \frac{(\min\{\ln t, c_k \ln n\})^{\alpha/s}}{(\max\{\ln t, c_k \ln n\})^{(\lambda+\alpha)/s}} \frac{1}{t \ln^{1-\lambda_1} t}\,dt \\ &= \int_0^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{\lambda_1-1}\,du = k_s(\lambda_1) \end{aligned}$$

and then (12) follows. □

Lemma 2 *We have the following inequality*:

$$\begin{aligned} I &:= \sum_{n=2}^\infty \sum_{m=2}^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} a_m b_n \\ &< k_s^{\frac{1}{p}}(\lambda - \lambda_2) k_s^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^\infty \frac{(\ln m)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=2}^\infty \frac{(\ln n)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Proof By Hölder's inequality (cf. [26]), we obtain

$$\begin{aligned} I &:= \sum_{n=2}^\infty \sum_{m=2}^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} \left[\frac{\ln^{(\lambda_2-1)p} n}{n^{1/p}} \frac{\ln^{(1-\lambda_1)/q} m}{m^{-1/q}} a_m \right] \\ &\quad \times \left[\frac{\ln^{(\lambda_1-1)/q} m}{m^{1/q}} \frac{\ln^{(1-\lambda_2)/p} n}{n^{-1/p}} b_n \right] \\ &\leq \left\{ \sum_{m=2}^\infty \left[\ln^{\lambda-\lambda_2} m \sum_{n=2}^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} \frac{\ln^{\lambda_2-1} n}{n} \right] \right. \\ &\quad \left. \times \left[\ln^{\lambda_1+\lambda_2-\lambda} m \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} \right] a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=2}^\infty \left[\ln^{\lambda-\lambda_1} n \sum_{m=2}^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} \frac{\ln^{\lambda_1-1} m}{m} \right] \right. \\ &\quad \left. \times \left[\ln^{\lambda_2+\lambda_1-\lambda} n \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}} \right] b_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{m=2}^\infty \omega_s(\lambda_2, m) \frac{(\ln m)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}} \end{aligned}$$

$$\times \left\{ \sum_{n=2}^{\infty} \varpi_s(\lambda_1, n) \frac{(\ln n)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}.$$

Then by (11) and (12), in view of (5), we have (13). □

Remark 1 By (13), for $\lambda_1 + \lambda_2 = \lambda$, we use the assumptions that $\lambda_i + \alpha \in (0,1] \cap (0, \lambda + 2\alpha)$ ($i = 1, 2$), which give

$$0 < \sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p < \infty, \qquad 0 < \sum_{n=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-p}} b_n^q < \infty,$$

$$k_s(\lambda_1) = \frac{c_1^{\lambda_1+\alpha}}{\lambda_1+\alpha} \prod_{k=1}^{s} c_k^{-\frac{\lambda+\alpha}{s}} + \frac{c_s^{-\lambda_2-\alpha}}{\lambda_2+\alpha} \prod_{k=1}^{s} c_k^{\frac{\alpha}{s}} + \sum_{i=1}^{s-1} \frac{c_{i+1}^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha} - c_i^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha}}{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha} \frac{\prod_{k=1}^{i} c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} \in \mathrm{R}_+, \tag{14}$$

and find the following inequality:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \prod_{k=1}^{s} \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} a_m b_n < k_s(\lambda_1) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{15}$$

Lemma 3 *The constant factor* $k_s(\lambda_1)$ *in* (15) *is the best possible.*

Proof For $0 < \varepsilon < \min\{p(\lambda_1+\alpha), q(\lambda_2+\alpha)\}$, we set

$$\tilde{a}_m := \frac{\ln^{\lambda_1-\frac{\varepsilon}{p}-1} m}{m}, \qquad \tilde{b}_n := \frac{\ln^{\lambda_2-\frac{\varepsilon}{q}-1} n}{n} \quad \big(m, n \in \mathrm{N}\backslash\{1\}\big).$$

If there exists a positive constant $M \le k_s(\lambda_1)$, such that (15) is valid when replacing $k_s(\lambda_1)$ by M, then in particular, we have

$$\tilde{I} := \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \prod_{k=1}^{s} \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} \tilde{a}_m \tilde{b}_n < M \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} \tilde{a}_m^p \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-p}} \tilde{b}_n^q \right]^{\frac{1}{q}}.$$

We obtain

$$\tilde{I} < M \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} \frac{\ln^{p\lambda_1-\varepsilon-p} m}{m^p} \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}} \frac{\ln^{q\lambda_2-\varepsilon-1} n}{n^q} \right]^{\frac{1}{q}} = M \left(\frac{\ln^{-\varepsilon-1} 2}{2} + \sum_{m=3}^{\infty} \frac{\ln^{-\varepsilon-1} m}{m} \right)^{\frac{1}{p}} \left(\frac{\ln^{-\varepsilon-1} 2}{2} + \sum_{n=3}^{\infty} \frac{\ln^{-\varepsilon-1} n}{n} \right)^{\frac{1}{q}}$$

$$< M\left(\frac{\ln^{-\varepsilon-1} 2}{2} + \int_2^\infty \frac{\ln^{-\varepsilon-1} t}{t}\,dt\right) = \frac{M}{\varepsilon \ln^\varepsilon 2}\left(\frac{\varepsilon}{2\ln 2} + 1\right).$$

Since $0 < (\lambda_1 - \frac{\varepsilon}{p}) + \alpha \leq 1, 0 < (\lambda_2 - \frac{\varepsilon}{q}) + \alpha \leq 1$, by the decreasingness property, we find

$$\begin{aligned}\tilde{I} &= \sum_{n=2}^\infty \left[\sum_{m=2}^\infty \prod_{k=1}^s \frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}} \frac{\ln^{(\lambda_1 - \frac{\varepsilon}{p})-1} m}{m}\right] \frac{\ln^{\lambda_2 - \frac{\varepsilon}{q} - 1} n}{n} \\ &> \int_2^\infty \left[\int_2^\infty \prod_{k=1}^s \frac{(\min\{\ln x, c_k \ln y\})^{\alpha/s}}{(\max\{\ln x, c_k \ln y\})^{(\lambda+\alpha)/s}} \frac{\ln^{(\lambda_1 - \frac{\varepsilon}{p})-1} x}{x}\,dx\right] \frac{\ln^{\lambda_2 - \frac{\varepsilon}{q} - 1} y}{y}\,dy.\end{aligned}$$

For fixed $x \geq 2$, setting $u = \frac{\ln x}{\ln y}$, by Fubini theorem (cf. [27]), we find

$$\begin{aligned}\tilde{I} &> \int_2^\infty \left[\int_{\frac{\ln 2}{\ln y}}^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{(\lambda_1 - \frac{\varepsilon}{p})-1}\,du\right] \frac{\ln^{-\varepsilon-1} y}{y}\,dy \\ &= \int_2^\infty \left[\int_{\frac{\ln 2}{\ln y}}^1 \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{(\lambda_1 - \frac{\varepsilon}{p})-1}\,du\right] \frac{\ln^{-\varepsilon-1} y}{y}\,dy \\ &\quad + \int_2^\infty \left[\int_1^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{(\lambda_1 - \frac{\varepsilon}{p})-1}\,du\right] \frac{\ln^{-\varepsilon-1} y}{y}\,dy \\ &= \int_0^1 \left(\int_{e^{\frac{\ln 2}{u}}}^\infty \frac{\ln^{-\varepsilon-1} y}{y}\,dy\right) \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{(\lambda_1 - \frac{\varepsilon}{p})-1}\,du \\ &\quad + \frac{1}{\varepsilon \ln^\varepsilon 2} \int_1^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{(\lambda_1 - \frac{\varepsilon}{p})-1}\,du \\ &= \frac{1}{\varepsilon \ln^\varepsilon 2}\left[\int_0^1 \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{\lambda_1 + \frac{\varepsilon}{q} - 1}\,du\right. \\ &\quad \left. + \int_1^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{\lambda_1 - \frac{\varepsilon}{p} - 1}\,du\right].\end{aligned}$$

Then we have

$$\begin{aligned}&\int_0^1 \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s} u^{\lambda_1 + \frac{\varepsilon}{q} - 1}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}}\,du + \int_1^\infty \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s} u^{\lambda_1 - \frac{\varepsilon}{p} - 1}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}}\,du \\ &\quad < \left(\varepsilon \ln^\varepsilon 2\right)\tilde{I} < M\left(\frac{\varepsilon}{2\ln 2} + 1\right).\end{aligned}$$

For $\varepsilon \to 0^+$, by Fatou lemma (cf. [27]), we find

$$\begin{aligned}k_s(\lambda_1) &= \int_0^1 \lim_{\varepsilon \to 0^+} \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{\lambda_1 + \frac{\varepsilon}{q} - 1}\,du \\ &\quad + \int_1^\infty \lim_{\varepsilon \to 0^+} \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{\lambda_1 - \frac{\varepsilon}{p} - 1}\,du \\ &\leq \underline{\lim}_{\varepsilon \to 0^+} \left[\int_0^1 \prod_{k=1}^s \frac{(\min\{u, c_k\})^{\alpha/s}}{(\max\{u, c_k\})^{(\lambda+\alpha)/s}} u^{\lambda_1 + \frac{\varepsilon}{q} - 1}\,du\right.\end{aligned}$$

$$+\int_{1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\alpha/s}}{(\max\{u,c_k\})^{(\lambda+\alpha)/s}}u^{\lambda_1-\frac{\varepsilon}{p}-1}\,du\Bigg]\le M.$$

Hence, $M = k_s(\lambda_1)$ is the best possible constant factor of (15). □

Remark 2 Setting $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, we find $\hat{\lambda}_1+\hat{\lambda}_2=\lambda$,

$$\begin{aligned}
&0<\hat{\lambda}_1+\alpha=\frac{\lambda-\lambda_2+\alpha}{p}+\frac{\lambda_1+\alpha}{q}\le\frac{1}{p}+\frac{1}{q}=1,\\
&0<\hat{\lambda}_2+\alpha=\frac{\lambda-\lambda_1+\alpha}{q}+\frac{\lambda_2+\alpha}{p}\le\frac{1}{q}+\frac{1}{p}=1,\\
&\hat{\lambda}_i+\alpha\in(0,1]\cap(0,\lambda+2\alpha)\quad(i=1,2),
\end{aligned}$$

and then we rewrite (13) as follows:

$$I<k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\Bigg[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\hat{\lambda}_1)-1}m}{m^{1-p}}a_m^p\Bigg]^{\frac{1}{p}}\Bigg[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\hat{\lambda}_2)-1}n}{n^{1-q}}b_n^q\Bigg]^{\frac{1}{q}}. \tag{16}$$

Lemma 4 *If the constant factor* $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ *in* (13) *is the best possible, then we have* $\lambda_1+\lambda_2=\lambda$.

Proof If the constant factor $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ in (13) is the best possible, then so is the constant factor in (16). By (15), the unique best possible constant factor must be $k_s(\hat{\lambda}_1)$, namely,

$$k_s(\hat{\lambda}_1)=k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1).$$

By Hölder's inequality (cf. [26]), we obtain

$$\begin{aligned}
k_s(\hat{\lambda}_1)&=k_s\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)\\
&=\int_0^{\infty}\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\alpha/s}}{(\max\{u,c_k\})^{(\lambda+\alpha)/s}}\left(u^{\frac{\lambda-\lambda_2-1}{p}}\right)\left(u^{\frac{\lambda_1-1}{q}}\right)du\\
&\le\Bigg[\int_0^{\infty}\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\alpha/s}}{(\max\{u,c_k\})^{(\lambda+\alpha)/s}}u^{\lambda-\lambda_2-1}\,du\Bigg]^{\frac{1}{p}}\\
&\quad\times\Bigg[\int_0^{\infty}\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\alpha/s}}{(\max\{u,c_k\})^{(\lambda+\alpha)/s}}u^{\lambda_1-1}\,du\Bigg]^{\frac{1}{q}}\\
&=k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1).
\end{aligned}\tag{17}$$

Observe that (17) keeps the form of equality if and only if there exist constants A and B such that they are not all zero and (cf. [26])

$$Au^{\lambda-\lambda_2-1}=Bu^{\lambda_1-1}\quad\text{a.e. in }\mathrm{R}_+.$$

Assuming that $A \neq 0$ (otherwise, $B = A = 0$), it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in R_+, and then $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$. □

3 Main results

Theorem 1 *Inequality* (13) *is equivalent to the following*:

$$J := \left\{\sum_{n=2}^{\infty} \frac{(\ln n)^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}}{n}\left[\sum_{m=2}^{\infty}\prod_{k=1}^{s}\frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}}a_m\right]^p\right\}^{\frac{1}{p}}$$
$$< k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^{\infty}\frac{(\ln m)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}}{m^{1-p}}a_m^p\right\}^{\frac{1}{p}}. \quad (18)$$

If the constant factor in (13) *is the best possible, then so is the constant factor in* (18).

Proof Suppose that (18) is valid. By Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty}\left[\frac{(\ln n)^{\frac{-1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}}{n^{1/p}}\sum_{m=2}^{\infty}\prod_{k=1}^{s}\frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}}a_m\right]\left[\frac{(\ln n)^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}}{n^{-1/p}}b_n\right]$$
$$\leq J\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}}{n^{1-q}}b_n^q\right\}^{\frac{1}{q}}. \quad (19)$$

Then by (18) we obtain (13).

On the other hand, assuming that (13) is valid, we set

$$b_n := \frac{(\ln n)^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}}{n}\left[\sum_{m=2}^{\infty}\prod_{k=1}^{s}\frac{(\min\{\ln m, c_k \ln n\})^{\alpha/s}}{(\max\{\ln m, c_k \ln n\})^{(\lambda+\alpha)/s}}a_m\right]^{p-1}, \quad n \in \mathrm{N}\backslash\{1\}.$$

If $J = 0$, then (18) is naturally valid; if $J = \infty$, then it is impossible to make (18) valid, hence $J < \infty$. Suppose that $0 < J < \infty$. By (13), we find

$$\sum_{n=2}^{\infty}\frac{(\ln n)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}}{n^{1-q}}b_n^q$$
$$= J^p = I < k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^{\infty}\frac{(\ln m)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}}{m^{1-p}}a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}}{n^{1-q}}b_n^q\right\}^{\frac{1}{q}},$$
$$J = \left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}}{n^{1-q}}b_n^q\right\}^{\frac{1}{p}}$$
$$< k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^{\infty}\frac{(\ln m)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}}{m^{1-p}}a_m^p\right\}^{\frac{1}{p}},$$

namely, (18) follows. Hence, inequality (13) is equivalent to (18).

If the constant factor in (13) is the best possible, then so is constant factor in (18). Otherwise, by (19), we would reach a contradiction that the constant factor in (13) is not the best possible. □

Theorem 2 *The following statements are equivalent*:

(i) $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ *is independent of* p,q;

(ii) $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ *is expressible as a single integral*;

(iii) $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ *in* (13) *is the best possible constant*;

(iv) $\lambda_1+\lambda_2=\lambda$.

If statement (iv) *is true, namely*, $\lambda_1+\lambda_2=\lambda$, *then we have* (15) *and the following equivalent inequality with the best possible constant factor* $k_s(\lambda_1)$:

$$\left\{\sum_{n=2}^{\infty}\frac{\ln^{p\lambda_2-1}n}{n}\left[\sum_{m=2}^{\infty}\prod_{k=1}^{s}\frac{(\min\{\ln m,c_k\ln n\})^{\alpha/s}}{(\max\{\ln m,c_k\ln n\})^{(\lambda+\alpha)/s}}a_m\right]^p\right\}^{\frac{1}{p}}$$
$$<k_s(\lambda_1)\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}. \tag{20}$$

Proof (i)$\Longrightarrow$(ii). If $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ is independent of p,q, then we find

$$k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)=\lim_{q\to1^+}k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)=k_s(\lambda_1),$$

namely, $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ is expressible by a single integral

$$k_s(\lambda_1)=\int_0^{\infty}\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\alpha/s}}{(\max\{u,c_k\})^{(\lambda+\alpha)/s}}u^{\lambda_1-1}\,du.$$

(ii)$\Longrightarrow$(iv). In (13), if $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral $k_s(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (17) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda_1+\lambda_2=\lambda$.

(iv)$\Longrightarrow$(i). If $\lambda_1+\lambda_2=\lambda$, then $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)=k_s(\lambda_1)$, which is independent of p,q. Hence, we have (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iv).

(iii)$\Longrightarrow$(iv). By Lemma 4, we have $\lambda_1+\lambda_2=\lambda$.

(iv)$\Longrightarrow$(iii). By Lemma 3, for $\lambda_1+\lambda_2=\lambda$,

$$k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)=k_s(\lambda_1)$$

is the best possible constant factor of (13). Therefore, we have (iii) $\Longleftrightarrow$(iv).

Hence, statements (i), (ii), (iii) and (iv) are equivalent. □

Remark 3 (i) For $\lambda=1,\alpha=0,\lambda_1=\frac{1}{q},\lambda_2=\frac{1}{p}$ in (15) and (20), we have the following equivalent inequalities with the best possible constant factor

$$\tilde{k}_s\left(\frac{1}{q}\right):=\frac{qc_1^{1/q}}{\prod_{k=1}^{s}c_k^{1/s}}+\frac{p}{c_s^{1/p}}+\sum_{i=1}^{s-1}\frac{c_{i+1}^{\frac{1}{q}-\frac{i}{s}}-c_i^{\frac{1}{q}-\frac{i}{s}}}{\frac{1}{q}-\frac{i}{s}}\frac{1}{\prod_{k=i+1}^{s}c_k^{1/s}}:$$

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\prod_{k=1}^{s}(\max\{\ln m, c_k \ln n\})^{1/s}} < \tilde{k}_s\left(\frac{1}{q}\right)\left(\sum_{m=2}^{\infty}\frac{1}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{1}{n^{1-q}}b_n^q\right)^{\frac{1}{q}}, \tag{21}$$

$$\left[\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(\max\{\ln m, c_k \ln n\})^{1/s}}\right)^p\right]^{\frac{1}{p}} < \tilde{k}_s\left(\frac{1}{q}\right)\left(\sum_{m=2}^{\infty}\frac{1}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}. \tag{22}$$

(ii) For $\lambda = 1, \alpha = 0, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$ in (15) and (20), we have the following equivalent dual inequalities with the best possible constant factor

$$\tilde{k}_s\left(\frac{1}{p}\right) := \frac{pc_1^{1/p}}{\prod_{k=1}^{s}c_k^{1/s}} + \frac{q}{c_s^{1/q}} + \sum_{i=1}^{s-1}\frac{c_{i+1}^{\frac{1}{p}-\frac{i}{s}} - c_i^{\frac{1}{p}-\frac{i}{s}}}{\frac{1}{p}-\frac{i}{s}}\frac{1}{\prod_{k=i+1}^{s}c_k^{1/s}}:$$

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\prod_{k=1}^{s}(\max\{\ln m, c_k \ln n\})^{1/s}} < \tilde{k}_s\left(\frac{1}{p}\right)\left(\sum_{m=2}^{\infty}\frac{\ln^{p-2} m}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{\ln^{q-2} n}{n^{1-q}}b_n^q\right)^{\frac{1}{q}}, \tag{23}$$

$$\left[\sum_{n=2}^{\infty}\frac{\ln^{p-2} n}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(\max\{\ln m, c_k \ln n\})^{1/s}}\right)^p\right]^{\frac{1}{p}} < \tilde{k}_s\left(\frac{1}{p}\right)\left(\sum_{m=2}^{\infty}\frac{\ln^{p-2} m}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}. \tag{24}$$

(iii) For $p = q = 2$, both (21) and (23) reduce to

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\prod_{k=1}^{s}(\max\{\ln m, c_k \ln n\})^{1/s}} < \tilde{k}_s\left(\frac{1}{2}\right)\left(\sum_{m=2}^{\infty}m a_m^2\sum_{n=2}^{\infty}n b_n^2\right)^{\frac{1}{2}}, \tag{25}$$

and both (22) and (24) reduce to the equivalent form of (25) as follows:

$$\left[\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(\max\{\ln m, c_k \ln n\})^{1/s}}\right)^2\right]^{\frac{1}{2}} < \tilde{k}_s\left(\frac{1}{2}\right)\left(\sum_{m=2}^{\infty}m a_m^2\right)^{\frac{1}{2}}, \tag{26}$$

where $\tilde{k}_s(\frac{1}{2})$ is the best possible constant factor given by

$$\tilde{k}_s\left(\frac{1}{2}\right) = \frac{2c_1^{1/2}}{\prod_{k=1}^{s}c_k^{1/s}} + \frac{2}{c_s^{1/2}} + \sum_{i=1}^{s-1}\frac{c_{i+1}^{\frac{1}{2}-\frac{i}{s}} - c_i^{\frac{1}{2}-\frac{i}{s}}}{\frac{1}{2}-\frac{i}{s}}\frac{1}{\prod_{k=i+1}^{s}c_k^{1/s}}.$$

In particular, for $s = 1$ in (25) and (26), we have the following equivalent inequalities with the best possible constant factor $\frac{4}{c_1^{1/2}}$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\max\{\ln m, c_1 \ln n\}} < \frac{4}{c_1^{1/2}}\left(\sum_{m=2}^{\infty}m a_m^2\sum_{n=2}^{\infty}n b_n^2\right)^{\frac{1}{2}}, \tag{27}$$

$$\left[\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}\frac{a_m}{\max\{\ln m, c_1\ln n\}}\right)^2\right]^{\frac{1}{2}} < \frac{4}{c_1^{1/2}}\left(\sum_{m=2}^{\infty} m a_m^2\right)^{\frac{1}{2}}. \tag{28}$$

4 Operator expressions

We set the following functions:

$$\varphi(m) := \frac{(\ln m)^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}}{m^{1-p}}, \qquad \psi(n) := \frac{(\ln n)^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}}{n^{1-q}},$$

where

$$\psi^{1-p}(n) = \frac{(\ln n)^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}}{n} \quad \bigl(m,n \in \mathrm{N}\backslash\{1\}\bigr).$$

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{a = \{a_m\}_{m=2}^{\infty}; \|a\|_{p,\varphi} := \left(\sum_{m=2}^{\infty}\varphi(m)|a_m|^p\right)^{\frac{1}{p}} < \infty\right\},$$

$$l_{q,\psi} := \left\{b = \{b_n\}_{n=2}^{\infty}; \|b\|_{q,\psi} := \left(\sum_{n=2}^{\infty}\psi(n)|b_n|^q\right)^{\frac{1}{q}} < \infty\right\},$$

$$l_{p,\psi^{1-p}} := \left\{c = \{c_n\}_{n=2}^{\infty}; \|c\|_{p,\psi^{1-p}} := \left(\sum_{n=2}^{\infty}(n)|c_n|^p\right)^{\frac{1}{p}} < \infty\right\}.$$

Assuming that $a = \{a_m\}_{m=2}^{\infty} \in l_{p,\varphi}$, and setting

$$c = \{c_n\}_{n=2}^{\infty}, \quad c_n := \sum_{m=2}^{\infty}\prod_{k=1}^{s}\frac{(\min\{\ln m, c_k\ln n\})^{\alpha/s}}{(\max\{\ln m, c_k\ln n\})^{(\lambda+\alpha)/s}}a_m, n \in \mathrm{N}\backslash\{1\},$$

we can rewrite (13) as

$$\|c\|_{p,\psi^{1-p}} < k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 2 Define a Mulholland-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta, b) := \sum_{n=2}^{\infty}\left[\sum_{m=2}^{\infty}\prod_{k=1}^{s}\frac{(\min\{\ln m, c_k\ln n\})^{\alpha/s}}{(\max\{\ln m, c_k\ln n\})^{(\lambda+\alpha)/s}}a_m\right]b_n,$$

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\varphi}}\frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\varphi}}.$$

By Theorems 1 and 2, we have

Theorem 3 *If* $a \in l_{p,\varphi}, b \in l_{q,\psi}, \|a\|_{p,\varphi}, \|b\|_{q,\psi} > 0$, *then we have the following equivalent inequalities*:

$$(Ta, b) < k_s^{\frac{1}{p}}(\lambda - \lambda_2) k_s^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\varphi} \|b\|_{q,\psi}, \tag{29}$$

$$\|Ta\|_{p,\psi^{1-p}} < k_s^{\frac{1}{p}}(\lambda - \lambda_2) k_s^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\varphi}. \tag{30}$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$, *if and only if*

$$\|T\| = k_s^{\frac{1}{p}}(\lambda - \lambda_2) k_s^{\frac{1}{q}}(\lambda_1) = k_s(\lambda_1). \tag{31}$$

5 Conclusions

In this paper, by means of the weight coefficients, using the idea of introduced parameters and techniques of real analysis, a Mulholland-type inequality with the homogeneous kernel and an equivalent form are given in Theorem 1. The equivalent statements of the best possible constant factor related to some parameters are considered in Theorem 2. Some particular cases and the operator expressions are obtained in Remark 3 and Theorem 3. The lemmas and theorems provide an extensive account of such inequalities.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. HM participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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