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Boundedness and essential norm of an integral-type operator on a Hilbert–Bergman-type spaces

Stevo Stević[1,2,3*] and Zhi-Jie Jiang[4]

*Correspondence: sstevic@ptt.rs
[1] Mathematical Institute of the Serbian Academy of Sciences, Beograd, Serbia
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, Republic of China
Full list of author information is available at the end of the article

Abstract

Let $\mathbb{D}$ be the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Let $A^2_{\gamma,\delta}(\mathbb{D})$ be the space of analytic functions that are L^2 with respect to the weight $\omega_{\gamma,\delta}(z) = (\ln\frac{1}{|z|})^{\gamma}[\ln(1-\frac{1}{\ln|z|})]^{\delta}$, where $-1 < \gamma < \infty$ and $\delta \le 0$. For given $g \in H(\mathbb{D})$, the integral-type operator I_g on $H(\mathbb{D})$ is defined as

$$I_g f(z) = \int_0^z f(\zeta)g(\zeta)\,d\zeta.$$

In this paper, we characterize the boundedness of I_g on $A^2_{\gamma,\delta}$, whereas in the main result we estimate the essential norm of the operator. Some basic results on the space $A^2_{\gamma,\delta}(\mathbb{D})$ are also presented.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane $\mathbb{C}$, $r\mathbb{D} = \{z \in \mathbb{D} : |z| < r\}$, $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$, and $dA(z) = \frac{1}{\pi} r\,dr\,d\theta$ be the normalized area measure on $\mathbb{D}$ (i.e., $A(\mathbb{D}) = 1$).

A positive continuous function on $\mathbb{D}$ is called weight. Let $\mu(z)$ be a weight function on $\mathbb{D}$. The weighted-type space $H^\infty_\mu(\mathbb{D}) = H^\infty_\mu$ consists of $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^\infty_\mu} := \sup_{z\in\mathbb{D}} \mu(z)\big|f(z)\big| < \infty.$$

The little weighted-type space on $\mathbb{D}$, $H^\infty_{\mu,0}(\mathbb{D}) = H^\infty_{\mu,0}$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1-0} \mu(z)\big|f(z)\big| = 0$$

(see, e.g., [1] and the related references therein). For $\mu(z) = (1-|z|)^\alpha$, $\alpha > 0$, the classical weighted-type space $H^\infty_\alpha(\mathbb{D}) = H^\infty_\alpha$ and the classical little weighted-type space $H^\infty_{\alpha,0}(\mathbb{D}) =$

$H^\infty_{\alpha,0}$ are obtained whose special cases frequently appear in the literature, which is also the case in this paper.

Let $-1 < \gamma < \infty$ and $\delta \le 0$. The logarithmic Hilbert–Bergman space, denoted by $A^2_{\gamma,\delta}(\mathbb{D}) = A^2_{\gamma,\delta}$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^2_{\gamma,\delta}} := \left(\int_{\mathbb{D}} \left|f(z)\right|^2 \omega_{\gamma,\delta}(z)\, dA(z)\right)^{1/2} < \infty,$$

where

$$\omega_{\gamma,\delta}(z) = \left(\ln \frac{1}{|z|}\right)^{\gamma} \left[\ln\left(1 - \frac{1}{\ln |z|}\right)\right]^{\delta}.$$

Under the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega_{\gamma,\delta}(z)\, dA(z),$$

$A^2_{\gamma,\delta}$ is a Hilbert space. This is why it is called the logarithmic Hilbert–Bergman space. The space $A^2_{\gamma,\delta}$ has been recently introduced in [2] as a special case of the logarithmic Bergman space $A^p_{\gamma,\delta}$, $p \in (0,\infty)$.

For a given function $g \in H(\mathbb{D})$, the integral-type operator I_g is defined by

$$I_g(f)(z) = \int_0^z f(\zeta)g(\zeta)\, d\zeta \tag{1}$$

for $f \in H(\mathbb{D})$.

As usual, throughout the paper, we will write frequently $I_g f$ instead of $I_g(f)$.

The operator (1) is clearly a natural generalization of the integral operator (the one obtained for $g(z) \equiv 1$). The operator can be regarded as a classical/folklore one. A variant of the operator can be found in [3], which could be one of the first papers studying such an operator on concrete spaces of analytic functions on $\mathbb{D}$ (see [3, Lemma 1]). The topic of studying integral-type operators on spaces of analytic functions has attracted some considerable recent attention. Much information on the topic, including a large list of references up to the end of 2006, can be found in [4]. Some product-type generalizations of the operator on the unit disc were later introduced and studied, for example, in [5], while the corresponding operator for the case of the unit ball was introduced in [6] and later studied in many papers to mention, for example, [7] (for the case of the polydisc, see, e.g., [8]). For some further generalizations, related operators, and related results, see also [9–11] and the references therein. We would like to point out that a great majority of these papers are devoted to characterizing some function-theoretic properties of these operators in terms of the involved symbols.

Let X, Y be two Banach spaces and $T : X \to Y$ be a bounded linear operator (an operator which maps bounded sets in X to bounded sets in Y). Recall that the essential norm of the bounded linear operator $T : X \to Y$, denoted by $\|T\|_e$, is defined as

$$\|T\|_e := \inf\bigl\{\|T - K\| : K \text{ is compact from } X \text{ to } Y\bigr\},$$

where $\|\cdot\|$ denotes the usual operator norm. For some results on the essential norms of concrete operators (such as the composition, multiplication, weighted composition, differentiation, integral, and their various products and relatives) see, for example, [7, 12–18] and the related references therein. From the definition of the essential norm and since the set of all compact operators is a closed subset of the space of bounded linear operators, it follows that the operator $T: X \to Y$ is compact if and only if $\|T\|_e = 0$.

In this paper, first, we characterize the boundedness and compactness of I_g on $A^2_{\gamma,\delta}$, when $\delta < 0$. As it is shown, when $\delta = 0$, the space is equivalent to the classical weighted Hilbert–Bergman space for which the corresponding results are known even in much more general settings [19, 20]. We also estimate essential norm of the operator, which is practically the main result in the paper. This paper, among others, can be regarded as a continuation of our investigations of integral-type operators (see [4–7] and the references therein), essential norms of concrete operators on spaces of analytic functions (see [7, 15, 16] and the references therein), as well as the investigation of concrete operators on $A^2_{\gamma,\delta}$. Before this work, composition operators on, or between, $A^2_{\gamma,\delta}$ were studied in [2, 18], and product-type operators from $A^2_{\gamma,\delta}$ to Zygmund–Orlicz spaces were studied in [21]. We also present some basic results on the space $A^2_{\gamma,\delta}$. For example, we give a completely analytic proof why the space $A^2_{\gamma,\delta}$ is the same as the space consisting of all $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} \left|f(z)\right|^2 (1-|z|)^{\gamma} \left(\ln \frac{e}{1-|z|}\right)^{\delta} dA(z) < +\infty. \tag{2}$$

It is well known that $f \in A^2_{\gamma}(\mathbb{D})$ if and only if $f \in \mathcal{D}^2_{\gamma+2}(\mathbb{D})$ (the classical weighted Dirichlet space); moreover,

$$\|f\|^2_{A^2_{\gamma}} \asymp \left|f(0)\right|^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-|z|^2\right)^{\gamma+2} dA(z).$$

Here, we prove a similar result for the space $A^2_{\gamma,\delta}$. We obtain pointwise estimates for functions in $A^2_{\gamma,\delta}$, as well as for their derivatives. We also give a complete orthonormal system in $A^2_{\gamma,\delta}$. These basic results on the space $A^2_{\gamma,\delta}$ seem new (we could not find them in the literature, so far).

In this paper, the letter C denotes a positive constant which may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant C independent of the essential variables in the quantities a and b such that $a \le Cb$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \asymp b$.

2 Some basic results on $A^2_{\gamma,\delta}$ and auxiliary ones

This section presents several auxiliary results which are employed in the proofs of the main ones, as well as several basic results on the space $A^2_{\gamma,\delta}$. First, we present a completely analytic proof of the fact that the space is the same as the space consisting of all $f \in H(\mathbb{D})$ satisfying (2), connecting it to a more familiar weight.

If we use the well-known asymptotic relation $\ln(1+x) = x + o(x)$, as $x \to 0$, we easily obtain

$$\omega_{\gamma,\delta}(z) = \left(\ln \frac{1}{|z|}\right)^{\gamma} \left[\ln\left(1 + \frac{1}{\ln \frac{1}{|z|}}\right)\right]^{\delta} \sim \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} \tag{3}$$

as $z \to 0$, and

$$\begin{aligned}\omega_{\gamma,\delta}(z) &= \left(\ln\left(1+\frac{1-|z|}{|z|}\right)\right)^{\gamma}\left[\ln\left(1+\frac{1}{\ln(1+\frac{1-|z|}{|z|})}\right)\right]^{\delta}\\ &= \left(\frac{1-|z|}{|z|}\big(1+O(1-|z|)\big)\right)^{\gamma}\left[\ln\left(1+\frac{1}{\frac{1-|z|}{|z|}(1+O(1-|z|))}\right)\right]^{\delta}\\ &\sim \big(1-|z|\big)^{\gamma}\left(\ln\left(\frac{1}{1-|z|}+O(1)\right)\right)^{\delta} \qquad (4)\end{aligned}$$

as $|z| \to 1-0$.

From (3), (4) and the continuity of the functions therein, it is easy to see that

$$\omega_{\gamma,\delta}(z) \asymp \left(\ln\frac{1}{|z|}\right)^{\gamma-\delta} \qquad (5)$$

for every $|z| \le r_1 < 1$ and each fixed $r_1 \in (0,1)$, while

$$\omega_{\gamma,\delta}(z) \asymp \big(1-|z|\big)^{\gamma}\left(\ln\frac{e}{1-|z|}\right)^{\delta}, \qquad (6)$$

for every $r_2 \le |z| < 1$ and each fixed $r_2 \in (0,1)$, where in the proof of relation (6) the fact that, for each $0 < m \le M$, there are $\alpha_1 \in (0,1]$ and $\alpha_2 \ge 1$ such that

$$\ln\left(\frac{e}{1-|z|}\right)^{\alpha_1} \le \ln\frac{m}{1-|z|} \le \ln\frac{M}{1-|z|} \le \ln\left(\frac{e}{1-|z|}\right)^{\alpha_2} \qquad (7)$$

for z sufficiently close to the unit circle is also used.

Proposition 1 *Let $-1 < \gamma < \infty$ and $\delta \le 0$. Then the asymptotic relation*

$$\|f\|_{A^2_{\gamma,\delta}} \asymp \int_{\mathbb{D}}\big|f(z)\big|^2\big(1-|z|\big)^{\gamma}\left(\ln\frac{e}{1-|z|}\right)^{\delta}dA(z) \qquad (8)$$

holds for every $f \in A^2_{\gamma,\delta}$.

Proof From (6), we have

$$\int_{\mathbb{D}\setminus r_0\mathbb{D}}\big|f(z)\big|^2\omega_{\gamma,\delta}(z)\,dA(z) \asymp \int_{\mathbb{D}\setminus r_0\mathbb{D}}\big|f(z)\big|^2\big(1-|z|\big)^{\gamma}\left(\ln\frac{e}{1-|z|}\right)^{\delta}dA(z) \qquad (9)$$

for each $r_0 \in (0,1)$.

It is easy to see that, for each fixed $r_0 \in (0,1)$, we have

$$\int_{r_0\mathbb{D}}\big|f(z)\big|^2\big(1-|z|\big)^{\gamma}\left(\ln\frac{e}{1-|z|}\right)^{\delta}dA(z) \asymp \int_{r_0\mathbb{D}}\big|f(z)\big|^2\,dA(z). \qquad (10)$$

From (5), we have

$$\int_{\frac{1}{e}\mathbb{D}}\big|f(z)\big|^2\omega_{\gamma,\delta}(z)\,dA(z) \asymp \int_{\frac{1}{e}\mathbb{D}}\big|f(z)\big|^2\left(\ln\frac{1}{|z|}\right)^{\gamma-\delta}dA(z). \qquad (11)$$

If $\gamma - \delta \geq 0$, then

$$\int_{\frac{1}{e}\mathbb{D}} |f(z)|^2 \, dA(z) \leq \int_{\frac{1}{e}\mathbb{D}} |f(z)|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z). \tag{12}$$

From (9) and (10) with $r_0 = 1/e$, (11) and (12), it follows that

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|)^\gamma \left(\ln \frac{e}{1-|z|}\right)^\delta dA(z) \lesssim \|f\|_{A^2_{\gamma,\delta}}. \tag{13}$$

Further, let

$$C_1 := \sup_{0<r<1/e} \left(\ln \frac{1}{r}\right)^{\gamma-\delta} \sqrt{r}.$$

It is easy to see that $C_1 \in (0, +\infty)$.

We have

$$\begin{aligned}
&\int_{\frac{1}{e}\mathbb{D}} |f(z)|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) \\
&\quad = 2\int_0^{1/e} M_2^2(f,r) \left(\ln \frac{1}{r}\right)^{\gamma-\delta} r \, dr \\
&\quad \leq 2C_1 \int_0^{1/e} M_2^2(f,r) \sqrt{r} \, dr \\
&\quad = 4C_1 \int_0^{1/\sqrt{e}} M_2^2(f,\rho^2) \rho^2 \, d\rho \\
&\quad \leq 4C_1 \int_0^{1/\sqrt{e}} M_2^2(f,\rho) \rho \, d\rho \\
&\quad \leq 2C_1 \int_{\frac{1}{\sqrt{e}}\mathbb{D}} |f(z)|^2 \, dA(z) \\
&\quad \leq 2C_1 \left(\int_{\frac{1}{e}\mathbb{D}} |f(z)|^2 \, dA(z) + \int_{\frac{1}{\sqrt{e}}\mathbb{D} \setminus \frac{1}{e}\mathbb{D}} |f(z)|^2 \, dA(z) \right) \\
&\quad \lesssim \int_{\frac{1}{e}\mathbb{D}} |f(z)|^2 \, dA(z) + \int_{\frac{1}{\sqrt{e}}\mathbb{D} \setminus \frac{1}{e}\mathbb{D}} |f(z)|^2 (1-|z|)^\gamma \left(\ln \frac{e}{1-|z|}\right)^\delta dA(z) \\
&\quad \lesssim \int_{\frac{1}{e}\mathbb{D}} |f(z)|^2 \, dA(z) + \int_{\mathbb{D} \setminus \frac{1}{e}\mathbb{D}} |f(z)|^2 (1-|z|)^\gamma \left(\ln \frac{e}{1-|z|}\right)^\delta dA(z),
\end{aligned} \tag{14}$$

where we have used the polar coordinates $z = re^{i\theta}$ and the monotonicity of the integral means

$$M_2^2(f,r) = \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta, \quad r \in [0,1).$$

From (9) and (10) with $r_0 = 1/e$, (11), and (14), it follows that

$$\|f\|_{A^2_{\gamma,\delta}} \lesssim \int_{\mathbb{D}} |f(z)|^2 (1-|z|)^\gamma \left(\ln \frac{e}{1-|z|}\right)^\delta dA(z). \tag{15}$$

From (13) and (15), we obtain that the asymptotic relation (8) holds in this case.

If $\gamma - \delta < 0$, then clearly

$$\int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) \le \int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 dA(z). \tag{16}$$

Hence, from (9) and (10) with $r_0 = 1/e$, (11), and (16), it follows that

$$\|f\|_{A^2_{\gamma,\delta}} \lesssim \int_{\mathbb{D}} \left|f(z)\right|^2 (1-|z|)^{\gamma} \left(\ln \frac{e}{1-|z|}\right)^{\delta} dA(z). \tag{17}$$

On the other hand, we have

$$C_2 := \sup_{0<r<1/e^{2/3}} r\left(\ln \frac{1}{r}\right)^{\delta-\gamma} \in (0,+\infty),$$

from which, along with the monotonicity of the integral means and use of the polar coordinates, it follows that

$$\begin{aligned}
\int_{\frac{1}{e^{2/3}}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) &= 2\int_0^{1/e^{2/3}} M_2^2(f,r)\left(\ln \frac{1}{r}\right)^{\gamma-\delta} r\,dr \\
&\ge \frac{2}{C_2}\int_0^{1/e^{2/3}} M_2^2(f,r) r^2\,dr \\
&\ge \frac{4}{3C_2}\int_0^{1/e} M_2^2\left(f,\rho^{2/3}\right)\rho\,d\rho \\
&\ge \frac{4}{3C_2}\int_0^{1/e} M_2^2(f,\rho)\rho\,d\rho \\
&\ge \frac{2}{3C_2}\int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 dA(z).
\end{aligned} \tag{18}$$

From (18), we have

$$\begin{aligned}
\int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 dA(z) &\lesssim \int_{\frac{1}{e^{2/3}}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) \\
&\lesssim \int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) + \int_{\frac{1}{e^{2/3}}\mathbb{D}\setminus\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) \\
&\lesssim \int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) + \int_{\frac{1}{e^{2/3}}\mathbb{D}\setminus\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 dA(z) \\
&\lesssim \int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) + \int_{\frac{1}{e^{2/3}}\mathbb{D}\setminus\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \omega_{\gamma,\delta}(z)\,dA(z) \\
&\lesssim \int_{\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \left(\ln \frac{1}{|z|}\right)^{\gamma-\delta} dA(z) + \int_{\mathbb{D}\setminus\frac{1}{e}\mathbb{D}} \left|f(z)\right|^2 \omega_{\gamma,\delta}(z)\,dA(z).
\end{aligned} \tag{19}$$

From (9) and (10) with $r_0 = 1/e$, (11), and (19), it follows that

$$\int_{\mathbb{D}} \left|f(z)\right|^2 (1-|z|)^{\gamma} \left(\ln \frac{e}{1-|z|}\right)^{\delta} dA(z) \lesssim \|f\|_{A^2_{\gamma,\delta}}. \tag{20}$$

From (17) and (20) the asymptotic relation (8) follows in this case, completing the proof of the proposition. □

Remark 1 In [18] it is said that the space $A^2_{\gamma,\delta}$ is the same as the space consisting of all $f \in H(\mathbb{D})$ satisfying condition (2) not giving a proof of the claim. Due to the estimates in (7), Proposition 1 gives a pure analytic proof of the equivalence of these two spaces. Note also that from Proposition 1 with $\delta = 0$ it is obtained that the space $A^2_{\gamma,0}$ is equivalent to the weighted Bergman space A^2_γ.

The proof of Proposition 1 can be shortened by applying the open mapping theorem, but we decided to present a complete analytic proof for the audience more interested in inequalities. However, in the proof of one of our next results (see Lemma 1) we will use the open mapping theorem.

Remark 2 Note that, by using (5), (6), and the polar coordinates, we have

$$\begin{aligned}
\|1\|_{A^2_{\gamma,\delta}} &= \int_{\mathbb{D}} \omega_{\gamma,\delta}(z)\,dA(z) = \int_{\frac{1}{e}\mathbb{D}} \omega_{\gamma,\delta}(z)\,dA(z) + \int_{\mathbb{D}\setminus\frac{1}{e}\mathbb{D}} \omega_{\gamma,\delta}(z)\,dA(z) \\
&\lesssim \int_{\frac{1}{e}\mathbb{D}} \left(\ln\frac{1}{|z|}\right)^{\gamma-\delta} dA(z) + \int_{\mathbb{D}\setminus\frac{1}{e}\mathbb{D}} (1-|z|)^\gamma \left(\ln\frac{e}{1-|z|}\right)^\delta dA(z) \\
&\le \frac{1}{\pi}\int_0^{1/e}\int_0^{2\pi} \left(\ln\frac{1}{\rho}\right)^{\gamma-\delta} d\theta\rho\,d\rho + \left(\ln\frac{e^2}{e-1}\right)^\delta \int_{\mathbb{D}\setminus\frac{1}{e}\mathbb{D}} (1-|z|)^\gamma\,dA(z) \\
&= 2\int_1^\infty \frac{s^{\gamma-\delta}}{e^{2s}}\,ds + 2\left(\ln\frac{e^2}{e-1}\right)^\delta \int_{1/e}^1 (1-\rho)^\gamma \rho\,d\rho < \infty
\end{aligned} \tag{21}$$

since $\gamma > -1$.

Proposition 2 *Let g be a nondecreasing integrable function on* $[0,1)$ *and h be a positive function on* $(0,1)$ *which is continuous on* $[0,1)$. *Then, for each fixed* $r_0 \in [0,1)$, *we have*

$$\int_0^1 g(r)h(r)\,dr \asymp \int_{r_0}^1 g(r)h(r)\,dr. \tag{22}$$

Proof If $r_0 = 0$, then the result is obvious. Hence, assume that $r_0 \in (0,1)$. We have

$$\int_0^1 g(r)h(r)\,dr = \int_0^{r_0} g(r)h(r)\,dr + \int_{r_0}^1 g(r)h(r)\,dr. \tag{23}$$

Now note that there is unique $n_0 \in \mathbb{N}_0$ such that

$$n_0\frac{1-r_0}{2} \le r_0 < (n_0+1)\frac{1-r_0}{2}.$$

Note that by the choice of n_0 we have

$$\frac{(n_0+1)(1-r_0)}{2} < \frac{1+r_0}{2} < 1.$$

Since g is a nondecreasing function and h is positive on $(0,1)$ and continuous on $[0,1)$, we have

$$\begin{aligned}
\int_0^{r_0} g(r)h(r)\,dr &\le \max_{0\le r\le r_0} h(r)\int_0^{r_0} g(r)\,dr\\
&\le \max_{0\le r\le r_0} h(r)\int_0^{(1-r_0)(n_0+1)/2} g(r)\,dr\\
&= \max_{0\le r\le r_0} h(r)\sum_{j=0}^{n_0}\int_{(1-r_0)j/2}^{(1-r_0)(j+1)/2} g(r)\,dr\\
&\le (n_0+1)\max_{0\le r\le r_0} h(r)\int_{r_0}^{(1+r_0)/2} g(r)\,dr\\
&\le (n_0+1)\frac{\max_{0\le r\le r_0} h(r)}{\min_{r_0\le r\le (1+r_0)/2} h(r)}\int_{r_0}^{(1+r_0)/2} g(r)h(r)\,dr\\
&\le C(r_0,h)\int_{r_0}^{1} g(r)h(r)\,dr. \qquad (24)
\end{aligned}$$

From (23) and (24) it follows that

$$\int_0^1 g(r)h(r)\,dr \lesssim \int_{r_0}^1 g(r)h(r)\,dr,$$

from which along with the obvious inequality

$$\int_{r_0}^1 g(r)h(r)\,dr \le \int_0^1 g(r)h(r)\,dr$$

the asymptotic relation (22) follows. □

Corollary 1 *Let g be an arbitrary nondecreasing integrable function on $[0,1)$ and h be a fixed positive function on $(0,1)$ which is continuous on $[0,1)$. Then, for each fixed $r_0\in[0,1)$, we have*

$$\int_{r_0}^1 g(r)h(r)\,dr \le \int_0^1 g(r)h(r)\,dr \le C(r_0,h)\int_{r_0}^1 g(r)h(r)\,dr$$

for some positive constant $C(r_0,h)$ depending on r_0 and h.

Lemma 1 *Let $-1<\gamma<\infty$ and $\delta\le 0$. Then*

$$\|f\|^2_{A^2_{\gamma,\delta}} \asymp \left|f(0)\right|^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\,dA(z) \qquad (25)$$

for every $f\in A^2_{\gamma,\delta}$.

Proof Since

$$\sqrt{|a|^2+|b|^2} \le |a|+|b| \le \sqrt{2\left(|a|^2+|b|^2\right)} \qquad (26)$$

for $a, b \in \mathbb{C}$, we have

$$\begin{aligned}&\left(\left|f(0)\right|^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\,dA(z)\right)^{1/2} \\ &\quad \asymp \left|f(0)\right| + \left(\int_{\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\,dA(z)\right)^{1/2}.\end{aligned} \tag{27}$$

Let

$$\|f\|_1 := \left|f(0)\right| + \left(\int_{\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\,dA(z)\right)^{1/2}.$$

It is clear that $\|f\|_1 \geq 0$ for every $f \in A^2_{\gamma,\delta}$, that $\|0\|_1 = 0$ and $\|\lambda f\|_1 = |\lambda|\|f\|_1$ for every $f \in A^2_{\gamma,\delta}$ and $\lambda \in \mathbb{C}$. If $\|f\|_1 = 0$, then obviously $f(0) = 0$, and since $\omega_{\gamma+2,\delta}(z) > 0$, $z \in \mathbb{D} \setminus \{0\}$, it follows that $f'(z) = 0$, $z \in \mathbb{D} \setminus \{0\}$, and since $f \in H(\mathbb{D})$, it must be $f'(z) = 0$, $z \in \mathbb{D}$, from which it follows that $f(z) \equiv f(0) = 0$. By using the triangle inequality and the Cauchy–Schwarz inequality it easily follows that $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$ for every $f, g \in A^2_{\gamma,\delta}$. Hence, $\|\cdot\|_1$ is a norm on $A^2_{\gamma,\delta}$.

By using the polar coordinates, we have

$$\int_{\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\,dA(z) = 2\int_0^1 M_2^2\big(f', r\big)\omega_{\gamma+2,\delta}(r)r\,dr. \tag{28}$$

Employing Proposition 2 with $g(r) = M_2^2(f', r)$ (here the well-known fact that the integral means of holomorphic functions are nondecreasing functions is used, which in this case is a simple statement due the easily checked equality $M_2^2(\hat{f}, r) = \sum_{j=0}^{\infty} |a_k|^2 r^{2k}$, if $\hat{f}(z) = \sum_{j=0}^{\infty} a_k z^k$) and $h(r) = r\omega_{\gamma+2,\delta}(r)$, which is positive and continuous on $(0,1)$ and if it is naturally defined by

$$h(0) := \lim_{r \to +0} h(r) = 0$$

it becomes continuous on $[0,1)$, we have

$$\int_0^1 M_2^2\big(f', r\big)\omega_{\gamma+2,\delta}(r)r\,dr \asymp \int_{r_0}^1 M_2^2\big(f', r\big)\omega_{\gamma+2,\delta}(r)r\,dr, \tag{29}$$

independently on f.

From (27)–(29) and by using the polar coordinates, it follows that

$$\begin{aligned}&\left(\left|f(0)\right|^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\,dA(z)\right)^{1/2} \\ &\quad \asymp \left|f(0)\right| + \left(\int_{\mathbb{D}\setminus r_0\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\,dA(z)\right)^{1/2}\end{aligned} \tag{30}$$

for each $r_0 \in [0,1)$.

Since

$$\|f\|^2_{A^2_{\gamma,\delta}} = 2\int_0^1 M_2^2(f, r)\omega_{\gamma,\delta}(r)r\,dr, \tag{31}$$

similar as above we obtain

$$\|f\|^2_{A^2_{\gamma,\delta}} \asymp \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f(z)|^2 \omega_{\gamma,\delta}(z)\,dA(z) \tag{32}$$

for each $r_0 \in [0,1)$.

By using the following known formula

$$M_2^2(f,r) = |f(0)|^2 + 2\int_{r\mathbb{D}} |f'(z)|^2 \ln\frac{r}{|z|}\,dA(z), \tag{33}$$

which can be easily checked by direct calculation, in (32) and employing polar coordinates and the Fubini theorem, we get

$$\begin{aligned}
\|f\|^2_{A^2_{\gamma,\delta}} &\asymp 2\int_{r_0}^1 M_2^2(f,r)\omega_{\gamma,\delta}(r)r\,dr \\
&\asymp 2|f(0)|^2\int_{r_0}^1 \omega_{\gamma,\delta}(r)r\,dr + 4\int_{r_0}^1 \omega_{\gamma,\delta}(r)r\int_{r\mathbb{D}} |f'(z)|^2 \ln\frac{r}{|z|}\,dA(z)\,dr \\
&= 2|f(0)|^2\int_{r_0}^1 \omega_{\gamma,\delta}(r)r\,dr + 8\int_{r_0}^1 \omega_{\gamma,\delta}(r)r\int_0^r M_2^2(f',\rho)\ln\frac{r}{\rho}\rho\,d\rho\,dr \\
&= 2|f(0)|^2\int_{r_0}^1 \omega_{\gamma,\delta}(r)r\,dr + 8\int_0^{r_0}\int_{r_0}^1 \omega_{\gamma,\delta}(r)r\ln\frac{r}{\rho}\,drM_2^2(f',\rho)\rho\,d\rho \\
&\quad + 8\int_{r_0}^1\int_\rho^1 \omega_{\gamma,\delta}(r)r\ln\frac{r}{\rho}\,drM_2^2(f',\rho)\rho\,d\rho.
\end{aligned} \tag{34}$$

By a slight modification of the proof of Lemma 3.1 in [2], we have that

$$\int_{|z|}^1 \omega_{\gamma,\delta}(r)r\ln\frac{r}{|z|}\,dr \asymp \omega_{\gamma+2,\delta}(z) \tag{35}$$

for $0 < r_0 \le |z| < 1$.

From (34) and (35), and by using (26), it follows that

$$\begin{aligned}
\|f\|^2_{A^2_{\gamma,\delta}} &\gtrsim |f(0)|^2 + \int_{r_0}^1 M_2^2(f',\rho)\omega_{\gamma+2,\delta}(\rho)\rho\,d\rho \\
&\gtrsim |f(0)|^2 + \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f'(z)|^2\omega_{\gamma+2,\delta}(z)\,dA(z) \\
&\gtrsim \left(|f(0)| + \left(\int_{\mathbb{D}\setminus r_0\mathbb{D}} |f'(z)|^2\omega_{\gamma+2,\delta}(z)\,dA(z)\right)^{1/2}\right)^2.
\end{aligned} \tag{36}$$

From (30) and (36), we have

$$\|f\|_1 \lesssim \|f\|_{A^2_{\gamma,\delta}} \tag{37}$$

for every $f \in A^2_{\gamma,\delta}$.

Applying the open mapping theorem to the identity map

$$I : (A^2_{\gamma,\delta}, \|\cdot\|_{A^2_{\gamma,\delta}}) \to (A^2_{\gamma,\delta}, \|\cdot\|_1),$$

we have that

$$\|f\|_{A^2_{\gamma,\delta}} \lesssim \|f\|_1,$$

from which along with (37) the asymptotic relation (25) follows. □

By repeating use of Lemma 1, it follows that the following corollary holds.

Corollary 2 *Let* $-1 < \gamma < \infty$, $\delta \le 0$, *and* $m \in \mathbb{N}$. *Then*

$$\|f\|^2_{A^2_{\gamma,\delta}} \asymp \sum_{j=0}^{m-1} \left|f^{(j)}(0)\right|^2 + \int_{\mathbb{D}} \left|f^{(m)}(z)\right|^2 \omega_{\gamma+2m,\delta}(z)\, dA(z)$$

for every $f \in A^2_{\gamma,\delta}$.

From Corollary 2, we obtain the following corollary.

Corollary 3 *Let* $-1 < \gamma < \infty$, $\delta \le 0$, $m \in \mathbb{N}$, *and* $(f_n)_{n\in\mathbb{N}} \subset A^2_{\gamma,\delta}$. *Then*

$$\lim_{n\to\infty} \|f_n - f\|_{A^2_{\gamma,\delta}} = 0$$

if and only if

$$\lim_{n\to\infty} \int_{\mathbb{D}} \left|f_n^{(m)}(z) - f^{(m)}(z)\right|^2 \omega_{\gamma+2m,\delta}(z)\, dA(z) = 0$$

and

$$\lim_{n\to\infty} f_n^{(j)}(0) = f^{(j)}(0)$$

for $j = \overline{0, m-1}$.

Note that from the proof of Lemma 1 we see that the following result was also proved.

Corollary 4 *Let* $-1 < \gamma < \infty$ *and* $\delta \le 0$. *Then, for fixed* $r_0 \in [0,1)$, *we have*

$$\|f\|^2_{A^2_{\gamma,\delta}} \asymp \left|f(0)\right|^2 + \int_{\mathbb{D}\setminus r_0\mathbb{D}} \left|f'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z).$$

We need also the following estimate for the functions in $A^2_{\gamma,\delta}$.

Lemma 2 *Let* $-1 < \gamma < \infty$, $\delta \le 0$, $r_0 \in (0,1)$, *and* $m \in \mathbb{N}_0$. *Then, if* $|z| \ge r_0$, *it follows that*

$$\left|f^{(m)}(z)\right| \lesssim \left[\omega_{\gamma+2m+2,\delta}(z)\right]^{-\frac{1}{2}} \|f\|_{A^2_{\gamma,\delta}}$$

for $f \in A^2_{\gamma,\delta}$.

Proof From Corollary 2, since

$$\omega_{\gamma+2m,\delta}(\zeta) \asymp \omega_{\gamma+2m,\delta}(z)$$

for $|\zeta - z| < \frac{1-|z|}{2}$, and by the subharmonicity of the function $|f^{(m)}(z)|^2$, we have

$$\begin{aligned}
\|f\|^2_{A^2_{\gamma,\delta}} &\gtrsim \int_{\mathbb{D}} \big|f^{(m)}(\zeta)\big|^2 \omega_{\gamma+2m,\delta}(\zeta)\, dA(\zeta) \\
&\gtrsim \int_{|\zeta-z|<\frac{1-|z|}{2}} \big|f^{(m)}(\zeta)\big|^2 \omega_{\gamma+2m,\delta}(\zeta)\, dA(\zeta) \\
&\gtrsim \omega_{\gamma+2m,\delta}(z) \int_{|\zeta-z|<\frac{1-|z|}{2}} \big|f^{(m)}(\zeta)\big|^2\, dA(\zeta) \\
&\gtrsim \omega_{\gamma+2m,\delta}(z)\big(1-|z|\big)^2 \big|f^{(m)}(z)\big|^2 \\
&\gtrsim \omega_{\gamma+2m+2,\delta}(z)\big|f^{(m)}(z)\big|^2,
\end{aligned}$$

from which the corollary follows. □

Remark 3 Lemma 2 with $m = 0$ shows that the point evaluations Λ_z at the point $z \in \mathbb{D} \setminus \{0\}$ are bounded linear functions on $A^2_{\gamma,\delta}$. Then there are reproducing kernels for these z, denoted by $K_{\gamma,\delta}(z,\cdot)$, such that

$$\Lambda_z f = \big\langle f, K_{\gamma,\delta}(z,\cdot)\big\rangle = \int_{\mathbb{D}} f(w)\overline{K_{\gamma,\delta}(z,w)}\omega_{\gamma,\delta}(w)\, dA(w)$$

and

$$\|\Lambda_z\| = \big\|K_{\gamma,\delta}(z,\cdot)\big\|_{A^2_{\gamma,\delta}}.$$

Therefore, from Lemma 2 with $m = 0$, we have

$$\sqrt{K_{\gamma,\delta}(z,z)} = \big\|K_{\gamma,\delta}(z,\cdot)\big\|_{A^2_{\gamma,\delta}} \lesssim \big[\omega_{\gamma+2,\delta}(z)\big]^{-\frac{1}{2}} \tag{38}$$

for $z \in \mathbb{D} \setminus \{0\}$.

Remark 4 From the proof of Lemma 2 we see that if $-1 < \gamma < \infty$, $\delta \le 0$, and $m \in \mathbb{N}_0$, then for each $r \in (0,1)$, there is a constant $c_m(r)$ such that

$$\big|f^{(m)}(z)\big| \le c_m(r)\big[\omega_{\gamma+2m+2,\delta}(z)\big]^{-\frac{1}{2}} \|f\|_{A^2_{\gamma,\delta}}$$

for $|z| \ge r$ and every $f \in A^2_{\gamma,\delta}$.

Lemma 3 *Let* $-1 < \gamma < \infty$, $\delta \le 0$, *and* $g \in H(\mathbb{D})$. *Then a bounded operator* K *is compact on* $A^2_{\gamma,\delta}$ *if and only if, for every bounded sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $A^2_{\gamma,\delta}$ *such that* $f_n \to 0$ *uniformly on every compact subset of* $\mathbb{D}$ *as* $n \to \infty$, *it follows that* $\lim_{n\to\infty} \|Kf_n\|_{A^2_{\gamma,\delta}} = 0$.

Proof We will apply Lemma 3 in [8]. Two cases are considered separately, when $\gamma \geq \delta$ and $\gamma < \delta$.

Case $\gamma < \delta$. We show that $\|f_n - f\|_{A^2_{\gamma,\delta}} \to 0$ as $n \to \infty$ implies $f_n \to f$ uniformly on compacts of $\mathbb{D}$.

Since $\gamma < \delta$, then there exists a positive integer m such that $\gamma + 2m \geq \delta$. Therefore, by Corollary 2, we have that

$$\left|f^{(j)}(0)\right| \leq c_j \|f\|_{A^2_{\gamma,\delta}} \tag{39}$$

for $j = \overline{0, m-1}$, for some nonnegative constants c_j, $j = \overline{0, m-1}$.

From Lemma 2, we see that for each fixed $r \in [1/2, 1)$ it follows that

$$\max_{1/2 \leq |z| \leq r} \left|f^{(m)}(z)\right| \lesssim \max_{1/2 \leq |z| \leq r} \left[\omega_{\gamma+2m+2,\delta}(z)\right]^{-\frac{1}{2}} \|f\|_{A^2_{\gamma,\delta}} = \widehat{c}_m(r)\|f\|_{A^2_{\gamma,\delta}}, \tag{40}$$

where

$$\widehat{c}_m(r) = \max_{1/2 \leq t \leq r} \left[\omega_{\gamma+2m+2,\delta}(t)\right]^{-\frac{1}{2}},$$

which is a finite constant due to the continuity of the function under the sign of maximum on a compact set.

On the other hand, by the subharmonicity of the function $|f^{(m)}(z)|^2$, we have that

$$\left|f^{(m)}(z)\right|^2 \leq 16 \int_{|\zeta - z| \leq 1/4} \left|f^{(m)}(\zeta)\right|^2 dA(\zeta),$$

from which it follows that

$$\max_{|z| \leq 1/2} \left|f^{(m)}(z)\right|^2 \leq 16 \int_{|\zeta| \leq 3/4} \left|f^{(m)}(\zeta)\right|^2 dA(\zeta) \tag{41}$$

for $m \in \mathbb{N}_0$.

Since $\gamma + 2m \geq \delta$, then there is a positive constant $\widehat{c}_1$ such that

$$0 < \widehat{c}_1 \leq \inf_{|z| \leq 3/4} \omega_{\gamma+2m,\delta}(z). \tag{42}$$

Hence, from (41) and (42), we have that

$$\max_{|z| \leq 1/2} \left|f^{(m)}(z)\right|^2 \leq \frac{16}{\widehat{c}_1} \int_{|\zeta| \leq 3/4} \left|f^{(m)}(\zeta)\right|^2 \omega_{\gamma+2m,\delta}(\zeta)\, dA(\zeta) \leq \frac{16}{\widehat{c}_1} \|f\|^2_{A^2_{\gamma,\delta}}. \tag{43}$$

From (40) and (43) we have that, for each fixed $r \in [0, 1)$, there is a constant $\widetilde{c}_m(r)$ such that

$$\max_{|z| \leq r} \left|f^{(m)}(z)\right| \leq \widetilde{c}_m(r)\|f\|_{A^2_{\gamma,\delta}}, \tag{44}$$

from which it further follows that

$$\max_{|z| \leq r} \left|\left(f^{(m)} - f_n^{(m)}\right)(z)\right| \leq \widetilde{c}_m(r)\|f - f_n\|_{A^2_{\gamma,\delta}},$$

and consequently $\|f_n - f\|_{A^2_{\gamma,\delta}} \to 0$ as $n \to \infty$ implies $f_n^{(m)} \to f^{(m)}$ uniformly on compacts of $\mathbb{D}$. Moreover, (44) means that the point evaluation functionals $f \mapsto f^{(m)}(z)$ are continuous.

Since

$$f^{(m-k)}(z) = \sum_{j=0}^{k-1} \frac{f^{(m-k+j)}(0)}{j!} z^j + \int_0^z \int_0^{\zeta_k} \cdots \int_0^{\zeta_2} f^{(m)}(\zeta_1)\, d\zeta_1 \cdots d\zeta_{k-1}\, d\zeta_k$$

for each $k \in \{1, \dots, m\}$, by using (39), (44), and some simple estimates, we obtain

$$\begin{aligned} \bigl|f^{(m-k)}(z)\bigr| &\le \sum_{j=0}^{k-1} \frac{|f^{(m-k+j)}(0)|}{j!} |z|^j + \sup_{z \in r\mathbb{D}} \left| \int_0^z \int_0^{\zeta_k} \cdots \int_0^{\zeta_2} f^{(m)}(\zeta_1)\, d\zeta_1 \cdots d\zeta_{k-1}\, d\zeta_k \right| \\ &\le \sum_{j=0}^{k-1} \frac{|f^{(m-k+j)}(0)|}{j!} r^j + \sup_{z \in r\mathbb{D}} \bigl|f^{(m)}(z)\bigr| \left| \frac{z^k}{k!} \right| \\ &\le \left(\sum_{j=0}^{k-1} \frac{c_{m-k+j}}{j!} r^j + \widetilde{c}_m(r) \frac{r^k}{k!} \right) \|f\|_{A^2_{\gamma,\delta}} \end{aligned} \tag{45}$$

for every $z \in r\mathbb{D}$.

If we take $k = m$ in (45), it follows that

$$\max_{|z| \le r} \bigl|f(z)\bigr| \le \left(\sum_{j=0}^{m-1} \frac{c_j}{j!} r^j + \widetilde{c}_m(r) \frac{r^m}{m!} \right) \|f\|_{A^2_{\gamma,\delta}}, \tag{46}$$

from which it follows that

$$\max_{|z| \le r} \bigl|(f - f_n)(z)\bigr| \le \left(\sum_{j=0}^{m-1} \frac{c_j}{j!} r^j + \widetilde{c}_m(r) \frac{r^m}{m!} \right) \|f - f_n\|_{A^2_{\gamma,\delta}},$$

and consequently $\|f_n - f\|_{A^2_{\gamma,\delta}} \to 0$ as $n \to \infty$ implies $f_n \to f$ uniformly on compacts of $\mathbb{D}$. Moreover, from (46) we see that $|f(z)| \lesssim \|f\|_{A^2_{\gamma,\delta}}$, which means that the point evaluation functionals are continuous. Hence, Lemma 3 in [8] can be applied in the case.

Case $\gamma \ge \delta$. From Lemma 2 we have that, for each fixed $r \in [1/2, 1)$, it follows that

$$\max_{1/2 \le |z| \le r} \bigl|f(z)\bigr| \lesssim \max_{1/2 \le |z| \le r} \bigl[\omega_{\gamma+2,\delta}(z)\bigr]^{-\frac{1}{2}} \|f\|_{A^2_{\gamma,\delta}} = \widehat{c}_0(r) \|f\|_{A^2_{\gamma,\delta}}, \tag{47}$$

where

$$\widehat{c}_0(r) := \max_{1/2 \le t \le r} \bigl[\omega_{\gamma+2,\delta}(t)\bigr]^{-\frac{1}{2}},$$

which is a finite constant due to the continuity of the function under the sign of maximum on a compact set.

Now note that (41) holds for $m = 0$, that is, we have

$$\max_{|z| \le 1/2} \bigl|f(z)\bigr|^2 \le 16 \int_{|\zeta| \le 3/4} \bigl|f(\zeta)\bigr|^2\, dA(\zeta). \tag{48}$$

Since $\gamma \geq \delta$, by using (3), it is not difficult to see that there is a positive constant c_2 such that

$$0 < c_2 \leq \inf_{|z|\leq 3/4} \omega_{\gamma,\delta}(z). \tag{49}$$

Hence, from (48) and (49), we have

$$\max_{|z|\leq 1/2} \left|f(z)\right|^2 \leq \frac{16}{c_2} \int_{|\zeta|\leq 3/4} \left|f(\zeta)\right|^2 \omega_{\gamma,\delta}(\zeta)\, dA(\zeta) \leq \frac{16}{c_2} \|f\|^2_{A^2_{\gamma,\delta}}. \tag{50}$$

From (47) and (50) we have that, for each fixed $r \in [0,1)$, there is a constant $\widetilde{c}_0(r)$ such that

$$\max_{|z|\leq r}\left|f(z)\right| \leq \widetilde{c}_0(r)\|f\|_{A^2_{\gamma,\delta}}, \tag{51}$$

from which it follows that

$$\max_{|z|\leq r}\left|(f - f_n)(z)\right| \leq \widetilde{c}_0(r)\|f - f_n\|_{A^2_{\gamma,\delta}},$$

and consequently $\|f_n - f\|_{A^2_{\gamma,\delta}} \to 0$ as $n \to \infty$ implies $f_n \to f$ uniformly on compacts of $\mathbb{D}$. Moreover, (51) means that the point evaluation functionals are continuous, which is the other necessary condition for applying Lemma 3 in [8], finishing the proof of the lemma. □

Remark 5 Note that the proof of Lemma 3 is considerably more complex than those for the case of some other spaces of analytic functions. For example, the corresponding result for the weighted Bloch space can be found in [8], for the Zygmund-type space in [12], while for the Besov and BMOA spaces in [22]. The first result of the type seems the one in [22], but it has some inaccuracies.

Lemma 4 *Let $-1 < \gamma < \infty$, $\delta \leq 0$, $r_0 \in (0,1)$, and $g \in H(\mathbb{D})$. If I_g is bounded on $A^2_{\gamma,\delta}$, then there exists a positive constant C independent of $f \in A^2_{\gamma,\delta}$ and $a \in \mathbb{D} \setminus r_0\mathbb{D}$ such that*

$$\left|f(a)\right|\left|g(a)\right| \leq C\|I_g f\|_{A^2_{\gamma,\delta}} \left\|K_{\gamma+2,\delta}(a,\cdot)\right\|_{A^2_{\gamma+2,\delta}}. \tag{52}$$

Proof Since I_g is bounded on $A^2_{\gamma,\delta}$, by Lemma 1 we have that $f(z)g(z) = (I_g f)'(z) \in A^2_{\gamma+2,\delta}$ for every $f \in A^2_{\gamma,\delta}$. Then, by the reproducing kernel of $A^2_{\gamma+2,\delta}$, for each $a \in \mathbb{D} \setminus r_0\mathbb{D}$ it follows that

$$f(a)g(a) = \int_{\mathbb{D}} f(z)g(z)\overline{K_{\gamma+2,\delta}(a,z)}\omega_{\gamma+2,\delta}(z)\, dA(z). \tag{53}$$

From (53), the Cauchy–Schwarz inequality, and Lemma 1, it follows that

$$\begin{aligned}
\left|f(a)g(a)\right| &\leq \int_{\mathbb{D}} \left|f(z)g(z)K_{\gamma+2,\delta}(a,z)\right|\omega_{\gamma+2,\delta}(z)\, dA(z)\\
&\leq \left(\int_{\mathbb{D}} \left|f(z)g(z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z)\right)^{\frac{1}{2}}\\
&\quad\times \left(\int_{\mathbb{D}} \left|K_{\gamma+2,\delta}(a,z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z)\right)^{\frac{1}{2}}
\end{aligned}$$

$$= \|(I_g f)'\|_{A^2_{\gamma+2,\delta}} \|K_{\gamma+2,\delta}(a,\cdot)\|_{A^2_{\gamma+2,\delta}}$$
$$\le C\|I_g f\|_{A^2_{\gamma,\delta}} \|K_{\gamma+2,\delta}(a,\cdot)\|_{A^2_{\gamma+2,\delta}}.$$

This finishes the proof of the lemma. □

For fixed $a \in \mathbb{D}$, let

$$k_a(z) = \frac{(1-|a|)^{-\delta}}{(1-\overline{a}z)^{\frac{\gamma}{2}+1-\delta}} \left[\ln\left(1-\frac{1}{\ln|a|}\right)\right]^{-\frac{\delta}{2}}, \quad z \in \mathbb{D}. \tag{54}$$

It is easy to see that

$$k_a(a) \asymp \frac{1}{(1-|a|)^{\frac{\gamma}{2}+1}} \left[\ln\left(1-\frac{1}{\ln|a|}\right)\right]^{-\frac{\delta}{2}}. \tag{55}$$

It was essentially proved in [2] that for each $r \in (0,1)$

$$\sup_{|a|\ge r} \|k_a\|_{A^2_{\gamma,\delta}} \lesssim 1. \tag{56}$$

When $\delta < 0$, it is easily seen that $k_a \to 0$ uniformly on compacts of $\mathbb{D}$ as $|a| \to 1$. Note that this is not true if $\delta = 0$, a claim which is stated in [2] as a minor oversight.

Let

$$\varphi_w(z) = \frac{w-z}{1-\overline{w}z}$$

and $D(w,\delta) = \{z : |\varphi_w(z)| < \delta\}$.

The following simple lemma is proved by a slight modification of the proof of Lemma 3.2 in [18], so the proof is omitted.

Lemma 5 *Let $r_0 \in (0,1)$, $a \in \mathbb{D} \setminus r_0\mathbb{D}$. Then*

$$\ln\left(1-\frac{1}{\ln|w|}\right) \asymp \ln\left(1-\frac{1}{\ln|a|}\right)$$

for $w \in D(a,1/2)$.

Now we present a lower bound estimate for the quantity $\|k_a\|_{A^2_{\gamma,\delta}}$.

Lemma 6 *Let $-1 < \gamma < \infty$, $\delta < 0$, $r_0 \in (0,1)$, and $a \in \mathbb{D} \setminus r_0\mathbb{D}$. Then $\|k_a\|_{A^2_{\gamma,\delta}} \gtrsim 1$.*

Proof By Lemma 1 and the expression of k_a, we have

$$\begin{aligned}\|k_a\|^2_{A^2_{\gamma,\delta}} &\ge C_1 \int_{\mathbb{D}} \left|k_a'(w)\right|^2 \omega_{\gamma+2,\delta}(w)\,dA(w) \\ &= c(\gamma,\delta,a) \int_{\mathbb{D}} \frac{1}{|1-\overline{a}w|^{\gamma-2\delta+4}} \omega_{\gamma+2,\delta}(w)\,dA(w),\end{aligned} \tag{57}$$

where

$$c(\gamma,\delta,a)=\left(\frac{\gamma}{2}+1-\delta\right)^2|a|^2\big(1-|a|^2\big)^{-2\delta}\left[\ln\left(1-\frac{1}{\ln|a|}\right)\right]^{-\delta}C_1.$$

On the other hand,

$$\int_{\mathbb{D}}\frac{1}{|1-\overline{a}u|^{\gamma-2\delta+4}}\omega_{\gamma+2,\delta}(u)\,dA(u)\gtrsim\frac{1}{(1-|a|^2)^{\gamma-2\delta+4}}\int_{D(a,1/2)}\omega_{\gamma+2,\delta}(u)\,dA(u). \tag{58}$$

Then, from Lemma 5 and since $1-|w|\asymp 1-|a|$ and $A(D(a,1/2))\asymp(1-|a|^2)^2$, when $w\in D(a,1/2)$, it follows that

$$\int_{D(a,1/2)}\omega_{\gamma+2,\delta}(w)\,dA(w)\geq C_2\big(1-|a|^2\big)^{\gamma+4}\left[\ln\left(1-\frac{1}{\ln|a|}\right)\right]^{\delta}. \tag{59}$$

Hence, from (57)–(59), we have that

$$\|k_a\|^2_{A^2_{\gamma,\delta}}\geq r_0^2C_1C_2\left(\frac{\gamma}{2}+1-\delta\right)^2$$

for $|a|\geq r_0$, from which the desired result follows. □

Remark 6 With the help of Lemma 6, we see that when $\delta<0$ the functions

$$f_a(z)=\frac{k_a(z)}{\|k_a\|_{A^2_{\gamma,\delta}}},\quad z\in\mathbb{D},$$

converge uniformly to zero on the compact subsets of $\mathbb{D}$ as $|a|\to 1$. The functions f_a will be used in the characterization of the compactness of I_g on $A^2_{\gamma,\delta}$.

3 Main results and proofs

First, we characterize the boundedness of the operator I_g on $A^2_{\gamma,\delta}$.

Theorem 1 *Let $-1<\gamma<\infty$, $\delta<0$, and $g\in H(\mathbb{D})$. Then the operator I_g is bounded on $A^2_{\gamma,\delta}$ if and only if $g\in H_1^\infty$.*

Proof Suppose that $g\in H_1^\infty$. First, note that

$$(I_gf)'(z)=f(z)g(z)\quad\text{and}\quad I_gf(0)=0. \tag{60}$$

Using (60), Corollary 4, and the fact that

$$\ln\frac{1}{x}\asymp 1-x \tag{61}$$

for $x \in (r,1)$, where r is a fixed number in the interval $(0,1)$, it follows that

$$\begin{aligned}\|I_g f\|^2_{A^2_{\gamma,\delta}} &\asymp \int_{\mathbb{D}\setminus r\mathbb{D}} \bigl|(I_g f)'(z)\bigr|^2 \omega_{\gamma+2,\delta}(z)\,dA(z)\\ &= \int_{\mathbb{D}\setminus r\mathbb{D}} \bigl|f(z)\bigr|^2\bigl|g(z)\bigr|^2 \omega_{\gamma+2,\delta}(z)\,dA(z)\\ &\le \sup_{\{z\in\mathbb{D}:|z|\ge r\}} \left(\ln\frac{1}{|z|}\right)^2 \bigl|g(z)\bigr|^2 \|f\|^2_{A^2_{\gamma,\delta}}\\ &\lesssim \sup_{\{z\in\mathbb{D}:|z|\ge r\}} \bigl(1-|z|\bigr)^2\bigl|g(z)\bigr|^2 \|f\|^2_{A^2_{\gamma,\delta}}\\ &\le \|g\|^2_{H_1^\infty}\|f\|^2_{A^2_{\gamma,\delta}}. \end{aligned} \tag{62}$$

From (62), it follows that I_g is bounded on $A^2_{\gamma,\delta}$, and moreover that $\|I_g\| \lesssim \|g\|_{H_1^\infty}$.

Conversely, suppose that I_g is bounded on $A^2_{\gamma,\delta}$ and that $r \in (0,1)$. By Lemma 4, we obtain

$$\bigl|f(a)\bigr|\bigl|g(a)\bigr| \le C\|I_g f\|_{A^2_{\gamma,\delta}} \bigl\|K_{\gamma+2,\delta}(a,\cdot)\bigr\|_{A^2_{\gamma+2,\delta}} \tag{63}$$

for $f \in A^2_{\gamma,\delta}$ and $a \in \mathbb{D} \setminus r\mathbb{D}$. Specially, for every $f \in A^2_{\gamma,\delta}$ with $f(a) \ne 0$, by (63) we have

$$\bigl(1-|a|\bigr)\bigl|g(a)\bigr| \le C\frac{(1-|a|)}{|f(a)|}\|I_g f\|_{A^2_{\gamma,\delta}} \bigl\|K_{\gamma+2,\delta}(a,\cdot)\bigr\|_{A^2_{\gamma+2,\delta}}. \tag{64}$$

Applying (64) to the function k_a, from (38), (55), and (56), it follows that

$$\bigl(1-|a|\bigr)\bigl|g(a)\bigr| \le C\frac{(1-|a|)}{|k_a(a)|}\|I_g\|\|k_a\|_{A^2_{\gamma,\delta}} \bigl\|K_{\gamma+2,\delta}(a,\cdot)\bigr\|_{A^2_{\gamma+2,\delta}} \lesssim \|I_g\|,$$

from which it follows that

$$\sup_{|a|\ge r}\bigl(1-|a|\bigr)\bigl|g(a)\bigr| < +\infty. \tag{65}$$

From (65) along with the obvious fact

$$\sup_{|a|\le r}\bigl(1-|a|\bigr)\bigl|g(a)\bigr| < +\infty,$$

we obtain that $g \in H_1^\infty$. □

Next we characterize the compactness of the operator I_g on $A^2_{\gamma,\delta}$. We will present a direct proof of it for the presentational reasons and an obvious connectedness with Theorem 1, although it will be also a consequence of our next theorem.

Theorem 2 *Let $-1 < \gamma < \infty$, $\delta < 0$, and $g \in H(\mathbb{D})$. Then the operator I_g is compact on $A^2_{\gamma,\delta}$ if and only if $g \in H^\infty_{1,0}(\mathbb{D})$.*

Proof First, assume that $g \in H^\infty_{1,0}(\mathbb{D})$. Then, for sufficiently small $\varepsilon > 0$, there exists $\delta_0 \in (0,1)$ such that

$$\bigl(1-|z|\bigr)\bigl|g(z)\bigr| < \varepsilon \tag{66}$$

for $z \in \mathbb{D} \setminus \delta_0\mathbb{D}$.

Now note that by (3) and since $\gamma - \delta > -1$, we have

$$\int_{\delta_0\mathbb{D}} \omega_{\gamma+2,\delta}(z)\,dA(z) \asymp \int_0^{\delta_0}\int_0^{2\pi}\left(\ln\frac{1}{\rho}\right)^{\gamma-\delta+2}\rho\,d\theta\,d\rho = 2\pi\int_{\ln\frac{1}{\delta_0}}^{\infty}\frac{s^{\gamma-\delta+2}}{e^{2s}}\,ds < \infty. \tag{67}$$

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $A^2_{\gamma,\delta}$ such that $\|f_n\|_{A^2_{\gamma,\delta}} \le M$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$.

For above chosen ε, choose a positive integer N such that

$$\bigl|f_n(z)\bigr| < \frac{\varepsilon}{\int_{\delta_0\mathbb{D}}|g(z)|^2\omega_{\gamma+2,\delta}(z)\,dA(z)} \tag{68}$$

for all $z \in \delta_0\mathbb{D}$ and $n \ge N$, which is possible since due to (67) we have

$$\int_{\delta_0\mathbb{D}}\bigl|g(z)\bigr|^2\omega_{\gamma+2,\delta}(z)\,dA(z) \le \max_{|z|\le\delta_0}\bigl|g(z)\bigr|^2\int_{\delta_0\mathbb{D}}\omega_{\gamma+2,\delta}(z)\,dA(z) < \infty.$$

By using (61), (66), and (68), we have

$$\begin{aligned}
\|I_gf_n\|^2_{A^2_{\gamma,\delta}} &= \int_{\mathbb{D}}\bigl|I_gf_n(z)\bigr|^2\omega_{\gamma,\delta}(z)\,dA(z)\\
&\le C\int_{\mathbb{D}}\bigl|f_n(z)g(z)\bigr|^2\omega_{\gamma+2,\delta}(z)\,dA(z)\\
&\lesssim \int_{\delta_0\mathbb{D}} + \int_{\mathbb{D}\setminus\delta_0\mathbb{D}}\bigl|f_n(z)g(z)\bigr|^2\omega_{\gamma+2,\delta}(z)\,dA(z)\\
&\lesssim \varepsilon^2 + \varepsilon^2\|f_n\|^2_{A^2_{\gamma,\delta}}.
\end{aligned}$$

Thus, the condition $g \in H^\infty_{1,0}(\mathbb{D})$ implies the compactness of I_g on $A^2_{\gamma,\delta}$.

Now suppose that I_g is compact on $A^2_{\gamma,\delta}$. Put

$$f_a(z) = \frac{k_a(z)}{\|k_a\|_{A^2_{\gamma,\delta}}}.$$

As we have seen in Remark 6, $f_a \to 0$ uniformly on compacts of $\mathbb{D}$ as $|a| \to 1$. Since I_g is compact on $A^2_{\gamma,\delta}$, by Lemma 3 we have

$$\lim_{|a|\to1^-}\|I_gf_a\|_{A^2_{\gamma,\delta}} = 0.$$

From (64) we have

$$\begin{aligned}\bigl(1-|a|\bigr)\bigl|g(a)\bigr| &\lesssim \frac{1-|a|}{|f_a(a)|}\|I_g f_a\|_{A^2_{\gamma,\delta}}\bigl\|K_{\gamma+2,\delta}(a,\cdot)\bigr\|_{A^2_{\gamma+2,\delta}}\\ &\lesssim \|I_g f_a\|_{A^2_{\gamma,\delta}} \to 0\end{aligned} \tag{69}$$

as $|a|\to 1$, from which it follows that $g\in H^\infty_{1,0}(\mathbb{D})$. □

We now consider the essential norm of the operator I_g on $A^2_{\gamma,\delta}$. Before this we formulate and prove an auxiliary result on a complete orthonormal system in $A^2_{\gamma,\delta}$. For some basics on the topic see, for example, [23] and [24].

Lemma 7 *Let*

$$\varphi_n(z)=\frac{z^n}{\|z^n\|_{A^2_{\gamma,\delta}}},\quad n\in\mathbb{N}_0. \tag{70}$$

Then the sequence $(\varphi_n(z))_{n\in\mathbb{N}_0}$ *is a complete orthonormal system in* $A^2_{\gamma,\delta}$.

Proof By using the polar coordinates, we have

$$\begin{aligned}\langle\varphi_n,\varphi_m\rangle &= \int_{\mathbb{D}}\varphi_n(z)\overline{\varphi_m(z)}\omega_{\gamma,\delta}(z)\,dA(z)\\ &= \frac{1}{\pi\|z^n\|_{A^2_{\gamma,\delta}}\|z^m\|_{A^2_{\gamma,\delta}}}\int_0^1\int_0^{2\pi} r^n e^{in\theta} r^m e^{-im\theta}\omega_{\gamma,\delta}(r)r\,dr\,d\theta\\ &= \frac{1}{\pi\|z^n\|_{A^2_{\gamma,\delta}}\|z^m\|_{A^2_{\gamma,\delta}}}\int_0^1 r^{n+m+1}\omega_{\gamma,\delta}(r)\,dr\int_0^{2\pi} e^{i(n-m)\theta}\,d\theta,\end{aligned} \tag{71}$$

from which it easily follows that

$$\bigl\langle z^n,z^m\bigr\rangle=\delta_{n,m},$$

where $\delta_{n,m}$ is the δ Kronecker symbol. Hence, the system is orthonormal.

To prove that the system is complete, we prove that the system span $A^2_{\gamma,\delta}$, for which it is enough to prove that the polynomials are dense in $A^2_{\gamma,\delta}$.

Let $f\in A^2_{\gamma,\delta}$. Then, for every $\varepsilon>0$, there is $r_0\in(0,1)$ such that

$$\int_{\mathbb{D}\setminus\overline{r_0\mathbb{D}}}\bigl|f(z)\bigr|^2\omega_{\gamma,\delta}(z)\,dA(z)<\varepsilon,$$

which is equivalent to

$$2\int_{r_0}^1 M_2^2(f,r)\omega_{\gamma,\delta}(r)\,dr<\varepsilon. \tag{72}$$

Since $M_2(f,r)$ is a nondecreasing function, we have also that

$$2\int_{r_0}^1 M_2^2(f_\rho,r)\omega_{\gamma,\delta}(r)\,dr<\varepsilon \tag{73}$$

for every $\rho\in[0,1)$, where $f_\rho(z)=f(\rho z)$.

Due to the uniform continuity of f on the compact $\overline{r_0\mathbb{D}}$, we have that, for every $\varepsilon > 0$, there is $\delta_0 \in (0,1)$ such that

$$\left|f_\rho(z) - f(z)\right| < \sqrt{\varepsilon} \tag{74}$$

for every $\rho \in (1-\delta_0, 1)$ and $z \in \overline{r_0\mathbb{D}}$.

By using (72)–(74), we have

$$\begin{aligned}\|f_\rho - f\|^2_{A^2_{\gamma,\delta}} &= \int_{\mathbb{D}} \left|f(\rho z) - f(z)\right|^2 \omega_{\gamma,\delta}(z)\,dA(z) \\ &= \int_{\overline{r_0\mathbb{D}}} \left|f(\rho z) - f(z)\right|^2 \omega_{\gamma,\delta}(z)\,dA(z) + \int_{\mathbb{D}\setminus\overline{r_0\mathbb{D}}} \left|f(\rho z) - f(z)\right|^2 \omega_{\gamma,\delta}(z)\,dA(z) \\ &< \varepsilon \int_{\overline{r_0\mathbb{D}}} \omega_{\gamma,\delta}(z)\,dA(z) + 2\int_{\mathbb{D}\setminus\overline{r_0\mathbb{D}}} \left(\left|f(\rho z)\right|^2 + \left|f(z)\right|^2\right)\omega_{\gamma,\delta}(z)\,dA(z) \\ &< \varepsilon\left(\int_{\mathbb{D}} \omega_{\gamma,\delta}(z)\,dA(z) + 2\right).\end{aligned} \tag{75}$$

Since $f \in H(\mathbb{D})$, the Taylor polynomials $s_n(f_\rho) = \sum_{j=0}^{n-1} a_j(f_\rho)z^j$ of f_ρ converge uniformly to the function on $\mathbb{D}$. Hence, for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}_0$ such that

$$\left|f_\rho(z) - s_n(f_\rho)(z)\right| < \sqrt{\varepsilon} \tag{76}$$

for $n \geq n_0$ and $z \in \mathbb{D}$.

From (76), we have

$$\begin{aligned}\left\|f_\rho - s_n(f_\rho)\right\|^2_{A^2_{\gamma,\delta}} &= \int_{\mathbb{D}} \left|f(\rho z) - s_n(f_\rho)(z)\right|^2 \omega_{\gamma,\delta}(z)\,dA(z) \\ &< \varepsilon \int_{\mathbb{D}} \omega_{\gamma,\delta}(z)\,dA(z).\end{aligned} \tag{77}$$

From (75) and (77), it follows that

$$\left\|f - s_n(f_\rho)\right\|_{A^2_{\gamma,\delta}} < \sqrt{\varepsilon}\left(\left(\int_{\mathbb{D}} \omega_{\gamma,\delta}(z)\,dA(z)\right)^{1/2} + \left(\int_{\mathbb{D}} \omega_{\gamma,\delta}(z)\,dA(z) + 2\right)^{1/2}\right) \tag{78}$$

for $\rho \in (1-\delta_0, 1)$, from which the result follows. □

Theorem 3 *Let* $-1 < \gamma < \infty$, $\delta < 0$, *and* $g \in H(\mathbb{D})$. *If the operator* I_g *is bounded on* $A^2_{\gamma,\delta}$, *then*

$$\|I_g\|_e \asymp A := \limsup_{|z|\to 1} \left(1 - |z|\right)\left|g(z)\right|.$$

Proof Let $K : A^2_{\gamma,\delta} \to A^2_{\gamma,\delta}$ be compact and k_a be the functions defined in (54). Then by Lemma 3 we have that $\|Kk_a\|_{A^2_{\gamma,\delta}} \to 0$ as $|a| \to 1-0$, from which it follows that

$$\begin{aligned} \|I_g - K\| &\geq \limsup_{|a|\to 1} \|I_g k_a - Kk_a\|_{A^2_{\gamma,\delta}} \\ &\geq \limsup_{|a|\to 1} \|I_g k_a\|_{A^2_{\gamma,\delta}} - \limsup_{|a|\to 1} \|Kk_a\|_{A^2_{\gamma,\delta}} \\ &= \limsup_{|a|\to 1} \|I_g k_a\|_{A^2_{\gamma,\delta}}. \end{aligned} \tag{79}$$

By (55) and Remark 3, we have that

$$\big\|K_{\gamma+2,\delta}(a,\cdot)\big\|_{A^2_{\gamma+2,\delta}} \lesssim \big(1-|a|\big)^{-1} k_a(a). \tag{80}$$

Hence, from Lemma 4 it follows that

$$\big|k_a(a)g(a)\big| \leq C\|I_g k_a\|_{A^2_{\gamma,\delta}} \big\|K_{\gamma+2,\delta}(a,\cdot)\big\|_{A^2_{\gamma+2,\delta}} \lesssim \|I_g k_a\|_{A^2_{\gamma,\delta}} \big(1-|a|\big)^{-1} k_a(a)$$

for $|a| \geq r > 0$, from which we obtain

$$\big(1-|a|\big)\big|g(a)\big| \lesssim \|I_g k_a\|_{A^2_{\gamma,\delta}}. \tag{81}$$

From (79) and (81), we have

$$\|I_g - K\| \gtrsim \limsup_{|a|\to 1} \big(1-|a|\big)\big|g(a)\big|,$$

which shows that

$$\|I_g\|_e \gtrsim \limsup_{|a|\to 1} \big(1-|a|\big)\big|g(a)\big|. \tag{82}$$

For a holomorphic function $f(z) = \sum_{m=0}^{\infty} a_m z^m$ on $\mathbb{D}$, let

$$P_j f(z) = \sum_{m=0}^{j} a_m z^m$$

and

$$R_j f(z) = \sum_{m=j+1}^{\infty} a_m z^m.$$

As a finite-rank operator, P_j is compact on $A^2_{\gamma,\delta}$, and

$$\|I_g\|_e = \big\|I_g(P_j + R_j)\big\|_e \leq \|I_g P_j\|_e + \|I_g R_j\|_e = \|I_g R_j\|_e \leq \|I_g R_j\| \tag{83}$$

for each $j \in \mathbb{N}$.

Thus

$$\|I_g\|_e \le \liminf_{j\to\infty} \|I_g R_j\|.$$

Hence, we have

$$\begin{aligned}
\|I_g\|_e^2 &\le \liminf_{j\to\infty} \|I_g R_j\|^2 = \liminf_{j\to\infty} \sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1} \|I_g R_j f\|^2_{A^2_{\gamma,\delta}} \\
&\asymp \liminf_{j\to\infty} \sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1} \int_{\mathbb{D}} \left|(I_g R_j f)'(z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z) \\
&= \liminf_{j\to\infty} \sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1} \int_{\mathbb{D}} \left|R_j f(z) g(z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z). \qquad (84)
\end{aligned}$$

For every $\varepsilon > 0$, there is $r_0 \in (0,1)$ such that

$$\bigl(1-|z|\bigr)\left|g(z)\right| < A + \varepsilon \qquad (85)$$

for $r_0 \le |z| < 1$.

From (84) and (85), it follows that

$$\begin{aligned}
\|I_g\|_e^2 &\le \liminf_{j\to\infty} \Biggl(\sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1} \int_{r_0\mathbb{D}} \left|R_j f(z) g(z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z) \\
&\quad + \sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1} \int_{\mathbb{D}\setminus r_0\mathbb{D}} \left|R_j f(z) g(z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z) \Biggr) \\
&\lesssim \liminf_{j\to\infty} \Biggl(\int_{r_0\mathbb{D}} \left|g(z)\right|^2 \omega_{\gamma+2,\delta}(z)\, dA(z) \sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1} M_\infty^2(R_j f, r_0) \\
&\quad + (A+\varepsilon)^2 \sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1} \int_{\mathbb{D}\setminus r_0\mathbb{D}} \left|R_j f(z)\right|^2 \omega_{\gamma,\delta}(z)\, dA(z) \Biggr). \qquad (86)
\end{aligned}$$

By Lemma 7 we know that the system $\varphi_n(z) = z^n/\|z^n\|_{A^2_{\gamma,\delta}}$, $n \in \mathbb{N}_0$, is a complete orthonormal system in the space $A^2_{\gamma,\delta}$. Then, by a known theory, we have that the reproducing kernel of $A^2_{\gamma,\delta}$ is

$$K_{\gamma,\delta}(z,w) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)} = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{\|z^n\|^2_{A^2_{\gamma,\delta}}}.$$

Now we estimate the quantity of $\|z^n\|^2_{A^2_{\gamma,\delta}}$. From Proposition 1, we have

$$\begin{aligned}
\|z^n\|^2_{A^2_{\gamma,\delta}} &\asymp \int_{\mathbb{D}} |z|^{2n} \bigl(1-|z|\bigr)^\gamma \left(\ln \frac{e}{1-|z|}\right)^\delta dA(z) \\
&= \int_0^1 \int_0^{2\pi} r^{2n+1} (1-r)^\gamma \left(\ln \frac{e}{1-r}\right)^\delta \frac{dr\, d\theta}{\pi}
\end{aligned}$$

$$\begin{aligned}
&\ge 2\int_0^{(1+r_0)/2} r^{2n+1}(1-r)^\gamma \left(\ln\frac{e}{1-r}\right)^\delta dr \\
&\gtrsim \int_0^{(1+r_0)/2} r^{2n+1}\,dr \\
&= \frac{1}{2n+2}\left(\frac{1+r_0}{2}\right)^{2n+2}.
\end{aligned} \tag{87}$$

Since $K_{\gamma,\delta}(z,w)$ is the kernel of $A^2_{\gamma,\delta}$ and $R_jf \in A^2_{\gamma,\delta}$ for $f \in A^2_{\gamma,\delta}$ (see (95) below), we have

$$R_jf(z) = \int_{\mathbb{D}} R_jf(w)\overline{K_{\gamma,\delta}(z,w)}\omega_{\gamma,\delta}(w)\,dA(w).$$

The orthogonality of w^n with respect to $\omega_{\gamma,\delta}(w)\,dA(w)$ shows that

$$R_jf(z) = \int_{\mathbb{D}} f(w)R_j\overline{K_{\gamma,\delta}(z,w)}\omega_{\gamma,\delta}(w)\,dA(w) \tag{88}$$

(for a related idea, see [25]).

From the expression of $K_{\gamma,\delta}(z,w)$, the Cauchy–Schwarz inequality, and by using (21), we have

$$\begin{aligned}
\left|R_jf(z)\right| &\le \int_{\mathbb{D}} \left|f(w)\right|\left|R_j\overline{K_{\gamma,\delta}(z,w)}\right|\omega_{\gamma,\delta}(w)\,dA(w) \\
&\le \|f\|_{A^2_{\gamma,\delta}}\left(\int_{\mathbb{D}}\left(\sum_{n=j+1}^{\infty}\frac{|z^n||w^n|}{\|z^n\|^2_{A_{\gamma,\delta}}}\right)^2\omega_{\gamma,\delta}(w)\,dA(w)\right)^{\frac{1}{2}} \\
&\le \|f\|_{A^2_{\gamma,\delta}}\left(\int_{\mathbb{D}}\omega_{\gamma,\delta}(w)\,dA(w)\right)^{1/2}\sum_{n=j+1}^{\infty}\frac{|z|^n}{\|z^n\|^2_{A_{\gamma,\delta}}} \\
&\lesssim \|f\|_{A^2_{\gamma,\delta}}\sum_{n=j+1}^{\infty}\frac{|z|^n}{\|z^n\|^2_{A_{\gamma,\delta}}}.
\end{aligned} \tag{89}$$

Therefore, from (87) and (89) it follows that

$$\left|R_jf(z)\right| \lesssim \|f\|_{A^2_{\gamma,\delta}}\sum_{n=j+1}^{\infty}(2n+2)\left(\frac{2}{1+r_0}\right)^{2n}|z|^n. \tag{90}$$

Now, from (90) we have

$$M_\infty(R_jf, r_0) = \max_{|z|=r_0}\left|R_jf(z)\right| \lesssim \|f\|_{A^2_{\gamma,\delta}}\sum_{n=j+1}^{\infty}(2n+2)\left(\frac{4r_0}{(1+r_0)^2}\right)^n. \tag{91}$$

Since for $r_0 \in (0,1)$ we have

$$\frac{4r_0}{(1+r_0)^2} \in (0,1),$$

the series

$$\sum_{n=0}^{\infty}(2n+2)\left(\frac{4r_0}{(1+r_0)^2}\right)^n$$

converges, which shows that

$$\lim_{j\to\infty}\sum_{n=j+1}^{\infty}(2n+2)\left(\frac{4r_0}{(1+r_0)^2}\right)^n=0. \tag{92}$$

Using (92) in (91), it follows that

$$\lim_{j\to\infty}M_\infty(R_jf,r_0)=0. \tag{93}$$

Employing (93) in (86), we obtain

$$\|I_g\|_e^2\le(A+\varepsilon)^2\liminf_{j\to\infty}\sup_{\|f\|_{A^2_{\gamma,\delta}}\le 1}\int_{\mathbb{D}\setminus r_0\mathbb{D}}\left|R_jf(z)\right|^2\omega_{\gamma,\delta}(z)\,dA(z). \tag{94}$$

On the other hand, since

$$\begin{aligned}\int_{\mathbb{D}\setminus r_0\mathbb{D}}\left|P_jf(z)\right|^2\omega_{\gamma,\delta}(z)\,dA(z)&\le\int_{\mathbb{D}}\left|P_jf(z)\right|^2\omega_{\gamma,\delta}(z)\,dA(z)\\&=\int_0^1\int_0^{2\pi}\left|\sum_{n=0}^{j}a_nr^ne^{in\theta}\right|^2\omega_{\gamma,\delta}(r)r\frac{dr\,d\theta}{\pi}\\&=2\sum_{n=0}^{j}|a_n|^2\int_0^1 r^{2n+1}\omega_{\gamma,\delta}(r)\,dr\\&\le 2\sum_{n=0}^{\infty}|a_n|^2\int_0^1 r^{2n+1}\omega_{\gamma,\delta}(r)\,dr\\&=\|f\|^2_{A^2_{\gamma,\delta}},\end{aligned}$$

we have

$$\|R_jf\|_{A^2_{\gamma,\delta}}=\|f-P_jf\|_{A^2_{\gamma,\delta}}\le 2\|f\|_{A^2_{\gamma,\delta}}. \tag{95}$$

From (94) and (95), we obtain

$$\|I_g\|_e^2\le 4(A+\varepsilon)^2.$$

From this and since ε is an arbitrary positive number, it follows that $\|I_g\|_e\le 2A$, finishing the proof of the theorem. □

Remark 7 If $\delta=0$, then by using the following test functions

$$k_a(z)=\frac{(1-|a|)^c}{(1-\overline{a}z)^{\frac{\gamma}{2}+1+c}},\quad z\in\mathbb{D},$$

for $a \in \mathbb{D}$ and some fixed $c > 0$, it can be proved in the same fashion that Theorems 1–3 also hold in the case.

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Author details
[1]Mathematical Institute of the Serbian Academy of Sciences, Beograd, Serbia. [2]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, Republic of China. [3]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan, Republic of China. [4]School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, P.R. China.

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