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On a new class of convex functions and integral inequalities

Shanhe Wu[1*], Muhammad Uzair Awan[2], Muhammad Aslam Noor[3], Khalida Inayat Noor[3] and Sabah Iftikhar[3]

*Correspondence:
shanhewu@163.com
[1]Department of Mathematics, Longyan University, Longyan, China
Full list of author information is available at the end of the article

Abstract
The aim of this paper is to introduce a new extension of convexity called σ-convexity. We show that the class of σ-convex functions includes several other classes of convex functions. Some new integral inequalities of Hermite–Hadamard type are established to illustrate the applications of σ-convex functions.

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1 Introduction

A set $\mathcal{K} \subset \mathbb{R}$ is said to be convex if $\forall x, y \in \mathcal{K}$, $t \in [0,1]$, we have

$$(1-t)x + ty \in \mathcal{K}.$$

A function $f : \mathcal{K} \to \mathbb{R}$ is said to be convex in the classical sense, if $\forall x, y \in \mathcal{K}$, $t \in [0,1]$, we have

$$f\big((1-t)x + ty\big) \leq (1-t)f(x) + tf(y).$$

The theory of convexity plays a vital role in different fields of pure and applied sciences. Consequently the classical concepts of convex sets and convex functions have been generalized in different directions. For more information, see [1–3]. Another aspect due to which the convexity theory has attracted many researchers is its close relation with theory of inequalities. Many famous inequalities can be obtained using the concept of convex functions. For details, interested readers are referred to [4–14]. Among these inequalities, Hermite–Hadamard's inequality, which provides us a necessary and sufficient condition for a function to be convex, is one of the most studied results. This result of Hermite and Hadamard reads as follows:

Theorem 1.1 *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

The main motivation of this paper is to introduce the notions of σ-convex sets and σ-convex functions by using the quasi-arithmetic means which can bring together all the power means $\mathcal{M}_p$ for $p \in \mathbb{R}$. More specifically, we define the σ-convex functions via the formula

$$\mathcal{M}_{[\sigma]}(x,y) := \sigma^{-1}\bigl[(1-t)\sigma(x) + t\sigma(y)\bigr],$$

which is associated with the strictly monotonic continuous function σ. As applications of the σ-convex functions, we derive some new Hermite–Hadamard-like inequalities. In the meantime some important special cases will also be discussed in detail.

2 σ-convex functions

Let us now introduce new classes of σ-convex sets and σ-convex functions.

Definition 2.1 A set $\mathcal{Q} \subset \mathbb{R}$ is said to be σ-convex set with respect to strictly monotonic continuous function σ if

$$\mathcal{M}_{[\sigma]}(x,y) := \sigma^{-1}\bigl((1-t)\sigma(x) + t\sigma(y)\bigr) \in \mathcal{Q}, \quad \forall x,y \in \mathcal{Q}, t \in [0,1].$$

Definition 2.2 A function $f : \mathcal{Q} \to \mathbb{R}$ is said to be σ-convex function with respect to strictly monotonic continuous function σ if

$$f\bigl(\mathcal{M}_{[\sigma]}(x,y)\bigr) \le (1-t)f(x) + tf(y), \quad \forall x,y \in \mathcal{Q}, t \in [0,1]. \tag{2.1}$$

Note that the function f is called strictly σ-convex on $\mathcal{Q}$ if the above inequality is true as a strict inequality for each distinct x and $y \in \mathcal{Q}$ and for each $t \in (0,1)$.

The function $f : \mathcal{Q} \to \mathbb{R}$ is called σ-concave (strictly σ-concave) on $\mathcal{Q}$, if $-f$ is σ-convex (strictly σ-convex) on $\mathcal{Q}$.

If we take $t = \frac{1}{2}$ in (2.1), then we have

$$f\left(\sigma^{-1}\left(\frac{\sigma(x)+\sigma(y)}{2}\right)\right) \le \frac{f(x)+f(y)}{2}, \quad \forall x,y \in \mathcal{Q}. \tag{2.2}$$

The function f is called σ-Jensen (or mid)-convex function.

We now discuss some special cases of Definition 2.2.

Case I. If we take $\sigma(x) = \ln x$, then condition (2.1) becomes

$$f\bigl(x^{1-t}y^t\bigr) \le (1-t)f(x) + tf(y), \quad \forall x,y \in [a,b] \subset (0,\infty), t \in [0,1],$$

which is the concept of geometric convexity as considered in [1].

Case II. If we take $\sigma(x) = \frac{1}{x}$, then condition (2.1) becomes

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le (1-t)f(x) + tf(y), \quad \forall x,y \in [a,b] \subset (0,\infty), t \in [0,1],$$

which is the concept of harmonic convexity as considered in [15].

Case III. If we take $\sigma(x) = x^p$ ($p > 0$), then condition (2.1) becomes

$$f\big(\big((1-t)x^p + ty^p\big)^{\frac{1}{p}}\big) \le (1-t)f(x) + tf(y), \quad \forall x, y \in [a,b] \subset (0,\infty), t \in [0,1],$$

which is the concept of p-convexity as considered in [16].

Case IV. If we take $\sigma(x) = e^x$, then condition (2.1) becomes

$$f\big(\ln\big((1-t)e^x + te^y\big)\big) \le (1-t)f(x) + tf(y), \quad \forall x, y \in [a,b], t \in [0,1],$$

which is the concept of log-exponential convex functions on $[a,b]$.

3 Applications of σ-convex functions to integral inequalities

In this section, we show a representative application of σ-convex functions. We will establish some new integral inequalities of Hermite–Hadamard type via σ-convex functions.

Let $\mathcal{I} = [a,b]$ unless otherwise specified and σ be a continuous differentiable and strictly monotonic function in its domain. We denote by $\mathbb{R}^+$ the set of positive real numbers.

Theorem 3.1 *Suppose that $f : \mathcal{I} \to \mathbb{R}$ is an integrable σ-convex function with respect to the function σ, then we have the following inequalities*:

$$\begin{aligned} f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right) &\le \frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)\sigma'(x)\,\mathrm{d}x \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.1}$$

Proof Since f is a σ-convex function, we have

$$f\left(\sigma^{-1}\left(\frac{\sigma(x)+\sigma(y)}{2}\right)\right) \le \frac{f(x)+f(y)}{2}.$$

Substituting $x = \sigma^{-1}((1-t)\sigma(a) + t\sigma(b))$ and $y = \sigma^{-1}(t\sigma(a) + (1-t)\sigma(b))$ in the above inequality, we have

$$\begin{aligned} &f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right) \\ &\quad\le \frac{f(\sigma^{-1}((1-t)\sigma(a)+t\sigma(b))) + f(\sigma^{-1}(t\sigma(a)+(1-t)\sigma(b)))}{2}. \end{aligned} \tag{3.2}$$

Integrating both sides of (3.2) with respect to t on $[0,1]$, we get

$$f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right) \le \frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)\sigma'(x)\,\mathrm{d}x. \tag{3.3}$$

Similarly, in light of the assumption in Theorem 3.1 that f is the σ-convex function, we have

$$f\big(\sigma^{-1}\big((1-t)\sigma(a) + t\sigma(b)\big)\big) \le (1-t)f(a) + tf(b).$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we obtain

$$\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)\sigma'(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{3.4}$$

Combining (3.3) and (3.4) completes the proof of Theorem 3.1. □

Theorem 3.2 *Let $f:\mathcal{I}\to\mathbb{R}^+$ be an integrable σ-convex function with respect to the function σ. Then*

$$\begin{aligned}
&\frac{2f(a)}{\sigma(b)-\sigma(a)}\int_a^b\left(\frac{\sigma(b)-\sigma(x)}{\sigma(b)-\sigma(a)}\right)f(x)\sigma'(x)\,\mathrm{d}x \\
&\quad+\frac{2f(b)}{\sigma(b)-\sigma(a)}\int_a^b\left(\frac{\sigma(x)-\sigma(a)}{\sigma(b)-\sigma(a)}\right)f(x)\sigma'(x)\,\mathrm{d}x \\
&\le \frac{1}{\sigma(b)-\sigma(a)}\int_a^b f^2(x)\sigma'(x)\,\mathrm{d}x+\frac{f^2(a)+f(a)f(b)+f^2(b)}{3} \\
&\le \frac{2[f^2(a)+f(a)f(b)+f^2(b)]}{3}.
\end{aligned} \tag{3.5}$$

Proof Using the arithmetic-geometric means inequality gives

$$\begin{aligned}
&2f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\bigl((1-t)f(a)+tf(b)\bigr) \\
&\quad\le \bigl(f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\bigr)^2+\bigl((1-t)f(a)+tf(b)\bigr)^2 \\
&\quad= \bigl(f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\bigr)^2+(1-t)^2f^2(a)+t^2f^2(b)+2t(1-t)f(a)f(b).
\end{aligned}$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we obtain

$$\begin{aligned}
&2f(a)\int_0^1(1-t)f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t \\
&\quad+2f(b)\int_0^1 tf\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t \\
&\quad\le \int_0^1 f^2\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t+f^2(a)\int_0^1(1-t)^2\,\mathrm{d}t+f^2(b)\int_0^1 t^2\,\mathrm{d}t \\
&\quad+2f(a)f(b)\int_0^1 t(1-t)\,\mathrm{d}t.
\end{aligned} \tag{3.6}$$

By making the change of variable, inequality (3.6) can be rewritten as

$$\begin{aligned}
&2f(a)\cdot\frac{1}{\sigma(b)-\sigma(a)}\int_a^b\left(\frac{\sigma(b)-\sigma(x)}{\sigma(b)-\sigma(a)}\right)f(x)\sigma'(x)\,\mathrm{d}x \\
&\quad+2f(b)\cdot\frac{1}{\sigma(b)-\sigma(a)}\int_a^b\left(\frac{\sigma(x)-\sigma(a)}{\sigma(b)-\sigma(a)}\right)f(x)\sigma'(x)\,\mathrm{d}x \\
&\quad\le \frac{1}{\sigma(b)-\sigma(a)}\int_a^b f^2(x)\sigma'(x)\,\mathrm{d}x+\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.
\end{aligned} \tag{3.7}$$

On the other hand, since f is a σ-convex function, we have

$$f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\le(1-t)f(a)+tf(b),\quad \forall t\in[0,1],$$

therefore

$$\begin{aligned}&\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f^2(x)\sigma'(x)\,\mathrm{d}x\\&=\int_0^1 f^2\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t\\&\le\int_0^1\bigl[(1-t)f(a)+tf(b)\bigr]^2\,\mathrm{d}t=\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.\end{aligned}\tag{3.8}$$

Combining (3.7) and (3.8) leads to the inequalities described in Theorem 3.2. □

Theorem 3.3 *Let $f:\mathcal{I}\to\mathbb{R}^+$ be an integrable σ-convex function with respect to the function σ. Then*

$$\begin{aligned}&\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)\sigma'(x)\,\mathrm{d}x\\&\le\frac{1}{2}f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\\&\quad+\frac{1}{4(\sigma(b)-\sigma(a))f(\sigma^{-1}(\frac{\sigma(a)+\sigma(b)}{2}))}\int_a^b f^2(x)\sigma'(x)\,\mathrm{d}x\\&\quad+\frac{1}{24f(\sigma^{-1}(\frac{\sigma(a)+\sigma(b)}{2}))}\bigl(f^2(a)+f^2(b)+4f(a)f(b)\bigr).\end{aligned}\tag{3.9}$$

Proof Using the arithmetic-geometric means inequality and the σ convexity of f, it follows that

$$\begin{aligned}&f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\bigl[f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)+f\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\bigr]\\&\le f^2\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\\&\quad+\frac{1}{4}\bigl[f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)+f\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\bigr]^2\\&=f^2\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\\&\quad+\frac{1}{4}\bigl[f^2\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)+f^2\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\\&\quad+2f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)f\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\bigr]\\&\le f^2\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\\&\quad+\frac{1}{4}\bigl[f^2\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)+f^2\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\\&\quad+2\bigl((1-t)f(a)+tf(b)\bigr)\bigl(tf(a)+(1-t)f(b)\bigr)\bigr].\end{aligned}$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we obtain

$$\begin{aligned}
&f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\left[\int_0^1 f\left(\sigma^{-1}\left((1-t)\sigma(a)+t\sigma(b)\right)\right)\mathrm{d}t\right.\\
&\quad\left.+\int_0^1 f\left(\sigma^{-1}\left(t\sigma(a)+(1-t)\sigma(b)\right)\right)\mathrm{d}t\right]\\
&\quad\le f^2\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\int_0^1 \mathrm{d}t\\
&\quad+\frac{1}{4}\left[\int_0^1 f^2\left(\sigma^{-1}\left((1-t)\sigma(a)+t\sigma(b)\right)\right)\mathrm{d}t\right.\\
&\quad+\int_0^1 f^2\left(\sigma^{-1}\left(t\sigma(a)+(1-t)\sigma(b)\right)\right)\mathrm{d}t\\
&\quad\left.+2\left(f^2(a)+f^2(b)\right)\int_0^1 t(1-t)\,\mathrm{d}t+2f(a)f(b)\int_0^1\left(t^2+(1-t)^2\right)\mathrm{d}t\right].
\end{aligned}$$

Performing the change of variable, we get

$$\begin{aligned}
&f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\frac{2}{\sigma(b)-\sigma(a)}\int_a^b f(x)\sigma'(x)\,\mathrm{d}x\\
&\quad\le f^2\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\\
&\quad+\frac{1}{2(\sigma(b)-\sigma(a))}\int_a^b f^2(x)\sigma'(x)\,\mathrm{d}x\\
&\quad+\frac{f^2(a)+f^2(b)+4f(a)f(b)}{12}.
\end{aligned}$$

After a simple computation, one can transform the above inequality to the required inequality of Theorem 3.3. □

Theorem 3.4 *Suppose that $f,h:\mathcal{I}\to\mathbb{R}^+$ are two similarly ordered integrable σ-convex functions with respect to the function σ, then hf is also the σ-convex function with respect to the function σ.*

Proof Since $f,h:\mathcal{I}\to\mathbb{R}^+$ are two similarly ordered integrable σ-convex functions, for $\forall x,y\in\mathcal{I}$, $t\in[0,1]$, we have

$$\begin{aligned}
&f\left(\sigma^{-1}\left((1-t)\sigma(x)+t\sigma(y)\right)\right)h\left(\sigma^{-1}\left((1-t)\sigma(x)+t\sigma(y)\right)\right)\\
&\quad\le\left[(1-t)f(x)+tf(y)\right]\left[(1-t)h(x)+th(y)\right]\\
&\quad=(1-t)^2f(x)h(x)+t(1-t)\left[f(x)h(y)+f(y)h(x)\right]+t^2f(y)h(y)\\
&\quad=(1-t)f(x)h(x)+tf(y)h(y)+(1-t)^2f(x)h(x)\\
&\qquad+t(1-t)\left[f(x)h(y)+f(y)h(x)\right]+t^2f(y)h(y)-(1-t)f(x)h(x)-tf(y)h(y)\\
&\quad=(1-t)f(x)h(x)+tf(y)h(y)-t(1-t)\left[\left(f(x)-f(y)\right)\left(h(x)-h(y)\right)\right]\\
&\quad\le(1-t)f(x)h(x)+tf(y)h(y).
\end{aligned}$$

Thus

$$f\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)$$
$$\le (1-t)f(x)h(x)+tf(y)h(y). \tag{3.10}$$

Using the definition of σ-convex function (Definition 2.2), we conclude that hf is the σ-convex function with respect to the function σ. The proof of Theorem 3.4 is complete. □

Theorem 3.5 *Let $f,h:\mathcal{I}\to\mathbb{R}^+$ be integrable σ-convex functions with respect to the function σ. Then*

$$\begin{aligned}&\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x\\&\quad\le\frac{1}{3}\mathbb{M}_2(a,b)+\frac{1}{6}\mathbb{N}_2(a,b)\\&\quad\le\frac{1}{6}\big[\big[\mathbb{M}_1(a,b)\big]^2+\big[\mathbb{N}_1(a,b)\big]^2-\big[f(a)f(b)+h(a)h(b)\big]\big],\end{aligned}$$

where

$$\begin{aligned}&\mathbb{M}_1(a,b)=f(a)+f(b),\\&\mathbb{N}_1(a,b)=h(a)+h(b),\\&\mathbb{M}_2(a,b)=f(a)h(a)+f(b)h(b),\\&\mathbb{N}_2(a,b)=f(a)h(b)+f(b)h(a).\end{aligned}$$

Proof Since $f,h:\mathcal{I}\to\mathbb{R}^+$ are integrable σ-convex functions, we have

$$\begin{aligned}&\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x\\&\quad=\int_0^1 f\big(\sigma^{-1}\big[(1-t)\sigma(a)+t\sigma(b)\big]\big)h\big(\sigma^{-1}\big[(1-t)\sigma(a)+t\sigma(b)\big]\big)\,\mathrm{d}t\\&\quad\le\int_0^1\big[(1-t)f(a)+tf(b)\big]\big[(1-t)h(a)+th(b)\big]\,\mathrm{d}t\\&\quad=f(a)h(a)\int_0^1(1-t)^2\,\mathrm{d}t+\big[f(a)h(b)+f(b)h(a)\big]\int_0^1 t(1-t)\,\mathrm{d}t+f(b)h(b)\int_0^1 t^2\,\mathrm{d}t\\&\quad=\frac{1}{3}\big[f(a)h(a)+f(b)h(b)\big]+\frac{1}{6}\big[f(a)h(b)+f(b)h(a)\big]\\&\quad=\frac{1}{3}\mathbb{M}_2(a,b)+\frac{1}{6}\mathbb{N}_2(a,b).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x\\&\quad\le\int_0^1\big[(1-t)f(a)+tf(b)\big]\big[(1-t)h(a)+th(b)\big]\,\mathrm{d}t\end{aligned}$$

$$\leq \int_0^1 \left[\frac{[(1-t)f(a)+tf(b)]^2+[(1-t)h(a)+th(b)]^2}{2}\right]\mathrm{d}t$$
$$= \frac{1}{2}\int_0^1 \bigl[(1-t)^2\bigl[f^2(a)+h^2(a)\bigr]+t^2\bigl[f^2(b)+h^2(b)\bigr]$$
$$+ 2t(1-t)\bigl[f(a)f(b)+h(a)h(b)\bigr]\bigr]\,\mathrm{d}t$$
$$= \frac{1}{6}\bigl[\bigl[f(a)+f(b)\bigr]^2+\bigl[h(a)+h(b)\bigr]^2-\bigl[f(a)f(b)+h(a)h(b)\bigr]\bigr]$$
$$= \frac{1}{6}\bigl[\bigl[\mathbb{M}_1(a,b)\bigr]^2+\bigl[\mathbb{N}_1(a,b)\bigr]^2-\bigl[f(a)f(b)+h(a)h(b)\bigr]\bigr].$$

This completes the proof of Theorem 3.5. □

Theorem 3.6 *Let $f,h:\mathcal{I}\to\mathbb{R}^+$ be two similarly ordered integrable σ-convex functions with respect to the function σ. Then*

$$\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x \leq \frac{1}{2}\mathbb{M}_2(a,b), \tag{3.11}$$

where $\mathbb{M}_2(a,b)=f(a)h(a)+f(b)h(b)$.

Proof Using inequality (3.10) gives

$$f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)$$
$$\leq (1-t)f(a)h(a)+tf(b)h(b).$$

Integrating both sides of the above inequality with respect to t on $[0,1]$ yields the inequality asserted by Theorem 3.6. □

Theorem 3.7 *Let $f,h:\mathcal{I}\to\mathbb{R}^+$ be two integrable σ-convex functions with respect to the function σ. Then*

$$2f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)$$
$$-\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x \leq \frac{1}{6}\mathbb{M}_2(a,b)+\frac{1}{3}\mathbb{N}_2(a,b), \tag{3.12}$$

where $\mathbb{M}_2(a,b)=f(a)h(a)+f(b)h(b)$, $\mathbb{N}_2(a,b)=f(a)h(b)+f(b)h(a)$.

Proof Since f and h are both integrable σ-convex functions, by the same way as in the proof of Theorem 3.1, we deduce that

$$f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)$$
$$\leq \frac{f(\sigma^{-1}((1-t)\sigma(a)+t\sigma(b)))+f(\sigma^{-1}(t\sigma(a)+(1-t)\sigma(b)))}{2},$$
$$h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)$$
$$\leq \frac{h(\sigma^{-1}((1-t)\sigma(a)+t\sigma(b)))+h(\sigma^{-1}(t\sigma(a)+(1-t)\sigma(b)))}{2}.$$

Hence, we have

$$
\begin{aligned}
&f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\\
&\quad\le \frac{1}{4}\big[f\big(\sigma^{-1}\big[(1-t)\sigma(a)+t\sigma(b)\big]\big)+f\big(\sigma^{-1}\big[t\sigma(a)+(1-t)\sigma(b)\big]\big)\big]\\
&\qquad\times\big[h\big(\sigma^{-1}\big[(1-t)\sigma(a)+t\sigma(b)\big]\big)+h\big(\sigma^{-1}\big[t\sigma(a)+(1-t)\sigma(b)\big]\big)\big]\\
&\quad=\frac{1}{4}\big[f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)\\
&\qquad+f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\\
&\qquad+f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\\
&\qquad+f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)\big]\\
&\quad\le\frac{1}{4}\big[f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)\\
&\qquad+f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\\
&\qquad+\big((1-t)f(a)+tf(b)\big)\big(th(a)+(1-t)h(b)\big)\\
&\qquad+\big(tf(a)+(1-t)f(b)\big)\big((1-t)h(a)+th(b)\big)\big].
\end{aligned}
$$

Integrating both sides of the above inequality with respect to t on $[0,1]$ gives

$$
\begin{aligned}
&f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\\
&\quad\le\frac{1}{4}\left[\int_0^1 f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)\,\mathrm{d}t\right.\\
&\qquad+\int_0^1 f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\,\mathrm{d}t\\
&\qquad+2\big(f(a)h(a)+f(b)h(b)\big)\int_0^1 t(1-t)\,\mathrm{d}t\\
&\qquad\left.+\big(f(a)h(b)+f(b)h(a)\big)\int_0^1\big(t^2+(1-t)^2\big)\,\mathrm{d}t\right]\\
&\quad=\frac{1}{4}\left[\int_0^1 f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)\,\mathrm{d}t\right.\\
&\qquad+\int_0^1 f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\,\mathrm{d}t\\
&\qquad\left.+\frac{1}{3}\big(f(a)h(a)+f(b)h(b)\big)+\frac{2}{3}\big(f(a)h(b)+f(b)h(a)\big)\right]\\
&\quad=\frac{1}{2}\left[\frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x+\frac{1}{6}\mathbb{M}_2(a,b)+\frac{1}{3}\mathbb{N}_2(a,b)\right].
\end{aligned}
$$

Theorem 3.7 is proved. □

Theorem 3.8 *Let $f,h:\mathcal{I}\to\mathbb{R}^+$ be two integrable σ-convex functions with respect to the function σ. Then*

$$f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\leq\frac{\mathbb{M}_2(a,b)+\mathbb{N}_2(a,b)}{4},\tag{3.13}$$

where $\mathbb{M}_2(a,b)=f(a)h(a)+f(b)h(b)$, $\mathbb{N}_2(a,b)=f(a)h(b)+f(b)h(a)$.

Proof From Theorem 3.6 and Theorem 3.7, one can obtain the required result. □

Theorem 3.9 *Let $f,h:\mathcal{I}\to\mathbb{R}^+$ be two integrable σ-convex functions with respect to the strictly increasing function σ. Then*

$$\begin{aligned}&\int_a^b\int_a^b\int_0^1 f\big(\sigma^{-1}\big[(1-t)\sigma(x)+t\sigma(y)\big]\big)h\big(\sigma^{-1}\big[(1-t)h(x)+th(y)\big]\big)\sigma'(x)\sigma'(y)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\\&\quad\leq\frac{2(\sigma(b)-\sigma(a))}{3}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x+\frac{(\sigma(b)-\sigma(a))^2}{12}\big[\mathbb{M}_2(a,b)+\mathbb{N}_2(a,b)\big],\end{aligned}\tag{3.14}$$

where $\mathbb{M}_2(a,b)=f(a)h(a)+f(b)h(b)$, $\mathbb{N}_2(a,b)=f(a)h(b)+f(b)h(a)$.

Proof Since f and h are σ-convex functions, then for $\forall x,y\in\mathcal{I}$, $t\in[0,1]$ we have

$$\begin{aligned}&f\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)\\&\quad\leq\big[(1-t)f(x)+tf(y)\big]\big[(1-t)h(x)+th(y)\big]\\&\quad=(1-t)^2f(x)h(x)+t(1-t)\big[f(x)h(y)+h(x)f(y)\big]+t^2f(y)h(y).\end{aligned}$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}&\int_0^1 f\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)\,\mathrm{d}t\\&\quad\leq f(x)h(x)\int_0^1(1-t)^2\,\mathrm{d}t+f(y)h(y)\int_0^1 t^2\,\mathrm{d}t\\&\qquad+\big[f(x)h(y)+h(x)f(y)\big]\int_0^1 t(1-t)\,\mathrm{d}t\\&\quad=\frac{1}{3}\big[f(x)h(x)+f(y)h(y)\big]+\frac{1}{6}\big[f(x)h(y)+h(x)f(y)\big].\end{aligned}$$

Again, integrating both sides of the above inequality over the plane domain $\{(x,y):x\in[a,b],y\in[a,b]\}$ and then using the right-hand side of the Hermite–Hadamard inequality (3.1), we deduce that

$$\begin{aligned}&\int_a^b\int_a^b\int_0^1 f\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(x)+t\sigma(y)\big)\big)\sigma'(x)\sigma'(y)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\\&\quad\leq\frac{2(\sigma(b)-\sigma(a))}{3}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x+\frac{1}{6}\int_a^b h(y)\sigma'(y)\,\mathrm{d}y\int_a^b f(x)\sigma'(x)\,\mathrm{d}x\\&\qquad+\frac{1}{6}\int_a^b f(y)\sigma'(y)\,\mathrm{d}y\int_a^b h(x)\sigma'(x)\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2(\sigma(b)-\sigma(a))}{3}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x \\
&\quad + \frac{(\sigma(b)-\sigma(a))^2}{6}\biggl[\biggl(\frac{h(a)+h(b)}{2}\biggr)\biggl(\frac{f(a)+f(b)}{2}\biggr) \\
&\quad + \biggl(\frac{f(a)+f(b)}{2}\biggr)\biggl(\frac{h(a)+h(b)}{2}\biggr)\biggr] \\
&= \frac{2(\sigma(b)-\sigma(a))}{3}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x + \frac{(\sigma(b)-\sigma(a))^2}{12}\bigl[\bigl(f(a)+f(b)\bigr)\bigl(h(a)+h(b)\bigr)\bigr] \\
&= \frac{2(\sigma(b)-\sigma(a))}{3}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x + \frac{(\sigma(b)-\sigma(a))^2}{12}\bigl[\mathbb{M}_2(a,b)+\mathbb{N}_2(a,b)\bigr],
\end{aligned}$$

which is the required result. The proof of Theorem 3.9 is complete. □

Theorem 3.10 *Let $f,h:\mathcal{I}\to\mathbb{R}$ be two integrable σ-convex functions with respect to the function σ. Then*

$$\begin{aligned}
&\frac{1}{\sigma(b)-\sigma(a)}\int_a^b \biggl(\frac{\sigma(b)-\sigma(x)}{\sigma(b)-\sigma(a)}\biggr)\bigl(f(a)h(x)+h(a)f(x)\bigr)\sigma'(x)\,\mathrm{d}x \\
&\quad + \frac{1}{\sigma(b)-\sigma(a)}\int_a^b \biggl(\frac{\sigma(x)-\sigma(a)}{\sigma(b)-\sigma(a)}\biggr)\bigl(f(b)h(x)+h(b)f(x)\bigr)\sigma'(x)\,\mathrm{d}x \\
&\leq \frac{1}{\sigma(b)-\sigma(a)}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x + \frac{\mathbb{M}_2(a,b)}{3} + \frac{\mathbb{N}_2(a,b)}{6},
\end{aligned} \tag{3.15}$$

where $\mathbb{M}_2(a,b)=f(a)h(a)+f(b)h(b)$, $\mathbb{N}_2(a,b)=f(a)h(b)+f(b)h(a)$.

Proof Since f and h are σ-convex functions, then for $\forall t\in[0,1]$ we have

$$\begin{aligned}
&f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\leq(1-t)f(a)+tf(b), \\
&h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\leq(1-t)h(a)+th(b).
\end{aligned}$$

Utilizing the rearrangement inequality, we obtain

$$\begin{aligned}
&f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\bigl((1-t)h(a)+th(b)\bigr) \\
&\quad + h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\bigl((1-t)f(a)+tf(b)\bigr) \\
&\leq \bigl((1-t)f(a)+tf(b)\bigr)\bigl((1-t)h(a)+th(b)\bigr) \\
&\quad + f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr).
\end{aligned}$$

Integrating both sides of the above inequality over the interval $[0,1]$, we find

$$\begin{aligned}
&h(a)\int_0^1 (1-t)f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t + h(b)\int_0^1 tf\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t \\
&\quad + f(a)\int_0^1 (1-t)h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t \\
&\quad + f(b)\int_0^1 th\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}&\le f(a)h(a)\int_0^1 (1-t)^2\,\mathrm{d}t + f(b)h(b)\int_0^1 t^2\,\mathrm{d}t + \bigl(f(a)h(b)+f(b)h(a)\bigr)\int_0^1 t(1-t)\,\mathrm{d}t\\&\quad+\int_0^1 f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\,\mathrm{d}t.\end{aligned}$$

Applying the change of variable to the integrations, we obtain the required result of Theorem 3.10. □

Theorem 3.11 *Let $f,h:\mathcal{I}\to\mathbb{R}^+$ be two integrable σ-convex functions with respect to the function σ. Then*

$$\begin{aligned}&\frac{1}{\sigma(b)-\sigma(a)}\int_a^b\biggl[f\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)h(x)+h\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)f(x)\biggr]\sigma'(x)\,\mathrm{d}x\\&\le\frac{1}{2(\sigma(b)-\sigma(a))}\int_a^b f(x)h(x)\sigma'(x)\,\mathrm{d}x\\&\quad+f\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)h\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)\\&\quad+\frac{1}{12}\mathbb{M}_2(a,b)+\frac{1}{6}\mathbb{N}_2(a,b),\end{aligned}$$

where $\mathbb{M}_2(a,b)=f(a)h(a)+f(b)h(b)$, $\mathbb{N}_2(a,b)=f(a)h(b)+f(b)h(a)$.

Proof Note that f and h are integrable σ-convex functions. Taking $t=\frac{1}{2}$ in (2.1) and letting $x=\sigma^{-1}((1-t)\sigma(a)+t\sigma(b))$, $y=\sigma^{-1}(t\sigma(a)+(1-t)\sigma(b))$, we have

$$\begin{aligned}&f\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)\\&\quad\le\frac{f(\sigma^{-1}((1-t)\sigma(a)+t\sigma(b)))+f(\sigma^{-1}(t\sigma(a)+(1-t)\sigma(b)))}{2},\\&h\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)\\&\quad\le\frac{h(\sigma^{-1}((1-t)\sigma(a)+t\sigma(b)))+h(\sigma^{-1}(t\sigma(a)+(1-t)\sigma(b)))}{2}.\end{aligned}$$

Utilizing the rearrangement inequality, we obtain

$$\begin{aligned}&\frac{1}{2}f\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)\bigl[h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)+h\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\bigr]\\&\quad+\frac{1}{2}h\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)\bigl[f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)\\&\quad+f\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\bigr]\\&\le\frac{1}{4}\bigl[f\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)+f\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\bigr]\\&\quad\times\bigl[h\bigl(\sigma^{-1}\bigl((1-t)\sigma(a)+t\sigma(b)\bigr)\bigr)+h\bigl(\sigma^{-1}\bigl(t\sigma(a)+(1-t)\sigma(b)\bigr)\bigr)\bigr]\\&\quad+f\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)h\biggl(\sigma^{-1}\biggl(\frac{\sigma(a)+\sigma(b)}{2}\biggr)\biggr)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\big[f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big) \\
&\quad + f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big) \\
&\quad + f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big) \\
&\quad + f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)\big] \\
&\quad + f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right) \\
&\leq \frac{1}{4}\big[f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big) \\
&\quad + f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big) \\
&\quad + \big((1-t)f(a)+tf(b)\big)\big(th(a)+(1-t)h(b)\big) \\
&\quad + \big(tf(a)+(1-t)f(b)\big)\big((1-t)h(a)+th(b)\big)\big] \\
&\quad + f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right).
\end{aligned}
$$

Integrating the first and last expressions among the above inequalities over the interval $[0,1]$, we have

$$
\begin{aligned}
&\frac{1}{2}f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\int_0^1\big[h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big) \\
&\quad + h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\big]\,\mathrm{d}t \\
&\quad + \frac{1}{2}h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)\int_0^1\big[f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big) \\
&\quad + f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\big]\,\mathrm{d}t \\
&\leq \frac{1}{4}\left[\int_0^1 f\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)h\big(\sigma^{-1}\big((1-t)\sigma(a)+t\sigma(b)\big)\big)\,\mathrm{d}t\right. \\
&\quad + \int_0^1 f\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)h\big(\sigma^{-1}\big(t\sigma(a)+(1-t)\sigma(b)\big)\big)\,\mathrm{d}t \\
&\quad \left. + \frac{1}{3}\big(f(a)h(a)+f(b)h(b)\big) + \frac{2}{3}\big(f(a)h(b)+f(b)h(a)\big)\right] \\
&\quad + f\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right)h\left(\sigma^{-1}\left(\frac{\sigma(a)+\sigma(b)}{2}\right)\right).
\end{aligned}
$$

From the above inequality, by simple computation and arrangement, we obtain the inequality asserted by Theorem 3.11. □

4 Conclusion

We have introduced the notions of σ-convex sets and σ-convex functions. Also, we have shown that the class of σ convexity includes several other classes of classical convexity. In fact, the σ-convex function is a unified generalization of convex functions related to various power means. Moreover, using the notion of σ convexity, we have derived some new integral inequalities of Hermite–Hadamard type. We expect that the ideas and techniques of the paper may stimulate further research in this field.

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The authors declare that they have no competing interests.

Authors' contributions
SW and MUA finished the proofs of the main results and the writing work. MAN, KIN, and SI gave lots of advice on the proofs of the main results and the writing work. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Longyan University, Longyan, China. [2]Department of Mathematics, Government College University, Faisalabad, Pakistan. [3]Department of Mathematics, COMSATS University Islamabad, Faisalabad, Pakistan.

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