Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fuglede–Putnam type theorems for (p,k)-quasihyponormal operators via hyponormal operators

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Abstract

For Hilbert space operators S, X, and T, $(S,X,T) \in FP$ means Fuglede–Putnam theorem holds for triplet (S,X,T), that is, $SX = XT$ ensures $S^*X = XT^*$. Similarly, $(S,T) \in FP$ means $(S,X,T) \in FP$ holds for each operator X. This paper is devoted to the study of Fuglede–Putnam type theorems for (p,k)-quasihyponormal operators via a class of operators based on hyponormal operators
$FP(H) := \{S|(S,T) \in FP$ holds for each hyponormal operator $T^*\}$. Fuglede–Putnam type theorems involving (p,k)-quasihyponormal, dominant, and w-hyponormal operators, which are extensions of the results by Tanahashi, Patel, Uchiyama, et al., are obtained.

MSC: 47B20; 47A63; 47B47

Keywords: Fuglede–Putnam theorem; Hyponormal operator; Dominant operator; w-hyponormal operator

1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators on $\mathcal{H}$ and the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, respectively.

Theorem 1.1 (Fuglede–Putnam theorem, [3, 10]) *Let $S \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{K})$, and $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. If S, T are normal operators, then $SX = XT$ ensures $S^*X = XT^*$.*

Theorem 1.2 ([14]) *Let $S \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{K})$. The following assertions are equivalent.*

(1) *If $SX = XT$, then $S^*X = XT^*$.*
(2) *If $SX = XT$, then $[R(X)]$ reduces S, where $[R(X)]$ means the closure of range $R(X)$ of X, $\ker X$ reduces T, $S|_{[R(X)]}$ and $T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

For $S \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{K})$, and $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $(S,X,T) \in FP$ means Fuglede–Putnam theorem holds for the triplet (S,X,T), that is, $SX = XT$ ensures $S^*X = XT^*$. Similarly, $(S,T) \in FP$ means $(S,X,T) \in FP$ holds for all $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$.

There are various extensions of Fuglede–Putnam theorem for non-normal operators including dominant operators (an operator T is called dominant if, for each complex number z, there exists $M_z > 0$ such that $(T-z)^*(T-z) \geq M_z^2(T-z)(T-z)^*$), (p,k)-quasihyponormal operators (defined by $T^{*k}|T|^{2p}T^k \geq T^{*k}|T^*|^{2p}T^k$, where $0 < p \leq 1$ and k is a nonnegative integer, a $(p,0)$-quasihyponormal operator means a p-hyponormal operator), w-hyponormal operators (defined by $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$, the class of w-hyponormal operators coincides with class $A(\frac{1}{2},\frac{1}{2})$), and so on. See [1, 2, 11–13, 15, 18].

Among others, Tanahashi, Patel, and Uchiyama [15] proved three kinds of Fuglede–Putnam type theorems with kernel conditions as follows.

(I) Fuglede–Putnam type theorems with restrictions on $\ker S$ or $\ker T^*$.

Theorem 1.3 ([15]) *Let S be (p,k)-quasihyponormal and T^* be (p,k)-quasihyponormal or dominant.*

(1) *If $\ker S = \{0\}$ or $\ker T^* = \{0\}$, then $(S,T) \in FP$.*
(2) *If $\ker S \subseteq \ker S^*$ and $\ker T^* \subseteq \ker T$, then $(S,T) \in FP$.*

It is known that every dominant operator has a reducing kernel, so the condition $\ker T^* \subseteq \ker T$ in (2) of the above theorem in the case when T^* is dominant holds.

(II) Fuglede–Putnam type theorems with restrictions on $\ker X$ or $\ker X^*$.

Theorem 1.4 ([12, 15]) *The following assertions hold.*

(1) *Let S be (p,k)-quasihyponormal and T be normal. If X has a dense range, then $(S,X,T) \in FP$ and S is normal.*
(2) *Let S be p-hyponormal and T^* be (p,k)-quasihyponormal. If $\ker X = \{0\}$, then $(S,X,T) \in FP$ and T is normal.*

(III) Fuglede–Putnam type theorems with restrictions on $\ker S$, $\ker S^*$, and $\ker X^*$.

Theorem 1.5 ([15]) *Let S and T^* be (p,k)-quasihyponormal. If $\ker S \subseteq \ker S^{*k}$ and $\ker S^{*k} \subseteq \ker X^*$, then $(S,X,T) \in FP$.*

In this paper, we shall show extensions of Theorems 1.3–1.5 via the following classes of operators based on hyponormal operators.

$$FP(N) := \{S | (S,T) \in FP \text{ holds for each normal operator } T^*\}.$$

$$FP(H) := \{S | (S,T) \in FP \text{ holds for each hyponormal operator } T^*\}.$$

$$FP(p\text{-}H) := \{S | (S,T) \in FP \text{ holds for each } p\text{-hyponormal operator } T^*\}.$$

It is clear that $FP(N) \supseteq FP(H) \supseteq FP(p\text{-}H)$.

A part of an operator is its restriction to a closed invariant subspace. A class of operators is called hereditary if each part of an operator in the class also belongs to the class.

Remark 1.6 It is well known that the class $FP(p\text{-}H)$ includes many classes of operators, such as dominant operators [11–13, 18], (p,k)-quasihyponormal operators with reducing kernels [15, 17], and w-hyponormal operators with reducing kernels [1]. Moreover, it is known that the classes above also belong to the class of hereditary $FP(H)$ (denote this class by $HFP(H)$), that is, every restriction of an operator to its closed invariant subspace also belongs to the class. See [1, 7, 13, 16, 18].

In Sect. 2, some elementary properties of $FP(H)$ are considered. For example, the reducibility of invariant subspaces of $FP(N)$ operators; the relations between $HFP(H)$ and

HFP(p-H); the relations between Fuglede–Putnam type theorems with $\ker S = \{0\}$ or $\ker T^* = \{0\}$ and Fuglede–Putnam type theorems with reducing kernels. Sections 3–5 are devoted to generalizations of Theorems 1.3–1.5, respectively. Among others, it is proved that Theorem 1.3 holds if T^* is a w-hyponormal operator, Theorem 1.4 holds if T^* in Theorem 1.4(1) and S in Theorem 1.4(2) are replaced with a (p,k)-quasihyponormal operator, and Theorem 1.5 holds without the restriction $\ker S \subseteq \ker S^{*k}$. Lastly, an example is given which says that some kernel conditions in Fuglede–Putnam type theorems are inevitable.

2 Elementary properties of *FP*(*H*)

By observation, the definitions of $FP(N)$, $FP(H)$, and $FP(p\text{-}H)$ are equivalent to the following assertions.

$$FP(N) := \{T \mid (S,T^*) \in FP \text{ holds for each normal operator } S\},$$
$$FP(H) := \{T \mid (S,T^*) \in FP \text{ holds for each hyponormal operator } S\},$$
$$FP(p\text{-}H) := \{T \mid (S,T^*) \in FP \text{ holds for each } p\text{-hyponormal operator } S\}.$$

In order to consider the reducibility of invariant subspaces of an operator, four properties are introduced in [20]. Let $\mathcal{M}$ be a nontrivial closed invariant subspace of T and $T|_{\mathcal{M}}$ be the restriction of T on $\mathcal{M}$.

R_1 If the restriction $T|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T.

R_2 If there exists a positive integer k such that for each $\mathcal{M} \subseteq [R(T^k)]$, the assertion that $T|_{\mathcal{M}}$ is normal ensures that $\mathcal{M}$ reduces T.

R_3 If $T|_{\mathcal{M}}$ is normal and injective, then $\mathcal{M}$ reduces T.

R_4 If $\lambda \neq 0$, then $\ker(T-\lambda)$ reduces T.

It is obvious that the property R_1 can be regarded as the case $k=0$ of R_2. An operator $T \in R_i$ means T has the property R_i, $i = 1,2,3,4$. It is known that, for each $i \in \{1,2,3\}$, $T \in R_i$ implies $T \in R_{i+1}$ [20, Lemma 2.2]. There exists an operator T such that $T \in R_3$ and $T \notin R_2$ (Example 5.3(4)).

Lemma 2.1 *The following assertions hold.*

(1) *If $T \in FP(N)$, then $T \in R_1$.*

(2) *If T is a (p,k)-quasihyponormal or w-hyponormal operator with reducing kernel, then $T \in R_1$.*

(3) *If T is (p,k)-quasihyponormal, then $T \in R_2$.*

Lemma 2.1 is a generalization of [15, Lemma 2.2].

Proof (1) Let $\mathcal{M}$ be a nontrivial closed invariant subspace of T, $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\mathcal{M} \oplus \mathcal{M}^\perp$, $T_{11} = T|_{\mathcal{M}}$ be normal, and $P = P_{\mathcal{M}}$ be a projection. Since $\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$ is normal and $TP = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} = P\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$, $T \in FP(N)$ implies $S^*P = \begin{pmatrix} (T_{11})^* & 0 \\ (T_{12})^* & 0 \end{pmatrix} = P\begin{pmatrix} (T_{11})^* & 0 \\ 0 & 0 \end{pmatrix}$. Then $T_{12} = 0$ and $\mathcal{M}$ reduces T.

(2) The assertion follows by Remark 1.6.

(3) Let $\mathcal{M} \subseteq [R(T^k)]$, $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\mathcal{M} \oplus \mathcal{M}^\perp$, $T_{11} = T|_{\mathcal{M}}$ be normal and $P = P_{\mathcal{M}}$. Then

$$\begin{aligned} TT^* &= \begin{pmatrix} T_{11}(T_{11})^* + T_{12}(T_{12})^* & T_{12}(T_{22})^* \\ T_{22}(T_{12})^* & T_{22}(T_{22})^* \end{pmatrix}, \\ P|T|^{2p}P &= PP_{[R(T^k)]}|T|^{2p}P_{[R(T^k)]}P \geq P\left|T^*\right|^{2p}P. \end{aligned} \tag{1}$$

By Hansen's inequality and Loewner–Heinz' inequality [5], [4, p.127],

$$\begin{aligned} \begin{pmatrix} (T_{11})^*T_{11} & 0 \\ 0 & 0 \end{pmatrix}^p &= \left(P|T|^2P\right)^p \geq P|T|^{2p}P \geq P\left|T^*\right|^{2p}P \\ &\geq P\left(TPT^*\right)^pP = \left(TPT^*\right)^p = \begin{pmatrix} T_{11}(T_{11})^*) & 0 \\ 0 & 0 \end{pmatrix}^p. \end{aligned}$$

The normality of T_{11} implies $(TT^*)^p = \begin{pmatrix} |T_{11}|^{2p} & A \\ A^* & B \end{pmatrix}$, where A is an operator and B is a positive semidefinite operator.

Let $(TT^*)^{\frac{p}{2}} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$, again, by Hansen's inequality and Loewner–Heinz's inequality,

$$\begin{pmatrix} |T_{11}|^{2p} & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}} = \left(P\left(TT^*\right)^pP\right)^{\frac{1}{2}} \geq P\left(TT^*\right)^{\frac{p}{2}}P \geq P\left(TPT^*\right)^{\frac{p}{2}}P = \begin{pmatrix} |T_{11}|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{p}{2}}.$$

So $X = |T_{11}|^p$, $(TT^*)^p = ((TT^*)^{\frac{p}{2}})^2 = \begin{pmatrix} |T_{11}|^{2p}+YY^* & * \\ ** & *** \end{pmatrix}$, where $*$ means some elements of the matrix.

Thus $Y = 0$, $(TT^*)^{\frac{p}{2}} = \begin{pmatrix} |T_{11}|^p & 0 \\ 0 & Z \end{pmatrix}$, and $TT^* = (TT^*)^{\frac{p}{2}\cdot\frac{2}{p}} = \begin{pmatrix} |T_{11}|^2 & 0 \\ 0 & Z^{\frac{2}{p}} \end{pmatrix}$. Then $T_{12} = 0$ follows by (1). □

Aluthge introduced Aluthge transform $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ where the polar decomposition of T is $T = U|T|$. For each $s > 0$ and $t > 0$, $T(s,t) = |T|^sU|T|^t$ is called generalized Aluthge transform.

Lemma 2.2 ([9]) *Let $s > 0$, $t > 0$, $T \in A(s,t)$. If $T(s,t)$ is quasinormal (normal), then T is quasinormal (normal).*

Lemma 2.3 ([6, 19]) *If T is p-hyponormal and $\alpha = \min\{p+s, p+t, s+t\}$, then*

$$\left(T(s,t)^*T(s,t)\right)^{\frac{\alpha}{s+t}} \geq \left(T(s,t)T(s,t)^*\right)^{\frac{\alpha}{s+t}}.$$

Lemma 2.4 *$HFP(H) = HFP(p\text{-}H)$.*

Proof It is sufficient to prove $HFP(H) \subseteq HFP(p\text{-}H)$.

Let $T^* \in HFP(H)$, S be p-hyponormal and $SX = XT$. Decompose S, T, X into

$$\begin{aligned} S &= \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \in \mathcal{B}\bigl(\bigl[R(X)\bigr] \oplus \ker X^*\bigr), \\ T &= \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{B}\bigl(\bigl[R\bigl(X^*\bigr)\bigr] \oplus \ker X\bigr), \\ X &= \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}\bigl(\bigl[R\bigl(X^*\bigr)\bigr] \oplus \ker X, \bigl[R(X)\bigr] \oplus \ker X^*\bigr). \end{aligned} \tag{2}$$

Then

$$SX = XT \iff \begin{pmatrix} S_{11}X_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}T_{11} & 0 \\ 0 & 0 \end{pmatrix} \iff S_{11}X_{11} = X_{11}T_{11}, \tag{3}$$

$$\begin{aligned} S^*X = XT^* &\iff \begin{pmatrix} (S_{11})^*X_{11} & 0 \\ (S_{12})^*X_{11} & 0 \end{pmatrix} = \begin{pmatrix} X_{11}(T_{11})^* & X_{11}(T_{21})^* \\ 0 & 0 \end{pmatrix} \\ &\iff (S_{11})^*X_{11} = X_{11}(T_{11})^*,\ (S_{12})^*X_{11} = 0 = X_{11}(T_{21})^*. \end{aligned} \tag{4}$$

Since $S, T^* \in R_1$ by Lemma 2.1 and X_{11} is quasiaffine, it is sufficient to prove $(S_{11}, T_{11}) \in FP$. By the assumption, S_{11} is p-hyponormal and $T_{11}^* \in FP(H)$.

If $\frac{1}{2} \le p \le 1$, by Lemma 2.3, the Aluthge transform $S_{11}(\frac{1}{2}, \frac{1}{2})$ of S_{11} is hyponormal and $(S_{11}(\frac{1}{2}, \frac{1}{2}), T_{11}) \in FP$. So

$$\begin{aligned} &S_{11}X_{11} = X_{11}T_{11} \\ &\quad\Longrightarrow\quad S_{11}\left(\frac{1}{2}, \frac{1}{2}\right)|S_{11}|^{\frac{1}{2}}X_{11} = |S_{11}|^{\frac{1}{2}}X_{11}T_{11} \\ &\quad\Longrightarrow\quad S_{11}\left(\frac{1}{2}, \frac{1}{2}\right) = \left(S_{11}\left(\frac{1}{2}, \frac{1}{2}\right)\right)\Big|_{[R(|S_{11}|^{\frac{1}{2}}X_{11})]} \oplus \left(S_{11}\left(\frac{1}{2}, \frac{1}{2}\right)\right)\Big|_{\ker(X_{11}^*|S_{11}|^{\frac{1}{2}})}, \end{aligned}$$

where $(S_{11}(\frac{1}{2}, \frac{1}{2}))|_{[R(|S_{11}|^{\frac{1}{2}}X_{11})]}$ is normal. The assertion "X_{11} is quasiaffine" implies that $[R(|S_{11}|^{\frac{1}{2}}X_{11})] = [R(|S_{11}|^{\frac{1}{2}})]$ and $\ker(X_{11}^*|S_{11}|^{\frac{1}{2}}) = \ker(|S_{11}|) \subseteq \ker S_{11}(\frac{1}{2}, \frac{1}{2})$. Then $S_{11}(\frac{1}{2}, \frac{1}{2}) = (S_{11}(\frac{1}{2}, \frac{1}{2}))|_{[R(|S_{11}|)]} \oplus 0$ is normal, S_{11} is normal by Lemma 2.2, and $(S_{11}, T_{11}) \in FP$ for $T_{11}^* \in FP(H)$.

If $0 < p \le \frac{1}{2}$, then $S_{11}(\frac{1}{2}, \frac{1}{2})$ is $(p + \frac{1}{2})$-hyponormal and $(S_{11}(\frac{1}{2}, \frac{1}{2}), T_{11}) \in FP$ in the case $\frac{1}{2} \le p \le 1$. Similar to the proof of the case $\frac{1}{2} \le p \le 1$, $S_{11}(\frac{1}{2}, \frac{1}{2}) = (S_{11}(\frac{1}{2}, \frac{1}{2}))|_{[R(|S_{11}|)]} \oplus 0$ is normal, S_{11} is normal and $(S_{11}, T_{11}) \in FP$ for $T_{11}^* \in FP(H)$. □

Lemma 2.5 *Let C_1, C_2 be two classes of operators with heredity. The following assertions* (1)–(2) *are equivalent to each other,* (1) *ensures* (4) *and* (3) *ensures* (4).

(1) *If $S \in C_1$ with $\ker S = \{0\}$ and $T^* \in C_2$, then $(S, T) \in FP$.*
(2) *If $S \in C_2$ and $T^* \in C_1$ with $\ker T^* = \{0\}$, then $(S, T) \in FP$.*
(3) *If $S \in C_1$ and $T^* \in C_2$ with $\ker T^* = \{0\}$, then $(S, T) \in FP$.*
(4) *If $S \in C_1$ with $\ker S \subseteq \ker S^*$ and $T^* \in C_2$ with $\ker T^* \subseteq \ker T$, then $(S, T) \in FP$.*

Proof Since

$$SX = XT \quad \Leftrightarrow \quad X^*S^* = T^*X^* \quad \text{and} \quad S^*X = XT^* \quad \Leftrightarrow \quad X^*S = TX^*,$$

we have

$$(S,X,T) \in FP \quad \Leftrightarrow \quad \left(T^*,X^*,S^*\right) \in FP \quad \text{and} \quad (S,T) \in FP \quad \Leftrightarrow \quad \left(T^*,S^*\right) \in FP. \quad (5)$$

By (5), it is sufficient to prove (1) ⇒ (4) and (3) ⇒ (4).

(1) ⇒ (4) Let $\ker S \subseteq \ker S^*$ and $\ker T^* \subseteq \ker T$. Decompose S, T, X into

$$\begin{aligned} S &= \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}\left(\left[R\left(S^*\right)\right] \oplus \ker S\right), \\ T &= \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}\left(\left[R(T)\right] \oplus \ker T^*\right), \\ X &= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathcal{B}\left(\left[R(T)\right] \oplus \ker T^*, \left[R\left(S^*\right)\right] \oplus \ker S\right). \end{aligned} \quad (6)$$

Then $\ker S_{11} = \{0\} = \ker T_{11}^*$,

$$\begin{aligned} SX = XT \quad &\Longleftrightarrow \quad \begin{pmatrix} S_{11}X_{11} & S_{11}X_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}T_{11} & 0 \\ X_{21}T_{11} & 0 \end{pmatrix} \\ &\Longleftrightarrow \quad S_{11}X_{11} = X_{11}T_{11},\ S_{11}X_{12} = 0 = X_{21}T_{11}, \end{aligned} \quad (7)$$

$$\begin{aligned} S^*X = XT^* \quad &\Longleftrightarrow \quad \begin{pmatrix} (S_{11})^*X_{11} & (S_{11})^*X_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}(T_{11})^* & 0 \\ X_{21}(T_{11})^* & 0 \end{pmatrix} \\ &\Longleftrightarrow \quad (S_{11})^*X_{11} = X_{11}(T_{11})^*,\ (S_{11})^*X_{12} = 0 = X_{21}(T_{11})^*. \end{aligned} \quad (8)$$

By heredity and (1), $(S_{11}, T_{11}) \in FP$. Since $\ker S_{11} = \{0\} = \ker T_{11}^*$, the assertion $S_{11}X_{12} = 0 = X_{21}T_{11}$ implies $X_{12} = 0 = X_{21}$. So that $(S,T) \in FP$.

(3) ⇒ (4) The assertion holds in a similar manner to (1) ⇒ (4). □

Lemma 2.6 *Let C be a class of operators with heredity. The following assertion* (1) *ensures* (2).

(1) *If $S \in C$ with* $\ker S = \{0\}$ *and* $T^* \in FP(N)$, *then* $(S,T) \in FP$.

(2) *If $S \in C$ with* $\ker S \subseteq \ker S^*$ *and* $T^* \in FP(N)$, *then* $(S,T) \in FP$.

Proof The proof is similar to the proof of [8, Theorem 7]. Let $\ker S \subseteq \ker S^*$. Decompose S, X into $S = S_n \oplus S_p$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where S_n and S_p are normal part and pure part of S, respectively,

$$X = \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix} : \mathcal{K} \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Then $\ker S_p = \{0\}$,

$$SX = XT \iff \begin{pmatrix} S_n X_{11} \\ S_p X_{21} \end{pmatrix} = \begin{pmatrix} X_{11} T \\ X_{21} T \end{pmatrix},$$

$$S^* X = X T^* \iff \begin{pmatrix} S_n^* X_{11} \\ S_p^* X_{21} \end{pmatrix} = \begin{pmatrix} X_{11} T^* \\ X_{21} T^* \end{pmatrix}.$$

Since $T^* \in FP(N)$, $(S_n, T) \in FP$ follows. By $\ker S_p = \{0\}$ and (1), $(S_p, T) \in FP$ and $(S, T) \in FP$. □

3 Extensions of Theorem 1.3

Theorem 3.1 *Let S be (p,k)-quasihyponormal and T^* be (p,k)-quasihyponormal, or dominant, or w-hyponormal.*

(1) *If* $\ker T^* = \{0\}$, *then* $(S,T) \in FP$.

(2) *If T^* is (p,k)-quasihyponormal*, $\ker S \subseteq \ker S^*$, *and* $\ker T^* \subseteq \ker T$, *then* $(S,T) \in FP$.

(3) *If T^* is dominant and* $\ker S \subseteq \ker S^*$, *then* $(S,T) \in FP$.

(4) *If T^* is w-hyponormal*, $\ker S \subseteq \ker S^*$, *and* $\ker T^* \subseteq \ker T$, *then* $(S,T) \in FP$.

Tanahashi et al. [15, Theorems 2.5, 2.7, 2.10–2.12] proved the case "T^* is (p,k)-quasihyponormal or dominant" of Theorem 3.1. Here we prove Theorem 3.1 by using the class $HFP(H)$ (Remark 1.6). Theorem 3.1 means that Theorem 1.3 holds if T^* is a w-hyponormal operator.

Lemma 3.2 ([7, 16]) *Let T be (p,k)-quasihyponormal.*

(1) *If $T^k\mathcal{H}$ is not dense and $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $[T^k\mathcal{H}] \oplus \ker T^{*k}$, then T_{11} is p-hyponormal, $T_{22}^k = 0$, and $\sigma(T) = \sigma(T_{11}) \cup \{0\}$.*

(2) *Each restriction $T|_{\mathcal{M}}$ of T to its invariant subspace $\mathcal{M}$ is also (p,k)-quasihyponormal.*

Lemma 3.3 *Let S be (p,k)-quasihyponormal and $T^* \in HFP(H)$.*

(1) *If* $\ker T^* = \{0\}$, *then* $(S,T) \in FP$.

(2) *If* $\ker S \subseteq \ker S^*$, *then* $(S,T) \in FP$.

Proof By Lemma 2.1(1), every $FP(N)$ operator has a reducing kernel. Thus, by Lemma 2.5, we only need to prove (1). Let $SX = XT$. As in the proof of Lemma 2.4, (2)–(4) hold. By $\ker T^* = \{0\}$ and Lemma 3.2(2),

$$S_{11}X_{11} = X_{11}T_{11} \implies \ker S_{11}^* = \{0\}, \quad S_{11} \text{ is } (p,k)\text{-quasihyponormal},$$

thus S_{11} is p-hyponormal follows by Lemma 3.2(1). Hence $(S_{11}, T_{11}) \in FP$ by $T_{11}^* \in HFP(H)$ and Lemma 2.4. So S_{11} is normal and injective. Lemma 2.1(3) ensures $S_{12} = 0$. Since X_{11} is quasiaffine, by Theorem 1.2 and Lemma 2.1(1), T_{11} is normal and $T_{21} = 0$ hold. So that the assertion holds by (4). □

Proof of Theorem 3.1 (1) If $\ker T^* = \{0\}$, then $T^* \in HFP(H)$ by Remark 1.6, and the assertion follows by Lemma 3.3(1).

(2) If T^* is (p,k)-quasihyponormal and $\ker T^* \subseteq \ker T$, then $T^* \in HFP(H)$ (Remark 1.6). So $\ker S \subseteq \ker S^*$ and Lemma 3.3(2) ensure $(S,T) \in FP$.

(3)–(4) hold in a similar manner to (2). □

4 Extensions of Theorem 1.4

Theorem 4.1 *The following assertions hold and they are equivalent to each other.*

(1) *Let S be (p,k)-quasihyponormal, let T^* be (p,k)-quasihyponormal with reducing kernel, or dominant, or w-hyponormal with reducing kernel. If $\ker X^* = \{0\}$, then $(S,X,T) \in FP$ and S is normal.*

(2) *Let S be (p,k)-quasihyponormal with reducing kernel, or dominant, or w-hyponormal with reducing kernel, let T^* be (p,k)-quasihyponormal. If $\ker X = \{0\}$, then $(S,X,T) \in FP$ and T is normal.*

Theorem 4.1 implies that the normal operator T^* in Theorem 1.4(1) can be replaced with a (p,k)-quasihyponormal operator with reducing kernel, or a dominant operator, or a w-hyponormal operator with reducing kernel; and the p-hyponormal operator S in Theorem 1.4(2) can be replaced with a (p,k)-quasihyponormal operator with reducing kernel, or a dominant operator, or a w-hyponormal operator with reducing kernel.

Lemma 4.2 *The following assertions hold and* (1) *is equivalent to* (2).

(1) *If S is (p,k)-quasihyponormal, $T^* \in HFP(H)$, and $\ker X^* = \{0\}$, then $(S,X,T) \in FP$ and S is normal.*

(2) *If $S \in HFP(H)$, T^* is (p,k)-quasihyponormal and X is injective, then $(S,X,T) \in FP$ and T is normal.*

Proof According to (5), it is sufficient to prove (1). Decompose S, T, X into

$$\begin{aligned} S &= \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \in \mathcal{B}\big([S\mathcal{H}] \oplus \ker S^*\big), \\ T &= \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}\big([T\mathcal{K}] \oplus \ker T^*\big), \\ X &= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathcal{B}\big([T\mathcal{K}] \oplus \ker T^*, [S\mathcal{H}] \oplus \ker S^*\big). \end{aligned} \tag{9}$$

Since X has a dense range,

$$\begin{aligned} SX = XT \quad &\Longrightarrow \quad [XT\mathcal{K}] = [SX\mathcal{K}] = [S\mathcal{H}] \\ &\Longrightarrow \quad X_{21} = 0,\ \ker X_{11}^* = \ker X_{22}^* = \{0\}. \end{aligned} \tag{10}$$

Then

$$\begin{aligned} SX = XT \quad &\Longleftrightarrow \quad \begin{pmatrix} S_{11}X_{11} & S_{11}X_{12} + S_{12}X_{22} \\ 0 & S_{22}X_{22} \end{pmatrix} = \begin{pmatrix} X_{11}T_{11} & 0 \\ 0 & 0 \end{pmatrix} \\ &\Longleftrightarrow \quad S_{11}X_{11} = X_{11}T_{11},\ S_{11}X_{12} + S_{12}X_{22} = S_{22}X_{22} = 0, \end{aligned} \tag{11}$$

$$
\begin{aligned}
S^*X = XT^* \quad &\Longleftrightarrow \quad \begin{pmatrix} (S_{11})^*X_{11} & (S_{11})^*X_{12} \\ (S_{12})^*X_{11} & (S_{12})^*X_{12} + (S_{22})^*X_{22} \end{pmatrix} = \begin{pmatrix} X_{11}(T_{11})^* & 0 \\ 0 & 0 \end{pmatrix} \\
&\Longleftrightarrow \quad (S_{11})^*X_{11} = X_{11}(T_{11})^*, \\
&\qquad\quad (S_{11})^*X_{12} = (S_{12})^*X_{11} = (S_{12})^*X_{12} + (S_{22})^*X_{22} = 0. \qquad (12)
\end{aligned}
$$

Since S_{11} is (p,k)-quasihyponormal, $T_{11}^* \in HFP(H)$ and $\ker T_{11}^* = \{0\}$, $(S_{11}, T_{11}) \in FP$ by Lemma 3.3. So $S_{11} = S_{11}|_{[R(X_{11})]}$ is normal, $\ker S_{11}^* = \{0\}$ follows by $S_{11}X_{11} = X_{11}T_{11}$ and $\ker X_{11}^* = \ker T_{11}^* = \{0\}$.

Then $S_{12} = 0$ holds by Lemma 2.1(3). Equation (11) and $\ker S_{11} = \ker X_{22}^* = \{0\}$ imply $X_{12} = S_{22} = 0$. The assertion holds by (12). □

According to Remark 1.6, Theorem 4.1 follows by Lemma 4.2 directly.

5 Extensions of Theorem 1.5

Theorem 5.1 *The following assertions hold and they are equivalent to each other.*

(1) *Let S be (p,k)-quasihyponormal, let T^* be (p,k)-quasihyponormal, or dominant, or w-hyponormal with reducing kernel. If $\ker S^{*k} \subseteq \ker X^*$, then $(S,X,T) \in FP$ and S is normal.*

(2) *Let S be (p,k)-quasihyponormal with reducing kernel, or dominant, or w-hyponormal with reducing kernel, let T^* be (p,k)-quasihyponormal. If $\ker T^k \subseteq \ker X$, then $(S,X,T) \in FP$ and T is normal.*

Theorem 5.1(1) holds for every (p,k)-quasihyponormal operator T^* and implies that the restriction $\ker S \subseteq \ker S^{*k}$ in Theorem 1.5 is redundant.

Lemma 5.2 *The following assertions hold and they are equivalent to each other.*

(1) *If S is (p,k)-quasihyponormal, $T^* \in HFP(H)$ and $\ker S^{*k} \subseteq \ker X^*$, then $(S,X,T) \in FP$.*

(2) *If $S \in HFP(H)$, T^* is (p,k)-quasihyponormal and $\ker T^k \subseteq \ker X$, then $(S,X,T) \in FP$.*

Proof By (5), it is sufficient to prove (1). Decompose S, T, X into

$$
\begin{aligned}
S &= \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \in \mathcal{B}\big(\big[S^k\mathcal{H}\big] \oplus \ker S^{*k}\big), \\
T &= \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{B}\big(\big[R\big(X^*\big)\big] \oplus \ker X\big), \qquad (13) \\
X &= \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} \in \mathcal{B}\big(\big[R\big(X^*\big)\big] \oplus \ker X, \big[S^k\mathcal{H}\big] \oplus \ker S^{*k}\big).
\end{aligned}
$$

The condition $\ker S^{*k} \subseteq \ker X^*$ implies that $R(X) \subseteq [S^k\mathcal{H}]$, $X_{21} = 0$ and $\ker X_{11} = \{0\}$. Thus

$$
SX = XT \quad \Longleftrightarrow \quad \begin{pmatrix} S_{11}X_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11}T_{11} & 0 \\ 0 & 0 \end{pmatrix} \quad \Longleftrightarrow \quad S_{11}X_{11} = X_{11}T_{11}, \qquad (14)
$$

$$
S^*X = XT^* \quad \Longleftrightarrow \quad (S_{11})^*X_{11} = X_{11}(T_{11})^*,\ (S_{12})^*X_{11} = X_{11}(T_{21})^* = 0. \qquad (15)
$$

The operator S_{11} is p-hyponormal follows by Lemma 3.2. Since each p-hyponormal operator has a reducing kernel and $T_{11}^* \in HFP(H)$, $(S_{11}, T_{11}) \in FP$ follows. Hence $S_{11}|_{[R(X_{11})]}$ $(= S|_{[R(X)]})$ and $T_{11}|_{[R((X_{11})^*)]}(= T_{11})$ are unitarily equivalent normal operators. So $T_{21} = 0$ holds by Lemma 2.1(1), $S_{12} = 0$ by $[R(X)] \subseteq [S^k\mathcal{H}]$, and (3) of Lemma 2.1. Therefore the assertion holds by (15). □

Proof of Theorem 5.1 It is sufficient to prove (1). If T^* is dominant or w-hyponormal with reducing kernel, the assertion is a direct result of Lemma 5.2.

If T^* is (p,k)-quasihyponormal, as in the proof of Lemma 5.2, (13)–(15) hold. Since S_{11} is p-hyponormal and $\ker X_{11} = \{0\}$, $(S_{11}, X_{11}, T_{11}) \in FP$ holds by Lemma 4.2(2). Then $S_{11}|_{[R(X_{11})]}(= S|_{[R(X)]})$ and T_{11} are normal operators, and $S_{12} = 0$ follows by $[R(X)] \subseteq [S^k\mathcal{H}]$ and Lemma 2.1(3).

Furthermore, let $P = P_{[R(S^k)]}$ and $x \in \ker S_{11}$, then $P(S^*S)^pP \geq P(SS^*)^pP$ and $S^*Sx = 0 = (S^*S)^px$. Hence $0 = \langle (S^*S)^pPx, Px\rangle \geq \langle (SS^*)^pPx, Px\rangle = \|(SS^*)^{\frac{p}{2}}x\|^2$, $x \in \ker(SS^*)^{\frac{p}{2}} \cap [R(S^k)] = \ker S^* \cap [R(S^k)] \subseteq \ker S^{*k} \cap [R(S^k)] = \{0\}$. Therefore $\ker S_{11} = \{0\}$. Thus, by Lemma 2.1,

$$S_{11}X_{11} = X_{11}T_{11} \quad \Longrightarrow \quad \ker T_{11} = \{0\} \quad \Longrightarrow \quad T_{21} = 0.$$

So $(S, X, T) \in FP$ follows. □

At the end, we give an example which implies that some kernel conditions in Fuglede–Putnam type theorems above are crucial.

Example 5.3 Let $k \geq 2$ be a positive integer, S be an operator such that $S^{k-1} \neq 0$ and $S^k = 0$.

(1) S and S^* are (p,k)-quasihyponormal with $\ker S \neq 0$ and $\ker S^* \neq 0$, and $(S, S, S) \notin FP$.
(2) Let $P = P_{[R(S^{k-1})]}$, then $\ker S \not\subseteq \ker S^*$ and $(S, P, 1-P) \notin FP$.
(3) Let $P = P_{[R(S^{k-1})]}$, then $\ker P^* \neq 0$ and $(S, P, 1-P) \notin FP$.
(4) If $k = 2$, then S is a quasiclass A operator, $S \in R_3$ and $S \notin R_2$.

Example 5.3(1)–(2) says that, if T^* is (p,k)-quasihyponormal, the kernel condition $\ker T^* = \{0\}$ in Theorem 3.1(1) is inevitable. Example 5.3(3) implies that the kernel condition $\ker X^* = \{0\}$ in Theorem 4.1(1) is crucial.

Lemma 5.4 ([20]) *If* $\ker(T-\lambda) \subseteq \ker(T-\lambda)^*$ *for a fixed number* λ, *then* $\ker(T-\lambda) = \ker(T-\lambda)^2$ *and* $\ker(T-\lambda) \perp \ker(T-\mu)$ *for each* $\mu \neq \lambda$.

Lemma 5.5 ([20]) *Let k be a positive integer,* $T \in k\text{-}QA(n)$, *and* $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ *on* $\mathcal{M} \oplus \mathcal{M}^\perp$.

(1) *If* $[R(T^k)] \subseteq \mathcal{M}$, *then* $T_{22}^k = 0$ *and* $\sigma(T) = \sigma(T_{11}) \cup \{0\}$.
(2) *If* $T \in k\text{-}QA(n)$ *and* $\mathcal{M} \subseteq [R(T^k)]$, *then* $T_{11}(= T|_{\mathcal{M}}) \in A(n)$.

Proof of Example 5.3 (1) By $S^k = 0 = S^{*k}$, S and S^* are (p,k)-quasihyponormal. If $\ker S = 0$, then $\ker S^k = \ker S = 0$ and it contradicts the condition $\ker S^{k-1} \neq 0$. So $\ker S \neq 0$, and $\ker S^* \neq 0$ holds in a similar manner.

If $\ker S \subseteq \ker S^*$, then Lemma 5.4 implies $\ker S^k = \ker S$. It also contradicts the condition $S^{k-1} \neq 0$. Hence $\ker S \not\subseteq \ker S^*$, $S^*S \neq SS^*$, and $(S, S, S) \notin FP$.

(2) The assumption $S^k = 0$ implies $SP = 0 = P(1-P)$. By (1), $\ker S$ does not reduce S. So $S^*P \neq 0 = P(1-P)$ and $(S, P, 1-P) \notin FP$.

(3) If $\ker P^* = \ker S^{*(k-1)} = 0$, then $\ker S^* = \ker S^{*k} = 0$. It contradicts the condition $S^k = 0$. Hence $\ker P^* \neq 0$ and $(S, P, 1-P) \notin FP$.

(4) Since

$$S^2 = 0 \implies R(S) \subseteq \ker S \subseteq \ker S^2 = \ker\left|S^2\right| \implies S^*\left|S^2\right|S = 0 = S^*|S|^2 S,$$

S is a quasiclass A operator. By Lemma 5.5 and $S^2 = 0$, $S = \begin{pmatrix} 0 & S_{12} \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = [S\mathcal{H}] \oplus \ker S^*$. The assumption $S \neq 0$ ensures $S_{12} \neq 0$, so $S|_{[S\mathcal{H}]} = 0$ is normal and $[S\mathcal{H}]$ does not reduce S. Hence $S \in R_3$ [20, Theorem 2.4] and $S \notin R_2$. □

Funding
This work was supported in part by the National Natural Science Foundation of China (11301155), Project of Science and Technology Department of Henan Province of China (182102210306).

Availability of data and materials
All data generated or analysed during this study are included in this published article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally. All authors read and approved the final manuscript.

Publisher's Note

Received: 8 September 2018 Accepted: 16 April 2019 Published online: 07 May 2019

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