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A note on degenerate Bernstein polynomials

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Abstract

Recently, degenerate Bernstein polynomials have been introduced by Kim and Kim. In this paper, we investigate some properties and identities for the degenerate Bernstein polynomials associated with special numbers and polynomials including degenerate Bernstein polynomials and central factorial numbers of the second kind.

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Keywords: Degenerate Bernstein polynomials; Degenerate Bernoulli polynomials; Central factorial numbers of the second kind

1 Introduction

Bernstein polynomials were first used by Bernstein in a constructive proof for the Stone–Weierstrass approximation theorem (see [2, 6, 21]). With the advent of computer graphics, Bernstein polynomials, restricted to the interval $[0, 1]$, became important in the form of Bézier curves (see [6]). The Bernstein polynomials are the mathematical basis for Bézier curves, which are frequently used in the mathematical field of numerical analysis (see [6, 24]). The study of degenerate versions of special numbers and polynomials began with the papers by Carlitz (see [3, 4]). Kim and his research colleagues have been studying various degenerate numbers and polynomials by means of generating functions, Fourier series, combinatorial methods, umbral calculus, p-adic analysis, and differential equations (see [10, 11, 13, 17–19]).

As a degenerate version of Bernstein polynomials, the degenerate Bernstein polynomials were introduced recently (see (1.9)). Here we will study for the degenerate Bernstein polynomials some fundamental properties and identities associated with special numbers and polynomials including degenerate Bernoulli polynomials and central factorial numbers of the second kind. Also, in the last section we will consider a matrix representation for those polynomials. For some recent works related to the present paper, the reader may want to see [14, 20, 22, 25, 27, 29]. The rest of this section is devoted to reviewing what we need in the following sections.

For $k, n \in \mathbb{Z}_{\geq 0}$, the Bernstein polynomials of degree n are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{(see [1, 2, 6, 21, 26, 28])}, \tag{1.1}$$

where $x \in [0, 1]$.

For $\lambda \in \mathbb{R}$, the degenerate Bernoulli polynomials of order r are defined by the generating function

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{1.2}$$

When $r=1$, $\beta_{n,\lambda}(x) = \beta_{n,\lambda}^{(1)}(x)$ $(n \geq 0)$ are called the degenerate Bernoulli polynomials (see [3, 4]). Further, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

The falling factorial sequences are defined by

$$(x)_0 = 1, \qquad (x)_n = x(x-1)(x-2)\cdots(x-n+1) \quad (n \geq 1) \quad (\text{see [15, 23]}). \tag{1.3}$$

The λ-analogue of the falling factorial sequences are given by

$$(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\bigl(x-(n-1)\lambda\bigr) \quad (n \geq 1) \quad (\text{see [7, 15, 16]}). \tag{1.4}$$

Note that $\lim_{\lambda\to 1}(x)_{n,\lambda} = (x)_n$, $\lim_{\lambda\to 0}(x)_{n,\lambda} = x^n$.

It is known that the degenerate exponential function is defined by

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} \quad (\text{see [16]}). \tag{1.5}$$

The λ-binomial coefficients are given by

$$\binom{x}{n}_{\lambda} = \frac{(x)_{n,\lambda}}{n!} = \frac{x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)}{n!} \quad (\text{see [8, 16]}). \tag{1.6}$$

From (1.6), we have

$$\binom{x+y}{n}_{\lambda} = \sum_{l=0}^{n}\binom{x}{l}_{\lambda}\binom{y}{n-l}_{\lambda} \quad (\text{see [8, 12]}), \tag{1.7}$$

which is equivalent to

$$(x+y)_{n,\lambda} = \sum_{l=0}^{n}\binom{n}{l}(x)_{l,\lambda}(y)_{n-l,\lambda} \quad (\text{see [12, 14]}). \tag{1.8}$$

Recently, Kim and Kim [12, 14] introduced the degenerate Bernstein polynomials of degree n, $B_{k,n}(x|\lambda)$ $(n,k \geq 0)$, which are given by

$$\frac{(x)_{k,\lambda}}{k!}t^k(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=k}^{\infty}B_{k,n}(x|\lambda)\frac{t^n}{n!} \quad (\text{see [12]}), \tag{1.9}$$

where k is a nonnegative integer.

From (1.9), we note that

$$B_{k,n}(x|\lambda) = \binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} \quad (n \geq k \geq 0) \quad (\text{see [12]}). \tag{1.10}$$

Thus, by (1.10), we easily get

$$(x)_{i,\lambda} = \frac{1}{(1-\lambda i)_{n-i,\lambda}} \sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x|\lambda) \quad (\text{see [12]}), \tag{1.11}$$

where $i, n \in \mathbb{N}$ with $i \leq n$, $x \in [0,1]$.

As is well known, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \quad (n \geq 0) \quad (\text{see [7, 12]}). \tag{1.12}$$

In [8], the degenerate Stirling numbers of the second kind are given by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \quad (n \geq 0). \tag{1.13}$$

Note that $\lim_{\lambda \to 0} S_{2,\lambda}(n,l) = S_2(n,l)$.

The degenerate Bernstein polynomials have been introduced recently by Kim and Kim. In this paper, we investigate some properties and identities for the degenerate Bernstein polynomials associated with special numbers and polynomials including degenerate Bernstein polynomials and central factorial numbers of the second kind.

2 Some fundamental properties of the degenerate Bernstein polynomials

First, we observe that

$$\begin{aligned}
&\bigl(1-x-(n-k-1)\lambda\bigr)B_{k,n-1}(x|\lambda) + \bigl(x-(k-1)\lambda\bigr)B_{k-1,n-1}(x|\lambda) \\
&\quad = \binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} + \binom{n-1}{k-1}(x)_{k,\lambda}(1-x)_{n-k,\lambda} \\
&\quad = \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right](x)_{k,\lambda}(1-x)_{n-k,\lambda} \\
&\quad = \binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} = B_{k,n}(x|\lambda) \quad (n,k \in \mathbb{N}).
\end{aligned} \tag{2.1}$$

Thus, by (2.1), we get the next theorem which already appeared in [12].

Theorem 2.1 *For $n, k \in \mathbb{N}$, we have*

$$\begin{aligned}
&\bigl(1-x-(n-k-1)\lambda\bigr)B_{k,n-1}(x|\lambda) + \bigl(x-(k-1)\lambda\bigr)B_{k-1,n-1}(x|\lambda) \\
&\quad = B_{k,n}(x|\lambda).
\end{aligned} \tag{2.2}$$

By (1.8) and (1.10), we easily get

$$\sum_{i=0}^{k} B_{i,k}(x|\lambda) = \sum_{i=0}^{k} \binom{k}{i}(x)_{i,\lambda}(1-x)_{k-i,\lambda} = (x+1-x)_{k,\lambda} = (1)_{k,\lambda}, \tag{2.3}$$

where k is a nonnegative integer.

From (1.10), we have

$$\begin{aligned}(x-i\lambda)B_{i,n}(x|\lambda) &= (x-i\lambda)\binom{n}{i}(x)_{i,\lambda}(1-x)_{n-i,\lambda}\\ &= \binom{n}{i}(x)_{i+1,\lambda}(1-x)_{n-i,\lambda}\\ &= \frac{\binom{n}{i}}{\binom{n+1}{i+1}}\binom{n+1}{i+1}(x)_{i+1,\lambda}(1-x)_{n+1-(i+1),\lambda}\\ &= \frac{\binom{n}{i}}{\binom{n+1}{i+1}}B_{i+1,n+1}(x|\lambda).\end{aligned} \tag{2.4}$$

Hence, by (2.2) and (2.4), we get the following theorem.

Theorem 2.2 *For $k \geq 0$, we have*

$$\sum_{i=0}^{k}B_{i,k}(x|\lambda)=\sum_{i=0}^{k}\binom{k}{i}(x)_{i,\lambda}(1-x)_{k-i,\lambda}=(x+1-x)_{k,\lambda}=(1)_{k,\lambda},$$
$$(x-i\lambda)B_{i,n}(x|\lambda)=\frac{i+1}{n+1}B_{i+1,n+1}(x|\lambda).$$

On the other hand, we have

$$\begin{aligned}&\bigl(1-x-(n-i)\lambda\bigr)B_{i,n}(x|\lambda)\\ &\quad= \binom{n}{i}(x)_{i,\lambda}(1-x)_{n-i,\lambda}\bigl(1-x-(n-i)\lambda\bigr)\\ &\quad= \binom{n}{i}(x)_{i,\lambda}(1-x)_{n+1-i,\lambda}=\frac{\binom{n}{i}}{\binom{n+1}{i}}\binom{n+1}{i}(x)_{i,\lambda}(1-x)_{n+1-i,\lambda}\\ &\quad= \frac{\binom{n}{i}}{\binom{n+1}{i}}B_{i,n+1}(x|\lambda)\quad (i,n\in\mathbb{N}).\end{aligned} \tag{2.5}$$

Thus, by (2.5), we get the next result.

Theorem 2.3 *For $0 \leq i \leq n+1$, we have*

$$\bigl(1-x-(n-i)\lambda\bigr)B_{i,n}(x|\lambda)=\frac{n+1-i}{n+1}B_{i,n+1}(x|\lambda).$$

We observe that

$$\begin{aligned}
&\frac{1}{\binom{n}{i}}B_{i,n}(x|\lambda)+\frac{1}{\binom{n}{i+1}}B_{i+1,n}(x|\lambda)\\
&\quad=(x)_{i,\lambda}(1-x)_{n-i,\lambda}+(x)_{i+1,\lambda}(1-x)_{n-i-1,\lambda}\\
&\quad=(x)_{i,\lambda}(1-x)_{n-i-1,\lambda}\bigl(1-x-(n-i-1)\lambda+x-i\lambda\bigr)\\
&\quad=(x)_{i,\lambda}(1-x)_{n-i-1,\lambda}\bigl(1-(n-1)\lambda\bigr)\\
&\quad=\frac{1-(n-1)\lambda}{\binom{n-1}{i}}\binom{n-1}{i}(x)_{i,\lambda}(1-x)_{n-i-1,\lambda}\\
&\quad=\frac{1-(n-1)\lambda}{\binom{n-1}{i}}B_{i,n-1}(x|\lambda). \qquad (2.6)
\end{aligned}$$

Hence, by (2.6), we obtain the following identity which appeared already in [12].

$$\begin{aligned}
\bigl(1-(n-1)\lambda\bigr)B_{i,n-1}(x|\lambda)&=\binom{n-1}{i}\biggl[\frac{B_{i,n}(x|\lambda)}{\binom{n}{i}}+\frac{B_{i+1,n}(x|\lambda)}{\binom{n}{i+1}}\biggr]\\
&=\frac{n-i}{n}B_{i,n}(x|\lambda)+\frac{i+1}{n}B_{i+1,n}(x|\lambda)\quad(n\in\mathbb{N},i\geq 0). \qquad (2.7)
\end{aligned}$$

Also, from (1.1) and (1.10), we have

$$\frac{B_{k,n}(x|\lambda)}{B_{k,n}(x)}=\prod_{l=0}^{k-1}\biggl(1-\lambda\frac{l}{x}\biggr)\prod_{l=0}^{n-k-1}\biggl(1-\lambda\frac{l}{1-x}\biggr). \qquad (2.8)$$

Hence, by (2.7) and (2.8), we get the following theorem.

Theorem 2.4 *For $n,k,i\in\mathbb{N}$ with $i\leq n$ and $k\leq n$, we have*

$$\begin{aligned}
&\bigl(1-(n-1)\lambda\bigr)B_{i,n-1}(x|\lambda)=\frac{n-i}{n}B_{i,n}(x|\lambda)+\frac{i+1}{n}B_{i+1,n}(x|\lambda),\\
&\frac{B_{k,n}(x|\lambda)}{B_{k,n}(x)}=\prod_{l=0}^{k-1}\biggl(1-\lambda\frac{l}{x}\biggr)\prod_{l=0}^{n-k-1}\biggl(1-\lambda\frac{l}{1-x}\biggr).
\end{aligned}$$

3 Some identities for degenerate Bernstein polynomials associated with special numbers and polynomials

Here in this section, we are going to derive some identities associated with special numbers and polynomials including the degenerate Bernoulli polynomials and central factorial numbers of the second kind.

From (1.2), we note that

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}=\sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!} \qquad (3.1)$$

and

$$\beta_{n,\lambda}(x)=\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}\beta_{l,\lambda}\quad\text{(see [3, 4])}.$$

Thus, by (3.1), we get

$$\begin{aligned} t &= \left(\sum_{l=0}^{\infty}\beta_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}-1\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\beta_{l,\lambda}-\beta_{n,\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{3.2}$$

By comparing the coefficients on both sides of (3.2), we get

$$\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\beta_{l,\lambda}-\beta_{n,\lambda} = \begin{cases} 1, & \text{if } n=1, \\ 0, & \text{if } n>1, \end{cases} \qquad \beta_{0,\lambda}=1 \quad (\text{see [3, 4]}). \tag{3.3}$$

When $n = 1$, we have

$$\beta_{1,\lambda}(1) = \sum_{l=0}^{1}\binom{1}{l}(1)_{1-l,\lambda}\beta_{l,\lambda} = 1+\beta_{1,\lambda}.$$

By (1.2), we easily get

$$\begin{aligned} \sum_{n=0}^{\infty}\beta_{n,\lambda}(1-x)\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{1-x}{\lambda}} \\ &= \frac{-t}{(1+(-\lambda)(-t))^{-\frac{1}{\lambda}}-1}\big(1+(-\lambda)(-t)\big)^{-\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty}\beta_{n,-\lambda}(x)(-1)^n\frac{t^n}{n!}. \end{aligned} \tag{3.4}$$

Comparing the coefficients on both sides of (3.4), we have

$$\beta_{n,\lambda}(1-x) = (-1)^n\beta_{n,-\lambda}(x) \quad (n\ge 0).$$

Taking $x = -1$, $\beta_{n,\lambda}(2) = (-1)^n\beta_{n,-\lambda}(-1)$ $(n \ge 0)$. We observe that

$$\begin{aligned} \sum_{n=0}^{\infty}\beta_{n,\lambda}(2)\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{2}{\lambda}} \\ &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+1\big)(1+\lambda t)^{\frac{1}{\lambda}} \\ &= t(1+\lambda t)^{\frac{1}{\lambda}} + \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{1}{\lambda}} \\ &= t\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!} + \sum_{n=0}^{\infty}\beta_{n,\lambda}(1)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\big(n(1)_{n-1,\lambda}+\beta_{n,\lambda}(1)\big)\frac{t^n}{n!}. \end{aligned} \tag{3.5}$$

By (3.5), we get

$$\beta_{n,\lambda}(2) = n(1)_{n-1,\lambda} + \beta_{n,\lambda}(1) \quad (n \geq 1). \tag{3.6}$$

It is not difficult to show that

$$\begin{aligned}
&\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x-1}{\lambda}}\bigl((1+\lambda t)^{\frac{1}{\lambda}} - 1 + 1\bigr) \\
&\quad = t(1+\lambda t)^{\frac{x-1}{\lambda}} + \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x-1}{\lambda}} \\
&\quad = t(1+\lambda t)^{\frac{x-1}{\lambda}} + t(1+\lambda t)^{\frac{x-2}{\lambda}} + \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x-2}{\lambda}} \\
&\quad = t(1+\lambda t)^{\frac{x-1}{\lambda}} + t(1+\lambda t)^{\frac{x-2}{\lambda}} + t(1+\lambda t)^{\frac{x-3}{\lambda}} + \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \\
&\qquad \times (1+\lambda t)^{\frac{x-3}{\lambda}}.
\end{aligned} \tag{3.7}$$

Continuing this process, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!} &= t\sum_{i=1}^{k}(1+\lambda t)^{\frac{x-i}{\lambda}} + \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x-k}{\lambda}} \\
&= t\sum_{i=1}^{k}\sum_{n=0}^{\infty}(x-i)_{n,\lambda}\frac{t^n}{n!} + \sum_{n=0}^{\infty}\beta_{n,\lambda}(x-k)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{i=1}^{k} n(x-i)_{n-1,\lambda}\right)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\beta_{n,\lambda}(x-k)\frac{t^n}{n!}.
\end{aligned} \tag{3.8}$$

Therefore, by comparing the coefficients on both sides of (3.8), we obtain the following proposition.

Proposition 1 *For $k \in \mathbb{N}$ and $n \geq 0$, we have*

$$\beta_{n,\lambda}(x) = \sum_{i=1}^{k} n(x-i)_{n-1,\lambda} + \beta_{n,\lambda}(x-k).$$

In particular,

$$\beta_{n,\lambda}(k) = \sum_{i=1}^{k} n(k-i)_{n-1,\lambda} + \beta_{n,\lambda}.$$

From (1.9), we have

$$\begin{aligned}(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} &= \frac{k!}{t^k}\frac{1}{k!}(x)_{k,\lambda}t^k(1+\lambda t)^{\frac{1-x}{\lambda}}\\ &= \frac{k!}{t^k}\sum_{n=k}^{\infty}B_{k,n}(x|\lambda)\frac{t^n}{n!} = k!\sum_{n=0}^{\infty}B_{k,n+k}(x|\lambda)\frac{n!}{(n+k)!}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{1}{\binom{n+k}{k}}B_{k,n+k}(x|\lambda)\frac{t^n}{n!}.\end{aligned} \tag{3.9}$$

On the other hand,

$$(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{k,\lambda}(1-x)_{n,\lambda}\frac{t^n}{n!}. \tag{3.10}$$

From (3.9) and (3.10), we have

$$(x)_{k,\lambda}(1-x)_{n,\lambda} = \frac{1}{\binom{n+k}{k}}B_{k,n+k}(x|\lambda) \quad (n,k \geq 0). \tag{3.11}$$

Note here that (3.11) also follows from (1.10) by replacing n by $n+k$.

From (1.2), we note that

$$\begin{aligned}(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} &= \frac{(x)_{k,\lambda}}{t}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{1-x}{\lambda}}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)\\ &= \frac{(x)_{k,\lambda}}{t}\left\{\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{2-x}{\lambda}} - \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{1-x}{\lambda}}\right\}\\ &= \frac{(x)_{k,\lambda}}{t}\left\{\sum_{n=1}^{\infty}\beta_{n,\lambda}(2-x)-\beta_{n,\lambda}(1-x)\right\}\frac{t^n}{n!}\\ &= (x)_{k,\lambda}\sum_{n=0}^{\infty}\left(\frac{\beta_{n+1,\lambda}(2-x)-\beta_{n+1,\lambda}(1-x)}{n+1}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.12}$$

By (3.12), we get

$$(x)_{k,\lambda}(1-x)_{n,\lambda} = (x)_{k,\lambda}\frac{\beta_{n+1,\lambda}(2-x)-\beta_{n+1,\lambda}(1-x)}{n+1}, \tag{3.13}$$

where $k \in \mathbb{N}$ and $n \geq 0$.

From (1.10) and (3.13), we have

$$B_{k,n+k}(x|\lambda) = \binom{n+k}{k}(x)_{k,\lambda}\frac{\beta_{n+1,\lambda}(2-x)-\beta_{n+1,\lambda}(1-x)}{n+1}.$$

Theorem 3.1 *For $k \in \mathbb{N}$ and $n \geq 0$, we have*

$$B_{k,n+k}(x|\lambda) = \binom{n+k}{k}(x)_{k,\lambda}\frac{\beta_{n+1,\lambda}(2-x)-\beta_{n+1,\lambda}(1-x)}{n+1}.$$

As is well known, the central factorial numbers of the second kind are defined by the generating function

$$\frac{1}{k!}\left(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}\right)^k=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!}\quad (k\geq 0)\quad \text{(see [5, 9])}. \tag{3.14}$$

Recently, the degenerate central factorial numbers $T_\lambda(n,k)$ of the second kind were introduced by the generating function

$$\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k=\sum_{n=k}^{\infty}T_\lambda(n,k)\frac{t^n}{n!}\quad \text{(see [16])}. \tag{3.15}$$

By (3.15), we get

$$\begin{aligned}\sum_{n=k}^{\infty}\lim_{\lambda\to 0}T_\lambda(n,k)\frac{t^n}{n!}&=\lim_{\lambda\to 0}\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k\\&=\frac{1}{k!}\left(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}\right)^k=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!}.\end{aligned} \tag{3.16}$$

Thus, by (3.16), we get

$$\lim_{\lambda\to 0}T_\lambda(n,k)=T(n,k)\quad (n,k\geq 0)\quad \text{(see [9])}.$$

Now, we observe that

$$\begin{aligned}\frac{(x)_{k,\lambda}}{k!}t^k(1+\lambda t)^{\frac{1-x}{\lambda}}&=\frac{(x)_{k,\lambda}}{k!}\left((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k\\&\quad\times\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^k(1+\lambda t)^{\frac{1-x+\frac{k}{2}}{\lambda}}\\&=(x)_{k,\lambda}\left(\sum_{m=k}^{\infty}T_\lambda(m,k)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(k)}\left(1-x+\frac{k}{2}\right)\frac{t^l}{l!}\right)\\&=\sum_{n=k}^{\infty}(x)_{k,\lambda}\sum_{m=k}^{n}\binom{n}{m}T_\lambda(m,k)\beta_{n-m,\lambda}^{(k)}\left(1-x+\frac{k}{2}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.17}$$

By combining the right-hand side of (1.9) with (3.17), we obtain the following theorem.

Theorem 3.2 *For $n,k\in\mathbb{N}\cup\{0\}$ with $n\geq k$, we have*

$$B_{k,n}(x|\lambda)=(x)_{k,\lambda}\sum_{m=k}^{n}\binom{n}{m}T_\lambda(m,k)\beta_{n-m,\lambda}^{(k)}\left(1-x+\frac{k}{2}\right).$$

4 A matrix representation for degenerate Bernstein polynomials

For $\lambda\in\mathbb{R}$, let

$$\mathbb{P}_{n,\lambda}=\left\{p_\lambda(x)\middle|p_\lambda(x)=\sum_{i=0}^{n}(x)_{i,\lambda}C_{i,\lambda}\in\mathbb{R}[x]\right\}.$$

Then $\mathbb{P}_{n,\lambda}$ is the $n+1$-dimensional vector space over $\mathbb{R}$.

For $B_\lambda(x) \in \mathbb{P}_{n,\lambda}$, we note that $B_\lambda(x)$ can be written as a linear combination of degenerate Bernstein basis functions:

$$\begin{aligned} B_\lambda(x) &= C_{0,\lambda} B_{0,n}(x|\lambda) + C_{1,\lambda} B_{1,n}(x|\lambda) + C_{2,\lambda} B_{2,n}(x|\lambda) + \cdots \\ &\quad + C_{n,\lambda} B_{n,n}(x|\lambda), \end{aligned} \tag{4.1}$$

where the constants $C_{i,\lambda}$ depend on λ for $i = 0, 1, 2, \ldots, n$.

Equation (4.1) can be written as the dot product of two vectors in the following:

$$B_\lambda(x) = \begin{pmatrix} B_{0,n}(x|\lambda) & B_{1,n}(x|\lambda) & \cdots & B_{n,n}(x|\lambda) \end{pmatrix} \begin{pmatrix} C_{0,\lambda} \\ C_{1,\lambda} \\ \vdots \\ C_{n,\lambda} \end{pmatrix}. \tag{4.2}$$

Now, we can convert (4.2) to

$$\begin{aligned} B_\lambda(x) &= \begin{pmatrix} 1 & x & \cdots & x^n \end{pmatrix} \\ &\quad \times \begin{pmatrix} b_{0,0}(\lambda) & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{1,0}(\lambda) & b_{1,1}(\lambda) & b_{1,2}(\lambda) & b_{1,3}(\lambda) & \cdots & b_{1,n-1}(\lambda) & b_{1,n}(\lambda) \\ b_{2,0}(\lambda) & b_{2,1}(\lambda) & b_{2,2}(\lambda) & b_{2,3}(\lambda) & \cdots & b_{2,n-1}(\lambda) & b_{2,n}(\lambda) \\ \vdots & & & & & & \vdots \\ b_{n-1,0}(\lambda) & b_{n-1,1}(\lambda) & b_{n-1,2}(\lambda) & b_{n-1,3}(\lambda) & \cdots & b_{n-1,n-1}(\lambda) & b_{n-1,n}(\lambda) \\ b_{n,0}(\lambda) & b_{n,1}(\lambda) & b_{n,2}(\lambda) & b_{n,3}(\lambda) & \cdots & b_{n,n-1}(\lambda) & b_{n,n}(\lambda) \end{pmatrix} \\ &\quad \times \begin{pmatrix} C_{0,\lambda} \\ C_{1,\lambda} \\ \vdots \\ C_{n,\lambda} \end{pmatrix}, \end{aligned}$$

where $b_{i,j}(\lambda)$ are the coefficients of the power basis that are used to determine the respective degenerate Bernstein polynomials.

For example, by (1.1), we get

$$\begin{aligned} B_{0,2}(x|\lambda) &= \binom{2}{0} (x)_{0,\lambda} (1-x)_{2,\lambda} = (1-x)(1-x-\lambda) = (1-x)^2 - \lambda(1-x) \\ &= x^2 + (\lambda - 2)x + 1 - \lambda, \\ B_{1,2}(x|\lambda) &= \binom{2}{1} (x)_{1,\lambda} (1-x)_{1,\lambda} = 2x(1-x) = 2x - 2x^2, \\ B_{2,2}(x|\lambda) &= \binom{2}{2} (x)_{2,\lambda} (1-x)_{0,\lambda} = x(x-\lambda) = x^2 - \lambda x. \end{aligned}$$

In the quadratic case ($n = 2$), $B_\lambda(x)$ can be represented in terms of matrices by

$$B_\lambda(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1-\lambda & 0 & 0 \\ \lambda-2 & 2 & -\lambda \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} C_{0,\lambda} \\ C_{1,\lambda} \\ C_{2,\lambda} \end{pmatrix}.$$

5 Conclusions

In Sect. 2, we investigated some fundamental properties for the degenerate Bernstein polynomials. In Sect. 3, we derived some identities for the degenerate Bernstein polynomials associated with special numbers and polynomials including degenerate Bernoulli polynomials and central factorial numbers of the second kind. In many applications, a matrix formulation for the Bernstein polynomials is useful. So, in Sect. 4, we studied some further properties of the matrix representation for degenerate Bernstein polynomials.

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