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Statistical inference for the new INAR(2) models with random coefficient

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Abstract

In this paper, we investigate a random coefficient INAR(2) process which may model the number of traded stocks, the number of infected people, the number of birds in some area, etc. We show that this process is a stationary and ergodic process under some mild conditions. Adopting the two-step conditional least-square estimation method, we give consistent estimations of the unknown parameters. Furthermore, the asymptotic distributions of the estimators are obtained and a simulation study is conducted for the evaluation of the developed approach.

Keywords: INAR process; Random coefficient models; Asymptotic distribution

1 Introduction

Integer-valued time series data have been studied a lot in the past three decades because its many applications in different fields. The integer-valued autoregressive models (INARs) defined through the thinning operator are the most popular model for describing such count data and have been extensively investigated by McKenzie [1], and investigated in detail by Al-Osh and Alzaid [2], Alzaid and Al-Osh [3], among others.

The classical INAR(1) model is defined as

$$X_t = \alpha \circ X_{t-1} + \epsilon_t, \quad t \in \mathbf{Z}, \quad (1.1)$$

where $\alpha \in (0, 1)$ is a constant, $\alpha \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} B_i$, $\{B_i\}$ is an i.i.d. Bernoulli random sequence with $P(B_i = 1) = 1 - P(B_i = 0) = \alpha$, and is independent of $\{X_{t-1}\}$, $\{\epsilon_t\}$ is a sequence of i.i.d. nonnegative integer-valued random variables with mean λ and variance σ_ϵ^2 , and is independent of $\{X_{t-1}\}$. Zheng et al. [4] extend the INAR(1) model to the random coefficient INAR(1) model, i.e. suppose that a random variable with cumulative distribution function (CDF) P_ϕ on $(0, 1)$. Since then, there were many authors to consider the INAR models with random coefficient. For example, Zheng et al. [5] consider statistical inference for the INAR(p) model with random coefficient, Zhang et al. [6, 7] investigate the INAR(1) and INAR(p) models using empirical likelihood method, Zhang and Wang [8] obtain some inference for random coefficient INAR(1) process based on frequency domain analysis; Nedényi and Pap [9] establish the iterated scaling limits for the aggregation of random coefficient INAR(1) processes, Ding and Wang [10] suppose that the random coefficient is incorporate with explanatory variables, Nastić and Ristić [11] introduce some geometric mixed INAR models.

In this paper we will study the new INAR(2) model with random coefficient (NINAR(2)) which is defined as follows:

$$X_t = \begin{cases} \alpha_1 \circ X_{t-1} + \epsilon_t & \text{with probability } p_1; \\ \alpha_2 \circ X_{t-2} + \epsilon_t & \text{with probability } p_2; \\ \epsilon_t & \text{with probability } 1 - p_1 - p_2, \end{cases} \quad (1.2)$$

where $\alpha_1, \alpha_2 \in (0, 1)$ are constant, $\{\epsilon_t\}$ is a sequence of i.i.d. nonnegative integer-valued random variables with mean λ and variance σ_ϵ^2 , and it is independent of $\{X_{t-1}\}$. It is easy to check that $\{X_t\}$ can be rewritten as

$$X_t = \theta_1 \circ X_{t-1} + \theta_2 \circ X_{t-2} + \epsilon_t,$$

where the random vector (θ_1, θ_2) is i.i.d. for different t and we have the joint distribution given by

$$\begin{aligned} P(\theta_1 = \alpha_1, \theta_2 = \alpha_2) &= 0, & P(\theta_1 = \alpha_1, \theta_2 = 0) &= p_1, \\ P(\theta_1 = 0, \theta_2 = \alpha_2) &= p_2, & P(\theta_1 = 0, \theta_2 = 0) &= 1 - p_1 - p_2. \end{aligned} \quad (1.3)$$

Here $p_1 + p_2 < 1$. That is why we call this model the INAR(2) model with random coefficient.

Lawrance and Lewis [12] investigate NEAR(2) models which are the nonlinear autoregressive time series in exponential variables, Dewald and Lewis [13] study the new Laplace second order autoregressive time series model, i.e. the NLAR(2) model, later, Karlsen and Tjøstheim [14] give the consistent estimates of the unknown parameters of the NEAR(2) and the NLAR(2) models. Inspired by the study of them we consider the NINAR(2) model defined by (1.2). There are many results about the inference of INAR(p) model with random coefficients; see [7] and [5] for more details. In general, one may assume that all the random coefficients are independent random variables. In this paper, we allow the dependence between random coefficients. Therefore, this model can be applied when we consider the dependence between random coefficients. Furthermore, the advantage of the proposed model over the general random coefficient INAR(2) model is that the unknown parameters can be estimated directly, therefore it can be applied easily.

The outline of this paper is as follows. In Sect. 2, we investigate the stationary and ergodic properties, present the estimations of parameters and give their asymptotic properties. In Sect. 3, we present some simulation results. In Sect. 4, we make conclusions. All proofs are postponed to Sect. 5.

2 Main results

The stationarity and ergodicity are important for time series, therefore we first consider the existence of unique stationary and ergodic solution of the random coefficient INAR(2) given in (1.2).

Theorem 2.1 *Suppose that $\alpha_1, \alpha_2 \in (0, 1)$ and $p_1, p_2 \in (0, 1)$, then there exists a strict stationary and ergodic integer-value random series satisfying model (1.2).*

Use the stationarity and ergodicity properties of the process, we can obtain the estimation of the unknown parameters. We use two-step condition least-square estimation to estimate the unknown parameters. Let \mathcal{F}_t be the σ -field generated by $\{X_s, s \leq t\}$. Note that $E[X_t|\mathcal{F}_{t-1}] = p_1\alpha_1X_{t-1} + p_2\alpha_2X_{t-2} + \lambda = \beta_1X_{t-1} + \beta_2X_{t-2} + \lambda$ with $\beta_1 = p_1\alpha_1, \beta_2 = p_2\alpha_2$. Use

$$S(\eta) = \sum_{t=3}^n (X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)^2, \tag{2.1}$$

denote the CLS criterion function, where $\eta = (\beta_1, \beta_2, \lambda)^T$. Then the CLS estimator of η is given by

$$\hat{\eta}_{CLS} = \arg \min_{\eta} S(\eta).$$

Denote $E_n(\eta) = (E_{n1}(\eta), E_{n2}(\eta), E_{n3}(\eta))$ the derivatives of $\partial S(\eta)/\partial \eta$, that is, $E_n(\eta) = \partial S(\eta)/\partial \eta$. By solving the equations $E_n(\eta) = 0$, i.e.,

$$\begin{aligned} E_{n1}(\eta) &= \frac{\partial S(\eta)}{\partial \beta_1} = \sum_{t=3}^n (X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)X_{t-1} = 0, \\ E_{n2}(\eta) &= \frac{\partial S(\eta)}{\partial \beta_2} = \sum_{t=3}^n (X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)X_{t-2} = 0, \\ E_{n3}(\eta) &= \frac{\partial S(\eta)}{\partial \lambda} = \sum_{t=3}^n (X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda) = 0, \end{aligned} \tag{2.2}$$

we obtain the estimator of η , which is as follows:

$$\hat{\eta} = M^{-1}b,$$

where $b = (\sum_{t=3}^n X_tX_{t-1}, \sum_{t=3}^n X_tX_{t-2}, \sum_{t=3}^n X_t)^T$ and

$$M = \begin{pmatrix} \sum_{t=3}^n X_{t-1}^2 & \sum_{t=3}^n X_{t-1}X_{t-2} & \sum_{t=3}^n X_{t-1} \\ \sum_{t=3}^n X_{t-1}X_{t-2} & \sum_{t=3}^n X_{t-2}^2 & \sum_{t=3}^n X_{t-2} \\ \sum_{t=3}^n X_{t-1} & \sum_{t=3}^n X_{t-2} & n-2 \end{pmatrix}.$$

In order to estimate the parameters $\alpha_1, \alpha_2, p_1, p_2$, we consider the conditional least-square estimation of the process $V_t = (X_t - E(X_t|\mathcal{F}_{t-1}))^2$. It is easy to verify that

$$\begin{aligned} E(V_t|\mathcal{F}_{t-1}) &= E(X_t^2|\mathcal{F}_{t-1}) - (E(X_t|\mathcal{F}_{t-1}))^2 \\ &= (\alpha_1\beta_1 - \beta_1^2)X_{t-1}^2 + (\alpha_2\beta_2 - \beta_2^2)X_{t-2}^2 + (\beta_1 - \alpha_1\beta_1)X_{t-1} + (\beta_2 - \alpha_2\beta_2)X_{t-2} \\ &\quad - 2\beta_1\beta_2X_{t-1}X_{t-2} + \sigma_\epsilon^2. \end{aligned}$$

Then the CLS (conditional least-square) criterion function for $\theta = (\alpha_1\beta_1 - \beta_1^2, \alpha_2\beta_2 - \beta_2^2, \beta_1 - \alpha_1\beta_1, \beta_2 - \alpha_2\beta_2, 2\beta_1\beta_2, \sigma_\epsilon^2)^T$ is given by

$$S(\theta) = \sum_{t=3}^n (V_t - E(V_t|\mathcal{F}_{t-1}))^2.$$

The CLS estimator of θ is given by

$$\hat{\theta}_{CLS} = \arg \min_{\theta} S(\theta).$$

Let $Z_t = (X_{t-1}^2, X_{t-2}^2, X_{t-1}, X_{t-2}, -X_{t-1}X_{t-2}, 1)^T$, thus by solving the equations $\partial S(\theta)/\partial \theta = 0$, we obtain the estimator of θ , which is as follows:

$$\hat{\theta}(\eta) = \left(\frac{1}{n-2} \sum_{t=3}^n Z_t Z_t^T \right)^{-1} \left(\frac{1}{n-2} \sum_{t=3}^n V_t Z_t \right).$$

Let $\bar{\theta}(\hat{\eta})$ be the estimator $\hat{\theta}(\eta)$ with η replaced by $\hat{\eta}$. Define $\hat{\eta}_1, \hat{\eta}_2$ as components of $\hat{\eta}$ and $\bar{\theta}_1(\hat{\eta}), \bar{\theta}_2(\hat{\eta})$ as components of $\bar{\theta}(\hat{\eta})$. We obtain the estimators of $\alpha_1, \alpha_2, p_1, p_2$ as follows:

$$\hat{\alpha}_1 = \frac{\bar{\theta}_1(\hat{\eta}) + \hat{\eta}_1^2}{\hat{\eta}_1}, \quad \hat{\alpha}_2 = \frac{\bar{\theta}_2(\hat{\eta}) + \hat{\eta}_2^2}{\hat{\eta}_2}, \quad \hat{p}_1 = \frac{\hat{\eta}_1^2}{\bar{\theta}_1(\hat{\eta}) + \hat{\eta}_1^2}, \quad \hat{p}_2 = \frac{\hat{\eta}_2^2}{\bar{\theta}_2(\hat{\eta}) + \hat{\eta}_2^2}.$$

About the consistency and asymptotic property of the estimators, we have the following theorems.

Theorem 2.2 *Assume that the process $\{X_t\}$ is a stationary ergodic process and $E|X_t|^8 < \infty$, then we see that $(\sqrt{n}(\bar{\theta}(\hat{\eta}) - \theta), \sqrt{n}(\hat{\eta} - \eta))$ converges to a normal distribution with mean zero and covariance matrix*

$$\Omega = (\omega_{ij})_{9 \times 9} = \begin{pmatrix} \Gamma^{-1}W\Gamma^{-1} & \Gamma^{-1}\Pi V^{-1} \\ V^{-1}\Pi\Gamma^{-1} & V^{-1}\Sigma V^{-1} \end{pmatrix},$$

where $V = \lim_{n \rightarrow \infty} (1/n)M, \Sigma = E(X_t - \eta^T D_t)^2 D_t D_t^T, \Gamma = E Z_t Z_t^T, W = E((V_t - Z_t^T \theta)^2 Z_t Z_t^T), \Pi = E((V_t - Z_t^T \theta)(X_t - \eta^T D_t) Z_t D_t^T), D_t = (X_{t-1}, X_{t-2}, 1)^T$.

Theorem 2.3 *Assume that the process $\{X_t\}$ is a stationary ergodic process and $E|X_t|^8 < \infty$, then the estimators $\hat{\alpha}_1, \hat{\alpha}_2, \hat{p}_1, \hat{p}_2$ are consistent estimators and have an asymptotic normal distribution with mean zero and variance given by (5.2) and (5.3), respectively.*

If $\alpha_2 = 0$, the model (1.2) is a specific first order INAR model with random coefficient. Zhao and Hu [15] give the estimators of the unknown parameters by using the least-square method. Therefore, we need to consider the following hypotheses:

$$H_0 : \alpha_2 = 0 \quad \text{vs.} \quad H_1 : \alpha_2 > 0.$$

Based on the asymptotical normality of the $\hat{\alpha}_2$ given by Theorem 2.3, we know that $\sqrt{n}(\hat{\alpha}_2 - \alpha_2)/\text{Var}(\hat{\alpha}_2)$ converges weakly to $N(0, 1)$, where $\text{Var}(\hat{\alpha}_2)$ is given by (5.2). There-

fore, we need to find an estimator of $\text{Var}(\hat{\alpha}_2)$. By Theorem 2.3, we know that

$$\text{Var}(\hat{\alpha}_2) = \frac{\beta_2^2 \omega_{22} + (\beta_2^2 - \theta_2)^2 \omega_{88} + \beta_2(\beta_2^2 - \theta_2) \omega_{28}}{\beta_2^4}.$$

In order to test this hypothesis, we need to estimate the unknown parameters in the above variance. Based on Theorem 2.3, we know that the estimators of β_2, θ_2 can be given by $\hat{\beta}_2 = \hat{\eta}_2, \hat{\theta}_2(\hat{\eta})$. From the stationary and ergodic properties, we can use the estimators

$$\hat{V} = \frac{1}{n-2} M, \quad \hat{\Gamma} = \frac{1}{n-2} \sum_{t=3}^n Z_t Z_t^T \tag{2.3}$$

to estimate V, Γ . We use the following estimators to estimate Σ, W, Π , respectively:

$$\begin{aligned} \hat{\Sigma} &= \frac{1}{n-2} \sum_{t=3}^n (X_t - \hat{\eta}^T D_t)^2 D_t D_t^T, \\ \hat{W} &= \frac{1}{n-2} \sum_{t=3}^n (V_t(\hat{\eta}) - Z_t^T \hat{\theta}_2(\hat{\eta}))^2 Z_t Z_t^T, \\ \hat{\Pi} &= \frac{1}{n-2} \sum_{t=3}^n (X_t - \hat{\eta}^T D_t)(V_t(\hat{\eta}) - Z_t^T \hat{\theta}_2(\hat{\eta})) Z_t D_t^T. \end{aligned}$$

Corollary 2.4 *Under the condition of Theorem 2.2, we conclude that*

$$\hat{\Sigma} \xrightarrow{P} \Sigma, \quad \hat{W} \xrightarrow{P} W, \quad \hat{\Pi} \xrightarrow{P} \Pi.$$

Thus we use the following statistic to test H_0 :

$$\frac{\sqrt{n}(\hat{\alpha}_2 - \alpha_2)}{\Upsilon}, \quad \text{here } \Upsilon = \frac{\hat{\beta}_2^2 \hat{\omega}_{22} + (\hat{\beta}_2^2 - \hat{\theta}_2)^2 \hat{\omega}_{88} + \hat{\beta}_2(\hat{\beta}_2^2 - \hat{\theta}_2) \hat{\omega}_{28}}{\hat{\beta}_2^4}.$$

Next we consider the one-step conditional expectation prediction of this process. Note that $E[X_t | \mathcal{F}_{t-1}] = \beta_1 X_{t-1} + \beta_2 X_{t-2} + \lambda$, we can use

$$\hat{X}_{t-1} = \hat{\beta}_1 X_{t-1} + \hat{\beta}_2 X_{t-2} + \hat{\lambda}$$

as the prediction value of X_t . From the asymptotic normality given in Theorem 2.2, we know $\sqrt{n}(\hat{\eta} - \eta) \xrightarrow{d} N(0, V^{-1} \Sigma V^{-1})$. Thus we have

$$\sqrt{n}(\hat{X}_{t-1} - E[X_t | \mathcal{F}_{t-1}]) | X_{t-1}, X_{t-2} \xrightarrow{d} N(0, A_t^T V^{-1} \Sigma V^{-1} A_t),$$

where $A_t^T = (X_{t-1}, X_{t-2}, 1)$. Then, by (2.3), we can obtain the confidence interval for the prediction value of X_t ,

$$\left[\hat{X}_{t-1} - \sqrt{\frac{A_t^T \hat{V}^{-1} \hat{\Sigma} \hat{V}^{-1} A_t}{n}} u_{\frac{\nu}{2}}, \hat{X}_{t-1} + \sqrt{\frac{A_t^T \hat{V}^{-1} \hat{\Sigma} \hat{V}^{-1} A_t}{n}} u_{\frac{\nu}{2}} \right],$$

where $u_{\nu/2}$ is the upper ν -quantile of the standard normal distribution.

Table 1 Bias and SE of the parameters in Models I with $\lambda = 1$

Sample size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$
50	(0.0236, 1.9775)	(0.3464, 1.5224)	(0.2219, 1.8419)	(0.0827, 0.9487)
100	(-0.0727, 0.8381)	(0.1041, 0.4088)	(0.0362, 0.6182)	(-0.0462, 0.9545)
200	(0.0344, 0.3609)	(0.0794, 0.2350)	(-0.03262, 0.9745)	(-0.1367, 0.2902)
500	(0.0181, 0.2290)	(0.0548, 0.1612)	(-0.1604, 0.8419)	(-0.0477, 0.1436)

Table 2 Bias and SE of the parameters in Models I with $\lambda = 2$

Sample size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$
50	(0.1352, 2.5840)	(0.3216, 3.4862)	(0.1786, 3.9142)	(-0.0954, 2.0482)
100	(0.0292, 1.7485)	(-0.0842, 1.6980)	(-0.0701, 1.1221)	(0.0716, 1.8540)
200	(-0.1227, 0.8058)	(0.0583, 0.2757)	(-0.0290, 1.5073)	(-0.0824, 0.3254)
500	(-0.0356, 0.2659)	(-0.0200, 0.1691)	(-0.0347, 0.4000)	(0.0131, 0.1199)

Table 3 Bias and SE of the parameters in Models II with $\lambda = 1$

Sample size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$
50	(0.1758, 1.5514)	(0.0972, 1.4666)	(-0.0179, 3.8898)	(0.0639, 1.7230)
100	(0.0902, 0.5844)	(0.0115, 1.3228)	(0.1757, 2.5644)	(-0.0125, 1.5973)
200	(0.0215, 0.5722)	(0.0361, 0.4129)	(0.0602, 0.5640)	(-0.0982, 0.7852)
500	(0.0416, 0.4463)	(0.0485, 0.2199)	(-0.0025, 0.8419)	(-0.0402, 0.1436)

Table 4 Bias and SE of the parameters in Models II with $\lambda = 2$

Sample size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$
50	(-0.0898, 1.7002)	(0.2452, 1.7146)	(0.0572, 1.2197)	(-0.1446, 2.2698)
100	(0.0773, 1.3076)	(0.05041, 1.3555)	(0.0303, 0.7952)	(0.0696, 1.9334)
200	(0.0570, 1.7427)	(0.0363, 0.3787)	(-0.0580, 1.5712)	(0.0172, 0.8405)
500	(-0.0443, 0.3132)	(0.0336, 0.1920)	(0.0409, 0.3914)	(-0.0248, 0.3106)

3 Simulation studies

In this section we present some simulation study.

3.1 Empirical results for unknown parameters

We consider the following two models:

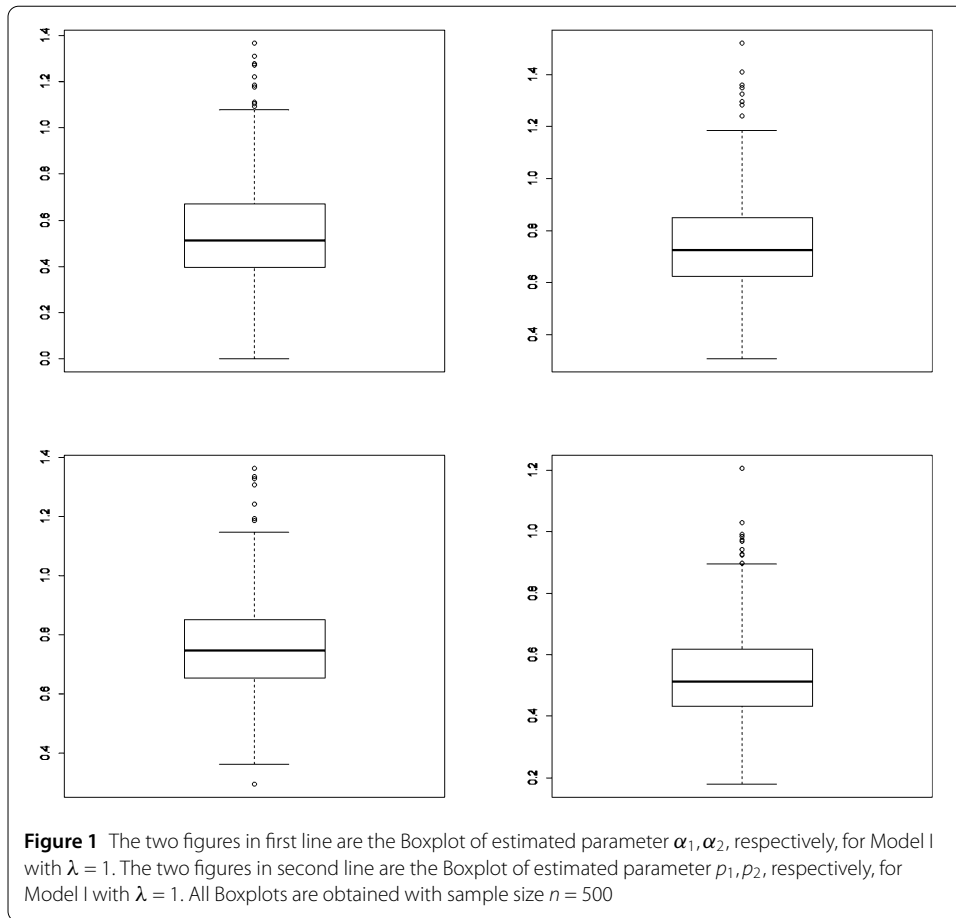
Model I: $\alpha_1 = 0.6, \alpha_2 = 0.8, p_1 = 0.4, p_2 = 0.5$ with $\lambda = 1$ and $\lambda = 2$.

Model II: $\alpha_1 = 0.7, \alpha_2 = 0.5, p_1 = 0.3, p_2 = 0.6$ with $\lambda = 1$ and $\lambda = 2$.

For both models, we obtain the empirical bias (Bias) and the standard error (SE) based on 500 replications for each parameter combination. These simulation studies are given in Table 1, 2, 3 and 4, respectively, where the format (Bias, SE) is used; for example, (-0.02130.0389) means that the bias is -0.0213, and SE is 0.0389.

From the simulation results, we can see that the bias and standard errors are getting smaller when the sample size increasing. For smaller sample size, the standard errors are a little bigger, this may be because the true values of parameters are small and may disappear in their own stand error.

In order to obtain a visualization of some of the distributional properties, we present the Box plot of estimated parameters in Figs. 1 and 2.



3.2 Test for parameters

In this subsection, we consider to test the following hypotheses for Model I and II, given in Sect. 3.1:

$$H_0 : \alpha_2 = 0 \quad \text{vs.} \quad H_1 : \alpha_2 > 0.$$

We report the empirical sizes for Model III and IV at a significance level 0.05 with sample size $n = 50, 100, 200, 500$, respectively:

Model III: $\alpha_1 = 0.6, \alpha_2 = 0, p_1 = 0.2, p_2 = 0.7$ with $\lambda = 1$ and $\lambda = 2$.

Model IV: $\alpha_1 = 0.3, \alpha_2 = 0, p_1 = 0.2, p_2 = 0.7$ with $\lambda = 1$ and $\lambda = 2$.

The results are presented in Table 5. From Table 5, we can see that the empirical sizes is closed to 0.05 when n increases.

In order to investigate the power of the test, we consider the alternative hypothesis with parameter $\alpha_2 = 0.2, 0.6, 0.8$ for Model III and IV, respectively. We report the empirical power at a significance level 0.05 with sample size $n = 50, 100, 200, 500$. The simulation results are given by Tables 6–9. From these tables, we can see that the power increases monotonically when the parameter α_2 increases.

4 Conclusion

In this paper we study the new INAR(2) model with random coefficient, this model is more practical than the classical INAR(2) model with random coefficient because the pa-

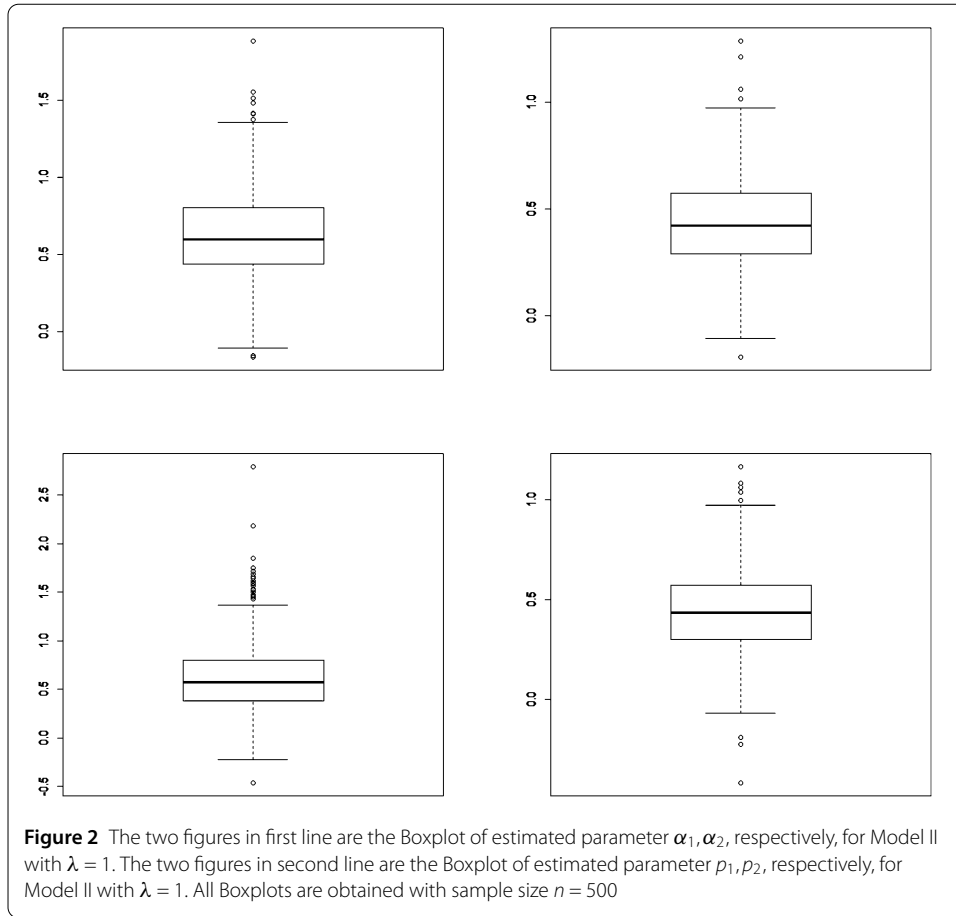


Table 5 Empirical sizes

Sample size	$\lambda = 1$ (III)	$\lambda = 2$ (III)	$\lambda = 1$ (IV)	$\lambda = 2$ (IV)
50	0.076	0.076	0.084	0.09
100	0.066	0.08	0.08	0.056
200	0.058	0.046	0.056	0.074
500	0.052	0.056	0.044	0.056

Table 6 Empirical powers for Model III with $\lambda = 1$

$\alpha_2 \setminus$ sample size	50	100	200	500
$\alpha_2 = 0.2$	0.59	0.608	0.706	0.842
$\alpha_2 = 0.4$	0.818	0.93	0.984	0.997
$\alpha_2 = 0.6$	0.808	0.942	0.996	0.999
$\alpha_2 = 0.8$	0.998	0.998	0.999	1

parameters can be estimated by the usual estimation method while for the classical INAR(2) model with random coefficient only the mean of the random coefficient can be estimated. We use the two-step conditional least squares estimate to estimate the unknown parameters. Also the stationary and ergodic properties of this model are established which guarantee the estimation method can be applied. The asymptotic properties of the estimators are investigated. The efficiency of our estimation method is illustrated by the simulation study.

Table 7 Empirical powers for Model III with $\lambda = 2$

$\alpha_2 \setminus$ sample size	50	100	200	500
$\alpha_2 = 0.2$	0.684	0.686	0.682	0.846
$\alpha_2 = 0.4$	0.864	0.936	0.976	0.999
$\alpha_2 = 0.6$	0.905	0.954	0.998	1
$\alpha_2 = 0.8$	0.978	0.997	0.999	1

Table 8 Empirical powers for Model IV with $\lambda = 1$

$\alpha_2 \setminus$ sample size	50	100	200	500
$\alpha_2 = 0.2$	0.636	0.604	0.696	0.824
$\alpha_2 = 0.4$	0.856	0.884	0.994	0.998
$\alpha_2 = 0.6$	0.964	0.996	0.999	0.998
$\alpha_2 = 0.8$	0.994	0.997	1	1

Table 9 Empirical powers for Model IV with $\lambda = 2$

$\alpha_2 \setminus$ sample size	50	100	200	500
$\alpha_2 = 0.2$	0.512	0.648	0.646	0.808
$\alpha_2 = 0.4$	0.816	0.916	0.992	0.999
$\alpha_2 = 0.6$	0.942	0.949	0.999	0.999
$\alpha_2 = 0.8$	0.994	0.997	1	1

5 The proofs of main results

Proof of Theorem 2.1 Define a random sequence $\{X_t^{(n)}\}_{n \in \mathbb{Z}}$ as follows:

$$X_t^{(n)} = \begin{cases} 0, & n < 0; \\ \epsilon_t, & n = 0; \\ \theta_1 \circ X_{t-1}^{n-1} + \theta_2 \circ X_{t-2}^{n-1} + \epsilon_t, & n > 0. \end{cases} \tag{5.1}$$

The random vector (θ_1, θ_2) has a joint distribution given by (1.3) and is independent of $\{\epsilon_t\}$, the random sequences used in the operator $\theta_1 \circ$ and $\theta_2 \circ$ are the same for fixed t . We first prove that the first two moments of $\{X_t^{(n)}\}$ are finite. It is easy to verify that

$$EX_t^{(0)} = \lambda, \quad EX_t^{(1)} = (\beta_1 + \beta_2)\lambda + \lambda,$$

where $\beta_1 = \alpha_1 p_1, \beta_2 = \alpha_2 p_2$. Using the method of induction, we conclude that

$$EX_t^{(n)} = \sum_{i=0}^n (\beta_1 + \beta_2)^i \lambda < \infty.$$

We have

$$\begin{aligned} E(X_t^{(0)})^2 &= E\epsilon_t^2, E(X_t^{(1)})^2 \\ &= (\alpha_1 \beta_1 + \alpha_2 \beta_2)E\epsilon_t^2 + E\epsilon_t^2 + \lambda[\beta_1(1 - \alpha_1) + \beta_2(1 - \alpha_2)] + 2\lambda^2(\beta_1 + \beta_2). \end{aligned}$$

For convenience, let $A = \beta_1(1 - \alpha_1) + \beta_2(1 - \alpha_2) + 2\lambda(\beta_1 + \beta_2)$. Then using the method of induction, we conclude that

$$E(X_t^{(n)})^2 = E\epsilon_t^2 \sum_{i=0}^n (\alpha_1\beta_1 + \alpha_2\beta_2)^i + A\lambda \left(\sum_{k=1}^{n-1} (\alpha_1\beta_1 + \alpha_2\beta_2)^{n-k} \sum_{i=0}^{k-1} (\beta_1 + \beta_2)^i \right) < \infty.$$

The last inequality is obtained by the fact that

$$\alpha_1\beta_1 + \alpha_2\beta_2 = \alpha_1^2 p_1 + \alpha_2^2 p_2 < p_1 + p_2 < 1, \quad \beta_1 + \beta_2 = \alpha_1 p_1 + \alpha_2 p_2 < p_1 + p_2 < 1.$$

Next we consider the convergence of the sequence $\{X_t^{(n)}\}$. By the definition of the sequence $\{X_t^n\}$, we have

$$\begin{aligned} E|X_t^{(n)} - X_t^{(n-1)}| &= E|\theta_1 \circ X_{t-1}^{(n-1)} + \theta_2 \circ X_{t-2}^{(n-1)} - \theta_1 \circ X_{t-1}^{(n-2)} - \theta_2 \circ X_{t-2}^{(n-2)}| \\ &\leq E|\theta_1 \circ X_{t-1}^{(n-1)} - \theta_1 \circ X_{t-1}^{(n-2)}| + E|\theta_2 \circ X_{t-2}^{(n-1)} - \theta_2 \circ X_{t-2}^{(n-2)}| \\ &= \beta_1 E|X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}| + \beta_2 E|X_{t-2}^{(n-1)} - X_{t-2}^{(n-2)}|, \end{aligned}$$

where the last equality is because the operator $\theta_{t1} \circ$ is the same for fixed t . Repeat the deduction and notice that $E|X_{t-1}^{(1)} - X_{t-1}^{(0)}| = E\epsilon_t$, we conclude that

$$E|X_t^{(n)} - X_t^{(n-1)}| \leq (\beta_1 + \beta_2)^n E\epsilon_t.$$

Then, by the triangle inequality, we obtain, for any integers $n > m$,

$$E|X_t^{(n)} - X_t^{(m)}| \leq E\epsilon_t (n - m) \sum_{j=m+1}^n (\beta_1 + \beta_2)^j,$$

which tends to zero as $n, m \rightarrow \infty$. Note that

$$\begin{aligned} E|X_t^{(n)} - X_t^{(n-1)}|^2 &= E|\theta_1 \circ X_{t-1}^{(n-1)} + \theta_2 \circ X_{t-2}^{(n-1)} - \theta_1 \circ X_{t-1}^{(n-2)} - \theta_2 \circ X_{t-2}^{(n-2)}|^2 \\ &= E|\theta_1 \circ X_{t-1}^{(n-1)} - \theta_1 \circ X_{t-1}^{(n-2)}|^2 + E|\theta_2 \circ X_{t-2}^{(n-1)} - \theta_2 \circ X_{t-2}^{(n-2)}|^2 \\ &\quad + 2E|\theta_1 \circ X_{t-1}^{(n-1)} - \theta_1 \circ X_{t-1}^{(n-2)}| |\theta_2 \circ X_{t-2}^{(n-1)} - \theta_2 \circ X_{t-2}^{(n-2)}| \\ &\leq [\beta_1(1 - \alpha_1) + 2\beta_1] E|X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}| + \alpha_1 \beta_1 E|X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}|^2 \\ &\quad + [\beta_2(1 - \alpha_2) + 2\beta_2] E|X_{t-2}^{(n-1)} - X_{t-2}^{(n-2)}| + \alpha_2 \beta_2 E|X_{t-2}^{(n-1)} - X_{t-2}^{(n-2)}|^2 \\ &\leq \dots \\ &\leq (\alpha_1\beta_1 + \alpha_2\beta_2)^n E\epsilon_t^2 + BE\epsilon_t \sum_{k=1}^{n-1} (\alpha_1\beta_1 + \alpha_2\beta_2)^{k-1} (\beta_1 + \beta_2)^{n-k}, \\ &\leq (\alpha_1\beta_1 + \alpha_2\beta_2)^n E\epsilon_t^2 + BE\epsilon_t n (\beta_1 + \beta_2)^n, \end{aligned}$$

where $B = \beta_1(1 - \alpha_1) + 2\beta_1 + \beta_2(1 - \alpha_2) + 2\beta_2$. Then we get, for any integers $n > m$,

$$E|X_t^{(n)} - X_t^{(m)}|^2 \leq [E\epsilon_t^2 + BE\epsilon_t](n - m) \sum_{j=m+1}^n [(\alpha_1\beta_1 + \alpha_2\beta_2)^j + j(\beta_1 + \beta_2)^j],$$

which tends to zero as $n, m \rightarrow \infty$. This implies that $\{X_t^{(n)}\}$ is a Cauchy sequence. Let $\{X_t\}$ be the limit process of $\{X_t^{(n)}\}$, then the first two moments of X_t exist. Now we verify that $\{X_t\}$ satisfies (1.2). Since $X_t^{(n)} \xrightarrow{L_2} X_t$, we have, for any t ,

$$E|\theta_1 \circ X_t^{(n)} - \theta_1 \circ X_t|^2 = \beta_1(1 - \alpha_1)E|X_t^{(n)} - X_t| + \alpha_1\beta_1E|X_t^{(n)} - X_t|^2 \rightarrow 0.$$

Similarly, we can prove that for any t

$$E|\theta_2 \circ X_t^{(n)} - \theta_2 \circ X_t|^2 \rightarrow 0.$$

By the uniqueness of the convergence in L_2 , we conclude that $\{X_t\}$ satisfies (1.2).

Notice that $X_t^{(1)} = \epsilon_t$, we know that $\{X_t^{(1)}\}$ is a strict stationary process, then, by the induction method and the definition (5.1), we can see that, for each n , the process $\{X_t^{(n)}\}$ is strict stationary. The ergodicity can be obtained similarly as in Zhang et al. [16], we omit the details here. □

Proof of Theorem 2.2 By the strict stationarity and ergodicity of the process $\{X_t\}$, adopting the standard martingale central limit theorem, we obtain

$$\sqrt{n}(\hat{\eta} - \eta) = \left(\frac{1}{n}M\right)^{-1} \frac{1}{\sqrt{n}}E_n(\eta) \xrightarrow{d} N(0, V^{-1}\Sigma V^{-1}),$$

where $V = \lim_{n \rightarrow \infty} (1/n)M, E_n(\eta) = (E_{n1}(\eta), E_{n2}(\eta), E_{n3}(\eta))^T, \Sigma = (\sigma_{ij})$ is a symmetric matrix with

$$\begin{aligned} \sigma_{11} &= E(X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)^2X_{t-1}^2, & \sigma_{22} &= E(X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)^2X_{t-2}^2, \\ \sigma_{33} &= E(X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)^2, & \sigma_{21} &= E(X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)^2X_{t-1}X_{t-2}, \\ \sigma_{23} &= E(X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)^2X_{t-2}, & \sigma_{31} &= E(X_t - \beta_1X_{t-1} - \beta_2X_{t-2} - \lambda)^2X_{t-1}. \end{aligned}$$

Note that

$$\sqrt{n-2}(\hat{\theta}(\eta) - \theta) = \left(\frac{1}{n-2} \sum_{t=3}^n Z_t Z_t^T\right)^{-1} \left(\frac{1}{\sqrt{n-2}} \sum_{t=3}^n Z_t (V_t - Z_t^T \theta)\right).$$

By simple calculation, we see that $Z_t(V_t - Z_t^T \theta)$ is a martingale. Then, by the condition $E|X_t|^8 < \infty$, stationarity and ergodicity of the process $\{X_t\}$, using the martingale central limit theorem as above, we have

$$\sqrt{n}(\hat{\theta}(\eta) - \theta) \xrightarrow{d} N(0, \Gamma^{-1}W\Gamma),$$

where $\Gamma = E(Z_t Z_t^T)$, $W = E((V_t - Z_t^T \theta)^2 Z_t Z_t^T)$. Observe that

$$\sqrt{n-2}(\bar{\theta}(\hat{\eta}) - \theta) = \sqrt{n-2}(\bar{\theta}(\hat{\eta}) - \hat{\theta}(\eta)) + \sqrt{n-2}(\hat{\theta}(\eta) - \theta)$$

and

$$\sqrt{n}(\bar{\theta}(\hat{\eta}) - \hat{\theta}(\eta)) = \left(\frac{1}{n} \sum_{t=1}^n Z_t Z_t^T \right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t (V_t(\hat{\eta}) - V_t(\eta)) \right),$$

where $V_t(\hat{\eta})$ is the $V_t(\eta)$ replaced by η with $\hat{\eta}$. By a Taylor expansion, we have

$$\begin{aligned} V_t(\hat{\eta}) - V_t(\eta) &= -2(X_t - E[X_t | \mathcal{F}_{t-1}]) [X_{t-1}(\hat{\beta}_1 - \beta_1) + X_{t-2}(\hat{\beta}_2 - \beta_2) + (\hat{\lambda} - \lambda)] \\ &\quad + o_P(\|\hat{\eta} - \eta\|). \end{aligned}$$

Note that

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t X_{t-1} (X_t - E[X_t | \mathcal{F}_{t-1}]) (\hat{\beta}_1 - \beta_1) \\ &= \sqrt{n}(\hat{\beta}_1 - \beta_1) \frac{1}{n} \sum_{t=1}^n Z_t X_{t-1} (X_t - E[X_t | \mathcal{F}_{t-1}]). \end{aligned}$$

Notice that $E[Z_t X_{t-1} (X_t - E[X_t | \mathcal{F}_{t-1}])] = 0$, then, by the ergodicity of the process $\{X_t\}$, we conclude that

$$\frac{1}{n} \sum_{t=1}^n Z_t X_{t-1} (X_t - E[X_t | \mathcal{F}_{t-1}]) \rightarrow 0, \quad \text{a.s.}$$

Combining with the fact that $\sqrt{n}(\hat{\beta}_1 - \beta_1)$ converges in distribution, we have

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t X_{t-1} (X_t - E[X_t | \mathcal{F}_{t-1}]) (\hat{\beta}_1 - \beta_1) = o_P(1).$$

Similarly, we can prove that

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t X_{t-2} (X_t - E[X_t | \mathcal{F}_{t-1}]) (\hat{\beta}_2 - \beta_2) = o_P(1), \\ &\frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t (X_t - E[X_t | \mathcal{F}_{t-1}]) (\hat{\lambda} - \lambda) = o_P(1). \end{aligned}$$

Therefore, we get

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t (V_t(\hat{\eta}) - V_t(\eta)) = o_P(1).$$

Again, by the ergodicity of the process $\{X_t\}$, we have

$$\frac{1}{n} \sum_{t=1}^n Z_t Z_t^T \rightarrow \Gamma, \quad \text{a.s.}$$

Thus, we can obtain the conclusion that

$$\sqrt{n}(\bar{\theta}(\hat{\eta}) - \hat{\theta}(\eta)) = o_p(1).$$

Then, by the Slutsky theorem, we have

$$\sqrt{n}(\bar{\theta}(\hat{\eta}) - \theta) = \sqrt{n}(\bar{\theta}(\hat{\eta}) - \bar{\theta}(\eta)) + \sqrt{n}(\bar{\theta}(\eta) - \theta) \rightarrow N(0, \Gamma^{-1}W\Gamma^{-1}).$$

Therefore, we conclude that the vector $(\sqrt{n}(\bar{\theta}(\hat{\eta}) - \theta), \sqrt{n}(\hat{\eta} - \eta))$ converges to a normal distribution with mean zero and covariance matrix

$$\Omega = (\omega_{ij})_{9 \times 9} = \begin{pmatrix} \Gamma^{-1}W\Gamma^{-1} & \Gamma^{-1}\Pi V^{-1} \\ V^{-1}\Pi\Gamma^{-1} & V^{-1}\Sigma V^{-1} \end{pmatrix},$$

where $\Pi = E((V_t - Z_t^T\theta)(X_t - \eta^T D_t)Z_t D_t^T)$, $D_t = (X_t, X_{t-1}, 1)^T$. □

Proof of Theorem 2.3 We can see that the estimators $\hat{\alpha}_1, \hat{\alpha}_2, \hat{p}_1, \hat{p}_2$ are consistent and have asymptotic normal distribution.

Specially, $\sqrt{n}(\hat{\alpha}_1 - \alpha_1), \sqrt{n}(\hat{\alpha}_2 - \alpha_2)$ converge to normal distribution with mean zero and variance

$$\begin{aligned} & \frac{\beta_1^2 \omega_{11} + (\beta_1^2 - \theta_1)^2 \omega_{77} + \beta_1(\beta_1^2 - \theta_1)\omega_{17}}{\beta_1^4}, \\ & \frac{\beta_2^2 \omega_{22} + (\beta_2^2 - \theta_2)^2 \omega_{88} + \beta_2(\beta_2^2 - \theta_2)\omega_{28}}{\beta_2^4}. \end{aligned} \tag{5.2}$$

And $\sqrt{n}(\hat{p}_1 - p_1), \sqrt{n}(\hat{p}_2 - p_2)$ converge to normal distribution with mean zero and variance

$$\begin{aligned} & \frac{\beta_1^2(2\theta_1^2 + \beta_1)^2 \omega_{77} + 4\beta_1^4 \theta_1 \omega_{11} - 4\beta_1^3 \theta_1(2\theta_1^2 + \beta_1)\omega_{17}}{(\theta_1^2 + \beta_1)^4}, \\ & \frac{\beta_2^2(2\theta_2^2 + \beta_2)^2 \omega_{88} + 4\beta_2^4 \theta_2 \omega_{22} - 4\beta_2^3 \theta_2(2\theta_2^2 + \beta_2)\omega_{28}}{(\theta_2^2 + \beta_2)^4}. \end{aligned} \tag{5.3}$$

Proof of Corollary 2.4 Observe that

$$\begin{aligned} \hat{W} - W &= \frac{1}{n-2} \sum_{t=3}^n [(V_t(\hat{\eta}) - Z_t^T \bar{\theta}_2(\hat{\eta}))^2 - (V_t - Z_t^T \theta_2)^2] Z_t Z_t^T \\ &= \frac{1}{n-2} \sum_{t=3}^n [V_t(\hat{\eta})^2 - V_t^2] Z_t Z_t^T + \frac{1}{n-2} \sum_{t=3}^n [(Z_t^T \bar{\theta}_2(\hat{\eta}))^2 - (Z_t^T \theta_2)^2] Z_t Z_t^T \\ &\quad + \frac{1}{n-2} \sum_{t=3}^n [2V_t Z_t^T \theta_2 - 2V_t(\hat{\eta}) Z_t^T \bar{\theta}_2(\hat{\eta})] Z_t Z_t^T \end{aligned}$$

$$:= T_1 + T_2 + T_3.$$

Note that

$$T_1 = \frac{1}{n-2} \sum_{t=3}^n (\hat{\eta}^T - \eta^T) F_t Z_t Z_t^T \quad \text{where } F_t = (X_{t-1}, X_{t-2}, 1).$$

We have

$$|T_1| \leq \|\hat{\eta}^T - \eta^T\| \frac{1}{n-2} \sum_{t=3}^n \|F_t\| \|Z_t Z_t^T\| \quad \text{where } F_t = (X_{t-1}, X_{t-2}, 1).$$

By ergodicity of the process $\{X_t\}$ and the fact that $\hat{\eta}^T \xrightarrow{P} \eta$, we have

$$T_1 = o_p(1).$$

Similarly, we have

$$T_2 = o_p(1), \quad T_3 = o_p(1).$$

Similarly, we can prove that $\hat{\Sigma} \xrightarrow{P} \Sigma, \hat{\Pi} \xrightarrow{P} \Pi$. The conclusion of this corollary follows. \square

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