Wang *Journal of Inequalities and Applications* (2019) 2019:87
https://doi.org/10.1186/s13660-019-2043-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A sharp reverse Bonnesen-style inequality and generalization

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Abstract

We investigate the isoperimetric deficit of the oval domain in the Euclidean plane. Via the kinematic formulae of Poincaré and Blaschke, and Blaschke's rolling theorem, we obtain a sharp reverse Bonnesen-style inequality for a plane oval domain, which improves Bottema's result. Furthermore, we extend the isoperimetric deficit to the symmetric mixed isoperimetric deficit for two plane oval domains, and we obtain two reverse Bonnesen-style symmetric mixed inequalities, which are generalizations of Bottema's result and its strengthened form.

MSC: Primary 52A10; secondary 52A22

Keywords: Convex domain; Isoperimetric deficit; Isoperimetric inequality; Reverse Bonnesen-style inequality; Symmetric mixed isoperimetric deficit; Reverse Bonnesen-style symmetric mixed inequality

1 Introduction and main results

Integral geometry originated from geometric probability. It is a very important branch of the global differential geometry, which investigates the global properties of manifolds and convex bodies. Geometric inequality is an important topic in integral geometry. Perhaps the classical isoperimetric inequality is the oldest geometric inequality, that is, the disc encloses the maximum area among all domains of fixed perimeter. Let K be a domain of area A with simple boundary of perimeter P in $\mathbb{R}^2$, then

$$P^2 - 4\pi A \geq 0, \tag{1.1}$$

the equality sign holds if and only if K is a disc.

The root of the classical isoperimetric problem can be traced back to ancient Greece. However, the rigorous mathematical proof of the isoperimetric inequality was obtained in the 19th century. Via the variational method, the first rigorous mathematical proof of the isoperimetric inequality was obtained by Weierstrass in 1870. By comparing a simple closed curve and a circle, Schmidt found a concise proof of the isoperimetric inequality in 1938. The isoperimetric inequality has been extended to the discrete case, the higher dimensions, and the surface of constant curvature (see [1, 2, 6, 9–11, 15, 18, 22, 31–35]).

The quantity of the isoperimetric inequality (1.1)

$$\Delta_2(K) = P^2 - 4\pi A \tag{1.2}$$

measures the deficit between K and a disc of radius $P/2\pi$, it is called the isoperimetric deficit of K.

During the 1920s, Bonnesen proved some inequalities of the following form:

$$\Delta_2(K) = P^2 - 4\pi A \geq B_K, \tag{1.3}$$

where B_K is a nonnegative invariant of geometric significance and $B_K = 0$ if and only if K is a disc. An inequality of the form (1.3) is called the Bonnesen-style inequality, and it is stronger than the isoperimetric inequality (1.1). Many Bonnesen-style inequalities have been found (see [1, 4, 12, 16, 19, 33]).

Conversely, we considered the upper bound of the isoperimetric deficit, that is,

$$\Delta_2(K) = P^2 - 4\pi A \leq U_K, \tag{1.4}$$

where U_K is a nonnegative invariant of geometric significance, it is called the reverse Bonnesen-style inequality.

For the oval domain K in $\mathbb{R}^2$, Bottema obtained the following reverse Bonnesen-style inequality (see [5]):

$$P^2 - 4\pi A \leq \pi^2(\rho_M - \rho_m)^2, \tag{1.5}$$

where ρ_m and ρ_M are the minimum and maximum of the continuous curvature radius ρ of the boundary ∂K, respectively. The equality holds if and only if $\rho_m = \rho_M$, that is, K is a disc. Howard, Gao, Pan, Zhang, and others (see [8, 17, 29]) obtained some reverse Bonnesen-style inequalities with the methods of analysis and curvature flow as follows:

$$P^2 - 4\pi A \leq c|\tilde{A}|, \tag{1.6}$$

where c is a constant and $\tilde{A}$ is the area of $\tilde{K}$, the domain $\tilde{K}$ is bounded by the locus of the curvature centers of ∂K, where the equality sign holds if and only if K is a disc, that is, $\tilde{K}$ is a point. Some reverse Bonnesen-style inequalities for surface X_ϵ^2 of constant curvature have been obtained in [13, 23, 27, 28]. Zhou et al. obtained some reverse Bonnesen-style inequalities for any convex domain in [33].

By comparing a simple closed curve and a circle, Schmidt proved the isoperimetric inequality in 1938. We were motivated by Schmidt's works, we compared the two simple closed curves directly and obtained the symmetric mixed isoperimetric inequality (see [14, 20, 21, 24–26, 30]). That is, let K_k $(k = 0, 1)$ be two domains of areas A_k with simple boundaries of perimeters P_k in $\mathbb{R}^2$. Then

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 0, \tag{1.7}$$

where the equality sign holds if and only if both K_0 and K_1 are discs. When one of the domains is a disc, inequality (1.7) immediately reduces to (1.1). That is, the symmetric mixed isoperimetric inequality (1.7) is a generalization of the isoperimetric inequality (1.1).

The quantity

$$\Delta_2(K_0,K_1) = P_0^2P_1^2 - 16\pi^2A_0A_1 \tag{1.8}$$

is called the symmetric mixed isoperimetric deficit of K_0 and K_1.

We were motivated by Bonnesen's works, we considered whether there is a nonnegative invariant B_{K_0,K_1} of geometric significance such that

$$\Delta_2(K_0,K_1) = P_0^2P_1^2 - 16\pi^2A_0A_1 \geq B_{K_0,K_1}, \tag{1.9}$$

where $B_{K_0,K_1} = 0$ if and only if both K_0 and K_1 are discs. An inequality of the form (1.9) is called the Bonnesen-style symmetric mixed inequality, it is stronger than the symmetric mixed isoperimetric inequality (1.7). Zhou, Xu, Zeng, and others (see [14, 20, 21, 24–26, 30]) obtained some Bonnesen-style symmetric mixed inequalities with the known kinematic formulae of Poincaré and Blaschke.

Conversely, we considered the upper bound of the symmetric mixed isoperimetric deficit of K_0 and K_1, that is,

$$\Delta_2(K_0,K_1) = P_0^2P_1^2 - 16\pi^2A_0A_1 \leq U_{K_0,K_1}, \tag{1.10}$$

where U_{K_0,K_1} is a nonnegative invariant of geometric significance, it is called the reverse Bonnesen-style symmetric mixed inequality. When one of the domains is a disc, an inequality of the form (1.10) reduces to a reverse Bonnesen-style inequality. For any convex domain K_k ($k = 0,1$) of areas A_k and perimeters P_k in $\mathbb{R}^2$, Zhou, Xu, Zeng, and others obtained the following reverse Bonnesen-style symmetric mixed inequalities (see [21, 25, 30]):

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \leq 4\pi^2P_0P_1\left(R_{01}R_1^2 - r_{01}r_1^2\right), \tag{1.11}$$

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \leq 16\pi^4\left(R_0^2R_1^2 - r_0^2r_1^2\right), \tag{1.12}$$

where $r_{01} = \max\{t : t(gK_1) \subseteq K_0; g \in G_2\}$ and $R_{01} = \min\{t : t(gK_1) \supseteq K_0; g \in G_2\}$ are the inradius of K_0 with respect to K_1 and the outradius of K_0 with respect to K_1, respectively. G_2 is a group of plane rigid motions. R_k and r_k are the radius of the minimum circumscribed disc and the radius of the maximum inscribed disc of K_k, respectively. Each equality sign holds if and only if both K_0 and K_1 are discs.

The purpose of this paper is to find some new reverse Bonnesen-style inequalities for the oval domain in $\mathbb{R}^2$, which generalize known reverse Bonnesen-style inequalities. Via the kinematic formulae of Poincaré and Blaschke, and Blaschke's rolling theorem, we obtain a sharp reverse Bonnesen-style inequality (3.10) in Theorem 3.2 as follows:

$$P^2 - 4\pi A \leq (2\pi\rho_M - P)(P - 2\pi\rho_m),$$

which improves Bottema's result. Furthermore, we obtain two reverse Bonnesen-style symmetric mixed inequalities (4.10) and (4.11) in Theorem 4.2 as follows:

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \leq 4\pi^2A_1^2\left(\rho_{01}^M - \rho_{01}^m\right)^2,$$

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \leq 16\pi^2A_1^2\left(\rho_{01}^M - \frac{P_0P_1}{4\pi A_1}\right)\left(\frac{P_0P_1}{4\pi A_1} - \rho_{01}^m\right).$$

When K_1 is a unit disc, (4.10) reduces to the known reverse Bonnesen-style inequality (1.5) of Bottema, inequality (4.11) reduces to (3.10).

2 Preliminaries

A set of points K in $\mathbb{R}^n$ is convex if the line segment $\lambda x + (1-\lambda)y \in K$ for all $x, y \in K$ and $0 \leq \lambda \leq 1$. A domain is a set with nonempty interior, and an oval domain is a convex domain of boundary at least C^2. A convex body is a compact convex domain. The Minkowski sum of convex sets K and L, the scalar product of convex set K with $\lambda \geq 0$ are, respectively, defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

and

$$\lambda K = \{\lambda x : x \in K\}.$$

A homothety of the convex set K is of the form $x + \lambda K$ for $x \in \mathbb{R}^n$, $\lambda > 0$.

For the proof of the main theorem, we cite Blaschke's rolling theorem in $\mathbb{R}^2$ from [3, 7, 13, 25].

Lemma 2.1 (Blaschke's rolling theorem) *Let K be an oval domain in $\mathbb{R}^2$, ρ_m and ρ_M be the minimum and maximum of the curvature radius of ∂K, respectively, B_t be a circle of radius t in $\mathbb{R}^2$.*

If $t \in (0, \rho_m]$ and B_t is tangent to ∂K inside, then B_t has no other common point with ∂K.

If $t \in [\rho_M, +\infty)$ and B_t is tangent to ∂K outside, then B_t has no other common point with ∂K.

By Lemma 2.1, we obtain the following corollary.

Corollary 2.1 *Let K be an oval domain in $\mathbb{R}^2$, ρ_m and ρ_M be the minimum and maximum of the curvature radius of ∂K, respectively, B_t be a circle of radius t in $\mathbb{R}^2$. When $t \in (0, \rho_m]$ or $t \in [\rho_M, +\infty)$, and $\partial K \cap \partial(B_t) \neq \emptyset$, then B_t has two common points with ∂K or B_t is tangent to ∂K.*

Proof Suppose that B_t has more than two common points with ∂K when $t \in (0, \rho_m]$ or $t \in [\rho_M, +\infty)$, then we can move B_t properly so that it is tangent to ∂K and has other common point with ∂K; this is inconsistent with Blaschke's rolling theorem. □

Corollary 2.2 *Let K_k $(k = 0, 1)$ be two oval domains in $\mathbb{R}^2$, $\rho_m(\partial K_k)$ and $\rho_M(\partial K_k)$ be the minimum and maximum of the curvature radius of ∂K_k, respectively. When $\rho_M(\partial K_1) \leq \rho_m(\partial K_0)$ or $\rho_m(\partial K_1) \geq \rho_M(\partial K_0)$, and $\partial K_0 \cap \partial K_1 \neq \emptyset$, then ∂K_0 has two common points with ∂K_1 or ∂K_0 is tangent to ∂K_1.*

Proof Suppose that ∂K_0 has more than two common points with ∂K_1, we can draw a circle B_t of radius t through three points among these common points. Therefore, we have $t \in (\rho_m(\partial K_0), \rho_M(\partial K_0))$ and $t \in (\rho_m(\partial K_1), \rho_M(\partial K_1))$; this is inconsistent with the conditions of Corollary 2.2. □

3 Reverse Bonnesen-style inequalities

Let K be an oval domain of area A and perimeter P in $\mathbb{R}^2$. Let $\rho(\partial K)$ be the curvature radius of boundary ∂K and $\rho_m = \min\{\rho(\partial K)\}$, $\rho_M = \max\{\rho(\partial K)\}$. Let dg denote the kinematic density of the group G_2 of plane rigid motions, and B_t be a circle of radius t in $\mathbb{R}^2$. Let $n\{\partial K \cap \partial(gB_t)\}$ denote the number of points of intersection $\partial K \cap \partial(gB_t)$ and $\chi\{K \cap gB_t\}$ be the Euler–Poincaré characteristics of the intersection $K \cap gB_t$. Then we have the following kinematic formula of Poincaré (see [18]):

$$\int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} n\bigl\{\partial K \cap \partial(gB_t)\bigr\}\,dg = 8\pi Pt \tag{3.1}$$

and the kinematic formula of Blaschke

$$\int_{\{g\in G_2:K\cap gB_t\neq\emptyset\}} \chi\{K \cap gB_t\}\,dg = 2\pi^2t^2 + 2\pi Pt + 2\pi A. \tag{3.2}$$

If μ denotes a set of all positions of B_t in which either $gB_t \subset K$ or $gB_t \supset K$, then the kinematic formula of Blaschke (3.2) can be rewritten as

$$\int_\mu dg = 2\pi^2t^2 + 2\pi Pt + 2\pi A - \int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} \chi\{K \cap gB_t\}\,dg. \tag{3.3}$$

Since K is an oval domain in $\mathbb{R}^2$, then

$$\int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} \chi\{K \cap gB_t\}\,dg = \int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} dg. \tag{3.4}$$

When $t \in (0, \rho_m]$ or $t \in [\rho_M, +\infty)$, by Corollary 2.1, we have $n\{\partial K \cap \partial(gB_t)\} = 2$ or gB_t is tangent to ∂K. When gB_t is tangent to ∂K, we have

$$\int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} n\bigl\{\partial K \cap \partial(gB_t)\bigr\}\,dg = 0, \tag{3.5}$$

therefore,

$$\int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} n\bigl\{\partial K \cap \partial(gB_t)\bigr\}\,dg = \int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} 2dg. \tag{3.6}$$

By (3.4) and (3.6), we have

$$\int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} \chi\{K \cap gB_t\}\,dg = \frac{1}{2}\int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} n\bigl\{\partial K \cap \partial(gB_t)\bigr\}\,dg. \tag{3.7}$$

Therefore, when $t \in (0, \rho_m]$ or $t \in [\rho_M, +\infty)$, by (3.3), (3.7), and (3.1), we obtain

$$\begin{aligned}\int_\mu dg &= 2\pi^2t^2 + 2\pi Pt + 2\pi A - \int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} \chi\{K \cap gB_t\}\,dg \\ &= 2\pi^2t^2 + 2\pi Pt + 2\pi A - \frac{1}{2}\int_{\{g\in G_2:\partial K\cap\partial(gB_t)\neq\emptyset\}} n\bigl\{\partial K \cap \partial(gB_t)\bigr\}\,dg \\ &= 2\pi^2t^2 + 2\pi Pt + 2\pi A - 4\pi Pt\end{aligned}$$

$$= 2\pi^2 t^2 - 2\pi Pt + 2\pi A$$
$$\geq 0. \tag{3.8}$$

Theorem 3.1 *Let K be an oval domain of area A and perimeter P in $\mathbb{R}^2$, then*

$$\pi t^2 - Pt + A \geq 0; \quad t \in (0, \rho_m] \text{ or } t \in [\rho_M, +\infty). \tag{3.9}$$

The inequality is strict whenever $t \in (0, \rho_m)$ or $t \in (\rho_M, +\infty)$. When $t = \rho_m$ or $t = \rho_M$, the equality holds if and only if K is a disc.

Proof We obtain inequality (3.9) directly from (3.8)

$$\int_\mu dg = 2\pi^2 t^2 - 2\pi Pt + 2\pi A \geq 0; \quad t \in (0, \rho_m] \text{ or } t \in [\rho_M, +\infty).$$

By Blaschke's rolling theorem (Lemma 2.1), we know B_t has no other common point with ∂K when B_t is tangent to ∂K inside with $t \in (0, \rho_m]$, or B_t is tangent to ∂K outside with $t \in [\rho_M, +\infty)$. Therefore, we have $gB_t \subset K$ when $t \in (0, \rho_m)$, $gB_t \supset K$ when $t \in (\rho_M, +\infty)$, and $\partial(gB_t)$ has no common point with ∂K, then $\int_\mu dg > 0$ when $t \in (0, \rho_m)$ or $t \in (\rho_M, +\infty)$. That is, inequality (3.9) is strict whenever $t \in (0, \rho_m)$ or $t \in (\rho_M, +\infty)$.

When $t = \rho_m$ or $t = \rho_M$, the equality holds clearly in inequality (3.9) if K is a disc. Conversely, if K is not a disc, by Blaschke's rolling theorem (Lemma 2.1), we know that B_{ρ_m} has no other common point with ∂K when B_{ρ_m} is tangent to ∂K inside, and B_{ρ_M} has no other common point with ∂K when B_{ρ_M} is tangent to ∂K outside. Therefore, if K is not a disc, we have $gB_{\rho_m} \subset K$ and $\partial(gB_{\rho_m})$ has no common point with ∂K, $gB_{\rho_M} \supset K$ and $\partial(gB_{\rho_M})$ has no common point with ∂K, then $\int_\mu dg > 0$ when K is not a disc. That is, K is a disc when $\int_\mu dg = 0$. Therefore, when $t = \rho_m$ or $t = \rho_M$, the equality holds in (3.9) if and only if K is a disc. □

Theorem 3.2 *Let K be an oval domain of area A and perimeter P in $\mathbb{R}^2$, then*

$$P^2 - 4\pi A \leq (2\pi\rho_M - P)(P - 2\pi\rho_m), \tag{3.10}$$

where ρ_m and ρ_M are the minimum and maximum of the continuous curvature radius ρ of the boundary ∂K, respectively. The equality holds if and only if K is a disc.

Proof By inequality (3.9),

$$\pi t^2 - Pt + A \geq 0; \quad t \in (0, \rho_m] \text{ or } t \in [\rho_M, +\infty),$$

we have

$$\pi {\rho_m}^2 - P\rho_m + A \geq 0,$$
$$\pi {\rho_M}^2 - P\rho_M + A \geq 0,$$

that is,

$$-4\pi A \leq 4\pi^2 {\rho_m}^2 - 4\pi P \rho_m,$$
$$-4\pi A \leq 4\pi^2 {\rho_M}^2 - 4\pi P \rho_M.$$

Therefore, we have

$$\begin{aligned} P^2 - 4\pi A &\leq P^2 - 4\pi P \rho_m + 4\pi^2 {\rho_m}^2 \\ &= (P - 2\pi\rho_m)^2, \end{aligned}$$

and

$$\begin{aligned} P^2 - 4\pi A &\leq P^2 - 4\pi P \rho_M + 4\pi^2 {\rho_M}^2 \\ &= (2\pi\rho_M - P)^2. \end{aligned}$$

Since $B(t) = \pi t^2 - Pt + A$ reaches the minimum when $t = \frac{P}{2\pi}$ and inequality (3.9), we have $\rho_m \leq \frac{P}{2\pi} \leq \rho_M$, that is, $2\pi\rho_m \leq P \leq 2\pi\rho_M$. Therefore,

$$\sqrt{P^2 - 4\pi A} \leq P - 2\pi\rho_m,$$
$$\sqrt{P^2 - 4\pi A} \leq 2\pi\rho_M - P.$$

By multiplying the last inequalities side by side, we have

$$P^2 - 4\pi A \leq (2\pi\rho_M - P)(P - 2\pi\rho_m).$$

The equality holds in (3.10) if and only if the two equalities hold in (3.9) when $t = \rho_m$ and $t = \rho_M$, that is, K is a disc. □

For all $a \geq 0$, $b \geq 0$, we have $4ab \leq (a + b)^2$, that is,

$$(2\pi\rho_M - P)(P - 2\pi\rho_m) \leq \pi^2(\rho_M - \rho_m)^2.$$

Therefore, the upper bound of the isoperimetric deficit in inequality (3.10) is better than the upper bound in inequality (1.5), that is, the reverse Bonnesen-style inequality (3.10) strengthens Bottema's result.

4 Reverse Bonnesen-style symmetric mixed inequalities

Let K_k $(k = 0, 1)$ be two oval domains in $\mathbb{R}^2$. Let $\rho(\partial K_k)$ be the curvature radii of boundaries ∂K_k, and let $\rho_m(\partial K_k) = \min\{\rho(\partial K_k)\}$, $\rho_M(\partial K_k) = \max\{\rho(\partial K_k)\}$. Let

$$\rho_m^g(K_0, K_1) = \max\bigl\{t : \rho_M\bigl(\partial\bigl(t(gK_1)\bigr)\bigr) \leq \rho_m(\partial K_0); g \in G_2\bigr\}$$

and

$$\rho_M^g(K_0, K_1) = \min\bigl\{t : \rho_m\bigl(\partial\bigl(t(gK_1)\bigr)\bigr) \geq \rho_M(\partial K_0); g \in G_2\bigr\}$$

be the inradius and the outradius of curvature, K_0 with respect to K_1, where G_2 is a group of plane rigid motions. It is obvious that $\rho_m^g(K_0, K_1) \le \rho_M^g(K_0, K_1)$. Since both $\rho_m^g(K_0, K_1)$ and $\rho_M^g(K_0, K_1)$ are rigid invariant, we simply denote them by ρ_{01}^m and ρ_{01}^M, respectively. Note that, if K_1 is a unit disc, then ρ_{01}^m and ρ_{01}^M are the minimum $\rho_m(\partial K_0)$ and the maximum $\rho_M(\partial K_0)$ of the continuous curvature radius of the boundary ∂K_0, respectively.

Let K_k $(k = 0, 1)$ be two oval domains of areas A_k and perimeters P_k in $\mathbb{R}^2$. Let dg denote the kinematic density of the group G_2 of plane rigid motions. Let $n\{\partial K_0 \cap \partial(t(gK_1))\}$ denote the number of points of intersection $\partial K_0 \cap \partial(t(gK_1))$, and let $\chi\{K_0 \cap t(gK_1)\}$ be the Euler–Poincaré characteristics of the intersection $K_0 \cap t(gK_1)$. Then we have the following kinematic formula of Poincaré (see [14, 20, 21, 24–26, 30]):

$$\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} n\bigl\{\partial K_0 \cap \partial\bigl(t(gK_1)\bigr)\bigr\}\, dg = 4tP_0P_1 \tag{4.1}$$

and the kinematic formula of Blaschke

$$\int_{\{g\in G_2:K_0\cap t(gK_1)\neq\emptyset\}} \chi\bigl\{K_0 \cap t(gK_1)\bigr\}\, dg = 2\pi\bigl(t^2A_1 + A_0\bigr) + tP_0P_1. \tag{4.2}$$

Let μ denote a set of all positions of K_1 in which either $t(gK_1) \subset K_0$ or $t(gK_1) \supset K_0$, then (4.2) can be rewritten as

$$\int_\mu dg = 2\pi\bigl(t^2A_1 + A_0\bigr) + tP_0P_1 - \int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} \chi\bigl\{K_0 \cap t(gK_1)\bigr\}\, dg. \tag{4.3}$$

Since K_k $(k = 0, 1)$ are two oval domains in $\mathbb{R}^2$, then

$$\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} \chi\bigl\{K_0 \cap t(gK_1)\bigr\}\, dg = \int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} dg. \tag{4.4}$$

When $t \in (0, \rho_{01}^m]$ or $t \in [\rho_{01}^M, +\infty)$, we can obtain $\rho_M(\partial(t(gK_1))) \le \rho_m(\partial K_0)$ or $\rho_m(\partial(t(gK_1))) \ge \rho_M(\partial K_0)$. By Corollary 2.2, we have $n\{\partial K_0 \cap \partial(t(gK_1))\} = 2$ or $\partial(t(gK_1))$ is tangent to ∂K_0. When $\partial(t(gK_1))$ is tangent to ∂K_0, we have

$$\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} n\bigl\{\partial K_0 \cap \partial\bigl(t(gK_1)\bigr)\bigr\}\, dg = 0, \tag{4.5}$$

therefore,

$$\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} n\bigl\{\partial K_0 \cap \partial\bigl(t(gK_1)\bigr)\bigr\}\, dg = \int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} 2dg. \tag{4.6}$$

By (4.4) and (4.6), we have

$$\begin{aligned} &\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} \chi\bigl\{K_0 \cap t(gK_1)\bigr\}\, dg \\ &\quad = \frac{1}{2}\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} n\bigl\{\partial K_0 \cap \partial\bigl(t(gK_1)\bigr)\bigr\}\, dg. \end{aligned} \tag{4.7}$$

Therefore, when $t \in (0, \rho_{01}^m]$ or $t \in [\rho_{01}^M, +\infty)$, by (4.3), (4.7), and (4.1), we obtain

$$\begin{aligned}\int_\mu dg &= 2\pi\left(t^2A_1 + A_0\right) + tP_0P_1 - \int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} \chi\left\{K_0 \cap t(gK_1)\right\} dg \\ &= 2\pi\left(t^2A_1 + A_0\right) + tP_0P_1 - \frac{1}{2}\int_{\{g\in G_2:\partial K_0\cap\partial(t(gK_1))\neq\emptyset\}} n\left\{\partial K_0 \cap \partial\left(t(gK_1)\right)\right\} dg \\ &= 2\pi A_1t^2 - P_0P_1t + 2\pi A_0 \\ &\geq 0. \end{aligned} \tag{4.8}$$

Theorem 4.1 *Let K_k $(k = 0, 1)$ be two oval domains of areas A_k and perimeters P_k in $\mathbb{R}^2$, then*

$$2\pi A_1t^2 - P_0P_1t + 2\pi A_0 \geq 0; \quad t \in \left(0, \rho_{01}^m\right] \text{ or } t \in \left[\rho_{01}^M, +\infty\right). \tag{4.9}$$

The inequality is strict whenever $t \in (0, \rho_{01}^m)$ or $t \in (\rho_{01}^M, +\infty)$. When $t = \rho_{01}^m$ or $t = \rho_{01}^M$, the equality holds if and only if both K_0 and K_1 are discs.

Proof We obtain inequality (4.9) directly from (4.8)

$$\int_\mu dg = 2\pi A_1t^2 - P_0P_1t + 2\pi A_0 \geq 0; \quad t \in \left(0, \rho_{01}^m\right] \text{ or } t \in \left[\rho_{01}^M, +\infty\right).$$

When $t \in (0, \rho_{01}^m)$, that is, $\rho_M(\partial(t(gK_1))) < \rho_m(\partial K_0)$, we have

$$t(gK_1) \subset B_{\rho_M(\partial(t(gK_1)))} \subset B_{\rho_m(\partial K_0)} \subset K_0; \quad t \in \left(0, \rho_{01}^m\right),$$

where $\partial(B_{\rho_M(\partial(t(gK_1)))})$ has no common point with $\partial(B_{\rho_m(\partial K_0)})$. Therefore, we have $t(gK_1) \subset K_0$, and $\partial(t(gK_1))$ has no common point with $\partial(K_0)$ when $t \in (0, \rho_{01}^m)$. When $t \in (\rho_{01}^M, +\infty)$, that is, $\rho_m(\partial(t(gK_1))) > \rho_M(\partial K_0)$, we have

$$t(gK_1) \supset B_{\rho_m(\partial(t(gK_1)))} \supset B_{\rho_M(\partial K_0)} \supset K_0; \quad t \in \left(\rho_{01}^M, +\infty\right),$$

where $\partial(B_{\rho_m(\partial(t(gK_1)))})$ has no common point with $\partial(B_{\rho_M(\partial K_0)})$. Therefore, we have $t(gK_1) \supset K_0$, and $\partial(t(gK_1))$ has no common point with $\partial(K_0)$ when $t \in (\rho_{01}^M, +\infty)$. In summary, we have $\int_\mu dg > 0$ when $t \in (0, \rho_{01}^m)$ or $t \in (\rho_{01}^M, +\infty)$. That is, inequality (4.9) is strict whenever $t \in (0, \rho_{01}^m)$ or $t \in (\rho_{01}^M, +\infty)$.

When $t = \rho_{01}^m$ or $t = \rho_{01}^M$, the equality holds clearly in inequality (4.9) if both K_0 and K_1 are discs. Conversely, if K_0 and K_1 of which at least one is not a disc, it includes the following two types: Only one of them is not a disc; K_0 and K_1 are not discs. When only one of K_0 and K_1 is not a disc, we have $\rho_{01}^m(gK_1) \subset K_0$ and $\partial(\rho_{01}^m(gK_1))$ has no common point with ∂K_0, $\rho_{01}^M(gK_1) \supset K_0$ and $\partial(\rho_{01}^M(gK_1))$ has no common point with ∂K_0, then $\int_\mu dg > 0$ when only one of K_0 and K_1 is not a disc. When K_0 and K_1 are not discs, we have

$$\rho_{01}^m(gK_1) \subset B_{\rho_M(\partial(\rho_{01}^m(gK_1)))} \subset B_{\rho_m(\partial K_0)} \subset K_0,$$

where $\partial(\rho_{01}^m(gK_1))$ has no common point with ∂K_0, and

$$\rho_{01}^M(gK_1) \supset B_{\rho_m(\partial(\rho_{01}^M(gK_1)))} \supset B_{\rho_M(\partial K_0)} \supset K_0,$$

where $\partial(\rho_{01}^M(gK_1))$ has no common point with ∂K_0, then $\int_\mu dg > 0$ when K_0 and K_1 are not discs. In summary, $\int_\mu dg > 0$ when K_0 and K_1 of which at least one is not a disc. That is, both K_0 and K_1 are discs when $\int_\mu dg = 0$. Therefore, when $t = \rho_{01}^m$ or $t = \rho_{01}^M$, the equality holds in inequality (4.9) if and only if both K_0 and K_1 are discs. □

When K_1 is a unit disc, inequality (4.9) immediately reduces to inequality (3.9).

We now obtain the following reverse Bonnesen-style symmetric mixed inequalities.

Theorem 4.2 *Let K_k $(k = 0,1)$ be two oval domains of areas A_k and perimeters P_k in $\mathbb{R}^2$, then*

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \le 4\pi^2A_1^2\bigl(\rho_{01}^M - \rho_{01}^m\bigr)^2, \tag{4.10}$$

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \le 16\pi^2A_1^2\left(\rho_{01}^M - \frac{P_0P_1}{4\pi A_1}\right)\left(\frac{P_0P_1}{4\pi A_1} - \rho_{01}^m\right), \tag{4.11}$$

where each equality holds if and only if both K_0 and K_1 are discs.

Proof By inequality (4.9),

$$2\pi A_1t^2 - P_0P_1t + 2\pi A_0 \ge 0; \quad t \in \bigl(0, \rho_{01}^m\bigr] \text{ or } t \in \bigl[\rho_{01}^M, +\infty\bigr),$$

we have

$$2\pi A_1\bigl(\rho_{01}^m\bigr)^2 - P_0P_1\rho_{01}^m + 2\pi A_0 \ge 0,$$
$$2\pi A_1\bigl(\rho_{01}^M\bigr)^2 - P_0P_1\rho_{01}^M + 2\pi A_0 \ge 0,$$

that is,

$$-16\pi^2A_0A_1 \le 16\pi^2A_1^2\bigl(\rho_{01}^m\bigr)^2 - 8\pi A_1P_0P_1\rho_{01}^m,$$
$$-16\pi^2A_0A_1 \le 16\pi^2A_1^2\bigl(\rho_{01}^M\bigr)^2 - 8\pi A_1P_0P_1\rho_{01}^M.$$

Therefore, we have

$$\begin{aligned} P_0^2P_1^2 - 16\pi^2A_0A_1 &\le P_0^2P_1^2 + 16\pi^2A_1^2\bigl(\rho_{01}^m\bigr)^2 - 8\pi A_1P_0P_1\rho_{01}^m \\ &= \bigl(P_0P_1 - 4\pi A_1\rho_{01}^m\bigr)^2 \end{aligned}$$

and

$$\begin{aligned} P_0^2P_1^2 - 16\pi^2A_0A_1 &\le P_0^2P_1^2 + 16\pi^2A_1^2\bigl(\rho_{01}^M\bigr)^2 - 8\pi A_1P_0P_1\rho_{01}^M \\ &= \bigl(P_0P_1 - 4\pi A_1\rho_{01}^M\bigr)^2. \end{aligned}$$

Since $B_{K_0,K_1}(t) = 2\pi A_1t^2 - P_0P_1t + 2\pi A_0$ reaches the minimum at $t = \frac{P_0P_1}{4\pi A_1}$, and inequality (4.9), we have $\rho_{01}^m \le \frac{P_0P_1}{4\pi A_1} \le \rho_{01}^M$, that is, $4\pi A_1\rho_{01}^m \le P_0P_1 \le 4\pi A_1\rho_{01}^M$. Therefore,

$$\sqrt{P_0^2P_1^2 - 16\pi^2A_0A_1} \le P_0P_1 - 4\pi A_1\rho_{01}^m,$$
$$\sqrt{P_0^2P_1^2 - 16\pi^2A_0A_1} \le 4\pi A_1\rho_{01}^M - P_0P_1.$$

By adding and multiplying the last inequalities side by side, we have

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \le 4\pi^2A_1^2\left(\rho_{01}^M - \rho_{01}^m\right)^2$$

and

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \le 16\pi^2A_1^2\left(\rho_{01}^M - \frac{P_0P_1}{4\pi A_1}\right)\left(\frac{P_0P_1}{4\pi A_1} - \rho_{01}^m\right).$$

Each equality holds in (4.10) and (4.11) if and only if the equalities hold in (4.9) when $t = \rho_{01}^m$ and $t = \rho_{01}^M$, that is, both K_0 and K_1 are discs. □

When K_1 is a unit disc, the reverse Bonnesen-style symmetric mixed inequality (4.10) immediately reduces to the known reverse Bonnesen-style inequality (1.5) of Bottema, inequality (4.11) reduces to inequality (3.10). For all $a \ge 0$, $b \ge 0$, we have $4ab \le (a+b)^2$, that is,

$$16\pi^2A_1^2\left(\rho_{01}^M - \frac{P_0P_1}{4\pi A_1}\right)\left(\frac{P_0P_1}{4\pi A_1} - \rho_{01}^m\right) \le 4\pi^2A_1^2\left(\rho_{01}^M - \rho_{01}^m\right)^2.$$

Therefore, the upper bound of the symmetric mixed isoperimetric deficit in inequality (4.11) is better than the upper bound in inequality (4.10), that is, the reverse Bonnesen-style symmetric mixed inequality (4.11) is stronger than inequality (4.10).

Acknowledgements
The author would like to express his deep gratitude to the anonymous referees for their valuable suggestions, which led to the improvement of the article.

Funding
The author is supported in part by the Chongqing Research Program of Basic Research and Frontier Technology (No. cstc2017jcyjAX0416), the Scientific and Technological Research Program of Chongqing Municipal Education Commission (No. KJ1706168), and the High Level Introduction of Talent Research Start-up Fund of Chongqing Technology and Business University (No. 1756005).

Availability of data and materials
Not applicable.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Received: 20 October 2018 Accepted: 26 March 2019 Published online: 01 April 2019

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