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The Minkowski inequality involving generalized k-fractional conformable integral

Shahid Mubeen[1*], Siddra Habib[2] and Muhammad Nawaz Naeem[2]

*Correspondence:
smjhanda@gmail.com
[1]Department of Mathematics, University of Sargodha, Sargodha, Pakistan
Full list of author information is available at the end of the article

Abstract

In the research paper, the authors exploit the definition of a new class of fractional integral operators, recently proposed by Jarad et al. (Adv. Differ. Equ. 2017:247, 2017), to define a new class of generalized k-fractional integral operators and develop a generalization of the reverse Minkowski inequality involving the newly introduced fractional integral operators. The two new theorems correlating with this inequality, including statements and verifications of other inequalities via the suggested k-fractional conformable integral operators, are presented.

Keywords: Generalized k-fractional conformable integrals; Fractional integral inequalities; Minkowski inequality

1 Introduction

The calculus of non-integer order, generally referred to as fractional calculus, deals with the generalization of integrals and derivatives operators, in particular inequalities involving fractional integrals. Many definitions of fractional integral operators exist in the literature, for example: Riemann–Liouville fractional integral, Hadamard integral, Liouville integral, Weyl, Erdélyi–Kober and Katugampola fractional integral [2–5]. Recently, Khalil et al. [6] and Adeljawad [7] presented a new class of fractional operators, namely the local fractional conformable integral and derivative operators. Using such fractional integral operators, one generalizes the fractional operators by involving new parameters and obtains the related inequalities: Hadamard, Hermite–Hadamard, Opial, Grüss, Ostrowski, among others [8–14]. For instance, Katugampola [15] suggested a generalized fractional integral operator unifying other well-known existing ones: Riemann–Liouville, Hadamard, Weyl, Liouville, and Erdélyi–Kober. Jarad et al. [1] presented the generalized conformable derivatives and integral operators by the standard fractional calculus iteration procedure on fractional conformable operators. Such generalizations motivate the upcoming research to present more innovative ideas to unify the fractional operators and obtain the inequalities involving such generalized fractional operators.

Applications of integral inequalities are important in numerous fields of science: mathematics, physics, engineering, among others; particularly we mention initial-value problems, stability of linear transformation, integral differential equations, and impulse equations [16, 17]. We refer the readers to [10, 18, 19] for such applications in several branches of mathematics and the references therein. Inequalities regarding fractional integral operators accumulate many functional applications in different areas of science. Moreover, the theory of fractional calculus plays an important role in solving differential equations, integral equations, and integral-differential equations, including many other special function problems.

Thus, the new results involving integral inequalities have been possible; consequently, some applications have been made [16, 17]. We mention a few of them, i.e., the inequalities of Minkowski, Holder, Hardy, Hermite–Hadamard, Jensen, among others [20–26]. Such applications of fractional integral operators motivate us to present the generalization of the existing fractional conformable operators and generalize the reverse Minkowski inequality [27–31] involving generalized k-fractional conformable integrals.

The paper is categorized as follows. In Sect. 2, we exhibit the notations and basic definitions of fractional integrals as well as our newly defined generalized k-fractional conformable integrals. We prove the theorems regarding the reverse Minkowski inequality as well as the appropriate spaces for such operators. In Sect. 3, we propose our main results consisting of the reverse Minkowski inequality via the generalized k-fractional conformable integral. In Sect. 4, we present the related inequalities using this fractional integral. The last section containing concluding remarks closes the article.

2 Notations and preliminaries

This section recalls some notations and useful definitions of classical fractional calculus as well as the reverse Minkowski inequality theorem using the classical Riemann integral addressed by Set et al. [21] and its relevant generalization via Riemann–Liouville and Hadamard fractional integrals which were motivation for the study. Furthermore, the fractional conformable integrals are discussed and a theorem is presented so as to recover particular cases.

Definition 1 A function $f(z)$ is said to be in $L_p[a,b]$ if

$$\left(\int_a^b \left|f(z)\right|^p dz\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

Theorem 1 *Let $f_1, f_2 \in L_p[a,b]$ be two positive functions, with $1 \leq p \leq \infty$, $0 < \int_a^b f_1^p(t)\,dt < \infty$, and $0 < \int_a^b f_2^p(t)\,dt < \infty$. If $0 < m \leq \frac{f_1(t)}{f_2(t)} \leq M$ for $m, M \in \mathbb{R}^+$ and $\forall t \in [a,b]$, then*

$$\left(\int_a^b f_1^p(t)\,dt\right)^{\frac{1}{p}} + \left(\int_a^b f_2^p(t)\,dt\right)^{\frac{1}{p}} \leq c_1\Big(\int_a^b \big(f_1^p + f_2^p(t)\,dt\big)^{\frac{1}{p}}, \tag{2.1}$$

with $c_1 = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ [21].

Theorem 2 *Let $f_1, f_2 \in L_p[a,b]$ be two positive functions, with $1 \leq p \leq \infty$, $0 < \int_a^b f_1^p(t)\,dt < \infty$, and $0 < \int_a^b f_2^p(t)\,dt < \infty$. If $0 < m \leq \frac{f_1(t)}{f_2(t)} \leq M$ for $m, M \in \mathbb{R}^+$ and $\forall t \in [a,b]$, then*

$$\left(\int_a^b f_1^p(t)\,dt\right)^{\frac{2}{p}} + \left(\int_a^b f_2^p(t)\,dt\right)^{\frac{2}{p}} \geq c_2\left(\int_a^b f_1^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_a^b f_2^p(t)\,dt\right)^{\frac{2}{p}}, \tag{2.2}$$

with $c_2 = \frac{(m+1)(M+1)}{M} - 2$ [21].

Definition 2 A function $f(z)$ is said to be in $L_{p,s}[a,b]$ if

$$\left(\int_a^b \left|f(z)\right|^p z^s\,dz\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty, s \ge 0.$$

Definition 3 The space $X_{c,p}(a,b)$ for $c \in \mathbb{R}$, $a < b$ and $1 \le p < \infty$ contains those complex valued Lebesgue measurable functions g on (a,b) with $\|g\|_{X_{c,p}}$, where

$$\|g\|_{X_{c,p}} = \left(\int_a^b \left|z^c f(z)\right|^p \frac{dz}{z}\right)^{\frac{1}{p}} \quad (1 \le p < \infty),$$

and for $p = \infty$,

$$\|g\|_{X_{c,\infty}} = \operatorname*{sup\,ess}_{z\in(a,b)}\left[z^c\left|g(z)\right|\right].$$

Particularly, for $c = \frac{1}{p}$, the function space $X_{c,p}(a,b)$ concurs with the space $L_p(a,b)$ [2].

Definition 4 For $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, the left Riemann–Liouville fractional integral [2, 4] of order α starting from a has the following form:

$$\left(\mathcal{I}_{a^+}^{\alpha} f\right)(\mathcal{T}) = \frac{1}{\Gamma(\alpha)} \int_a^{\mathcal{T}} f(x) \frac{dx}{(\mathcal{T} - x)^{1-\alpha}}, \tag{2.3}$$

while the right Riemann–Liouville fractional integral of [2, 4] order $\alpha > 0$ ending at $b > a$ is defined by

$$\left(\mathcal{I}_{b^-}^{\alpha} f\right)(\mathcal{T}) = \frac{1}{\Gamma(\alpha)} \int_{\mathcal{T}}^{b} f(x) \frac{dx}{(x - \mathcal{T})^{1-\alpha}}. \tag{2.4}$$

Definition 5 For $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$, the left Riemann–Liouville fractional derivative [1] of order α starting from a is defined below:

$$\left(\mathcal{D}_{a^+}^{\alpha} f\right)(\mathcal{T}) = \left(\frac{d}{dt}\right)^n \left(\mathcal{I}_{a^+}^{n-\alpha} f\right)(\mathcal{T}), \quad n = [\alpha] + 1. \tag{2.5}$$

Meanwhile the right Riemann–Liouville fractional derivative [1] of order $\alpha > 0$ ending at $b > a$ takes the form

$$\left(\mathcal{D}_{b^-}^{\alpha} f\right)(\mathcal{T}) = \left(-\frac{d}{dt}\right)^n \left(\mathcal{I}_{b^-}^{n-\alpha} f\right)(\mathcal{T}). \tag{2.6}$$

Definition 6 The left Caputo fractional derivative [1] of order α, $\mathrm{Re}(\alpha) > 0$ starting from a has the form

$$\left({}^{C}\mathcal{D}_{a^+}^{\alpha} f\right)(\mathcal{T}) = \left(\mathcal{I}_{a^+}^{n-\alpha} f^{(n)}\right)(\mathcal{T}), \quad n = [\alpha] + 1, \tag{2.7}$$

while the right Caputo fractional derivative [1] of order $\alpha > 0$ ending at $b > a$ takes the form

$$\left({}^{C}\mathcal{D}_{b^-}^{\alpha} f\right)(\mathcal{T}) = \left(\mathcal{I}_{b^-}^{n-\alpha} (-1)^n f^{(n)}\right)(\mathcal{T}). \tag{2.8}$$

Definition 7 The left Hadamard fractional integral [1] of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ starting from a is given below:

$$\big(\mathcal{F}_{a^+}^{\alpha} f\big)(\mathcal{T}) = \frac{1}{\Gamma(\alpha)} \int_a^{\mathcal{T}} (\ln \mathcal{T} - \ln x)^{\alpha-1} f(x) \frac{dx}{x}, \tag{2.9}$$

and the right Hadamard fractional integral [1] of order α ending at $b > a$ takes the form

$$\big(\mathcal{F}_{b^-}^{\alpha} f\big)(\mathcal{T}) = \frac{1}{\Gamma(\alpha)} \int_{\mathcal{T}}^{b} (\ln x - \ln \mathcal{T})^{\alpha-1} f(x) \frac{dx}{x}. \tag{2.10}$$

Definition 8 The left Hadamard fractional derivative [1] of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ starting from a can be defined as

$$\big(\mathcal{G}_{a^+}^{\alpha} f\big)(\mathcal{T}) = \left(t \frac{d}{dt}\right)^n \big(\mathcal{F}_{a^+}^{n-\alpha} f\big)(\mathcal{T}), \quad n = [\alpha] + 1, \tag{2.11}$$

and the right Hadamard fractional derivative [1] of order α ending at $b > a$ takes the form

$$\big(\mathcal{G}_{b^-}^{\alpha} f\big)(\mathcal{T}) = \left(-t \frac{d}{dt}\right)^n \big(\mathcal{F}_{b^-}^{n-\alpha} f\big)(\mathcal{T}). \tag{2.12}$$

Definition 9 For a real function $f \in X_{c,p}(a,b)$, the generalized left and right Katugampola fractional integrals [28] of order $\alpha \in \mathbb{R}$, $\rho > 0$, $\mathrm{Re}(\alpha) > 0$ take the form

$$\big(\mathcal{K}_{a^+}^{\alpha,\rho} f\big)(\mathcal{T}) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^{\mathcal{T}} f(x) \frac{x^{\rho-1}\, dx}{(\mathcal{T}^{\rho} - x^{\rho})^{1-\alpha}} \tag{2.13}$$

and

$$\big(\mathcal{K}_{b^-}^{\alpha,\rho} f\big)(\mathcal{T}) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathcal{T}}^{b} f(x) \frac{x^{\rho-1}\, dx}{(x^{\rho} - \mathcal{T}^{\rho})^{1-\alpha}}, \tag{2.14}$$

respectively.

Definition 10 The generalized left and right Katugampola fractional derivatives [29] of order $\alpha \in \mathbb{R}$, $\rho > 0$, $\mathrm{Re}(\alpha) > 0$ are defined below:

$$\big(\mathcal{L}_{a^+}^{\alpha,\rho} f\big)(\mathcal{T}) = \gamma^n \big(\mathcal{K}_{a^+}^{n-\alpha,\rho} f\big)(\mathcal{T}) = \frac{\gamma^n \rho^{n-\alpha}}{\Gamma(n-\alpha)} \int_a^{\mathcal{T}} f(x) \frac{x^{\rho-1}\, dx}{(\mathcal{T}^{\rho} - x^{\rho})^{1+\alpha-n}} \tag{2.15}$$

and

$$\big(\mathcal{L}_{b^-}^{\alpha,\rho} f\big)(\mathcal{T}) = (-\gamma)^n \big(\mathcal{K}_{b^-}^{n-\alpha,\rho} f\big)(\mathcal{T}) = \frac{(-\gamma)^n \rho^{n-\alpha}}{\Gamma(n-\alpha)} \int_{\mathcal{T}}^{b} f(x) \frac{x^{\rho-1}\, dx}{(x^{\rho} - \mathcal{T}^{\rho})^{1+\alpha-n}}, \tag{2.16}$$

respectively, where $\gamma = t^{1-\rho} \frac{d}{dt}$.

Dahmani [30] verified the reverse Minkowski inequality together with a relevant result to the inequality associated with Riemann–Liouville fractional integral corresponding to the following two theorems.

Theorem 3 *For $\alpha > 0$ and $p \geq 1$. Let $f_1, f_2 \in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t > a$, $\mathcal{I}^{\alpha,s}_{a^+,k} f_1^p(t) < \infty$ and $\mathcal{I}^{\alpha,s}_{a^+,k} f_2^p(t) < \infty$. If $0 < m \leq \frac{f_1(x)}{f_2(x)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $x \in [a,t]$, then*

$$\left(\mathcal{I}^{\alpha,s}_{a^+,k} f_1^p(t)\right)^{\frac{1}{p}} + \left(\mathcal{I}^{\alpha,s}_{a^+,k} f_2^p(t)\right)^{\frac{1}{p}} \leq c_1 \left(\mathcal{I}^{\alpha,s}_{a^+,k} (f_1 + f_2)^p(t)\right)^{\frac{1}{p}}, \tag{2.17}$$

with $c_1 = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ [30].

Theorem 4 *For $\alpha > 0$ and $p \geq 1$. Let $f_2, f_2 \in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t > a$, $\mathcal{I}^{\alpha,s}_{a^+,k} f_1^p(t) < \infty$ and $\mathcal{I}^{\alpha,s}_{a^+,k} f_2^p(t) < \infty$. If $0 < m \leq \frac{f_1(x)}{f_2(x)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $x \in [a,t]$, then*

$$\left(\mathcal{I}^{\alpha,s}_{a^+,k} f_1^p(t)\right)^{\frac{2}{p}} + \left(\mathcal{I}^{\alpha,s}_{a^+,k} f_2^p(t)\right)^{\frac{2}{p}} \geq c_2 \left(\mathcal{I}^{\alpha,s}_{a^+,k} f_1^p(t)\right)^{\frac{1}{p}} \left(\mathcal{I}^{\alpha,s}_{a^+,k} f_2^p(t)\right)^{\frac{1}{p}}, \tag{2.18}$$

with $c_2 = \frac{(m+1)(M+1)}{M} - 2$ [30].

Chinchane et al. [31] and Sabrina et al. [32] established the following two theorems for the reverse Minkowski inequality involving Hadamard fractional integral operator.

Theorem 5 *For $\alpha > 0$ and $p \geq 1$. Let $f_1, f_2 \in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t > a$, $\mathcal{F}^{\alpha,s}_{a^+,k} f_1^p(t) < \infty$ and $\mathcal{F}^{\alpha,s}_{a^+,k} f_2^p(t) < \infty$. If $0 < m \leq \frac{f_1(x)}{f_2(x)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $x \in [a,t]$, then*

$$\left(\mathcal{F}^{\alpha,s}_{a^+,k} f_1^p(t)\right)^{\frac{1}{p}} + \left(\mathcal{F}^{\alpha,s}_{a^+,k} f_2^p(t)\right)^{\frac{1}{p}} \leq c_1 \left(\mathcal{F}^{\alpha,s}_{a^+,k} (f_1 + f_2)^p(t)\right)^{\frac{1}{p}}, \tag{2.19}$$

with $c_1 = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ [31, 32].

Theorem 6 *For $\alpha > 0$ and $p \geq 1$. Let $f_2, f_2 \in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t > a$, $\mathcal{F}^{\alpha,s}_{a^+,k} f_1^p(t) < \infty$ and $\mathcal{F}^{\alpha,s}_{a^+,k} f_2^p(t) < \infty$. If $0 < m \leq \frac{f_1(x)}{f_2(x)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $x \in [a,t]$, then*

$$\left(\mathcal{F}^{\alpha,s}_{a^+,k} f_1^p(t)\right)^{\frac{2}{p}} + \left(\mathcal{F}^{\alpha,s}_{a^+,k} f_2^p(t)\right)^{\frac{2}{p}} \geq c_2 \left(\mathcal{F}^{\alpha,s}_{a^+,k} f_1^p(t)\right)^{\frac{1}{p}} \left(\mathcal{F}^{\alpha,s}_{a^+,k} f_2^p(t)\right)^{\frac{1}{p}}, \tag{2.20}$$

with $c_2 = \frac{(m+1)(M+1)}{M} - 2$ [31, 32].

Chinchane et al. [33] presented the reverse Minkowski inequality by fractional integral of Saigo, and recently Chinchane [34] proved the same inequality via the k-fractional integral. In 2017, Jarad et al. [1] introduced a new fractional integral that generalizes the above mentioned pre-existing fractional integrals. Conclusively, we define the generalization of this integral in the k-analogue form in addition to a theorem for the study of their particular cases.

Definition 11 If $f \in L_1[a,b]$, then the left fractional conformable integral of order $\alpha \geq 0$ defined by Abdeljawad [7] is given by

$$I^{\alpha}_{a} f(x) = \int_a^x (t-a)^{\alpha-1} f(t)\,dt, \quad 0 \leq a < x < b \leq \infty, 0 < \alpha < 1. \tag{2.21}$$

Definition 12 If $f \in L_1[a,b]$, then the right fractional conformable integral of order $\alpha \geq 0$ defined by Abdeljawad [7] is given by

$$I_b^\alpha f(x) = \int_x^b (b-t)^{\alpha-1} f(t)\,dt, \quad 0 \leq a < x < b \leq \infty, 0 < \alpha < 1. \tag{2.22}$$

Definition 13 If $f \in L_{1,s}[a,b]$, then the generalized left fractional conformable integral $\mathfrak{T}_a^{\alpha,s}$ of order $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ and $s > 0$, introduced by Jarad et al. [1], is defined by

$$\begin{aligned} &\mathfrak{T}_a^{\alpha,s} f(t) = \frac{s^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \bigl((t-a)^s - (x-a)^s\bigr)^{\alpha-1} (x-a)^{s-1} f(x)\,dx, \\ &\quad 0 \leq a < t < b \leq \infty, \end{aligned} \tag{2.23}$$

where Γ is the Euler gamma function.

Definition 14 If $f \in L_{1,s}[a,b]$, then the generalized right fractional conformable integral $\mathfrak{T}_b^{\alpha,s}$ of order $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ and $s > 0$, introduced by Jarad et al. [1], is defined by

$$\begin{aligned} &\mathfrak{T}_b^{\alpha,s} f(t) = \frac{s^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \bigl((b-x)^s - (b-t)^s\bigr)^{\alpha-1} (b-x)^{s-1} f(x)\,dx, \\ &\quad 0 \leq a < t < b \leq \infty, \end{aligned} \tag{2.24}$$

where Γ is the Euler gamma function.

Definition 15 The (k,s)-fractional conformable integrals (left and right) of order $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ of a continuous function $f(x)$ on $[0,\infty)$, are given as follows:

$$\begin{aligned} &\mathfrak{F}_{a^+,k}^{\alpha,s} f(t) = \frac{(s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \bigl((t-a)^s - (x-a)^s\bigr)^{\frac{\alpha}{k}-1} (x-a)^{s-1} f(x)\,dx, \\ &\quad 0 \leq a < t < b \leq \infty, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &\mathfrak{F}_{b^-,k}^{\alpha,s} f(t) = \frac{(s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_t^b \bigl((b-x)^s - (b-t)^s\bigr)^{\frac{\alpha}{k}-1} (b-x)^{s-1} f(x)\,dx, \\ &\quad 0 \leq a < t < b \leq \infty, \end{aligned} \tag{2.26}$$

respectively, if integrals exist, where $k > 0$, $s \in \mathbb{R} \setminus \{0\}$.

Theorem 7 *For $k > 0$, $s \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$ and $p \geq 1$. Then, for $f \in L_{1,s}[a,t]$, for all $t > a$, we have* [15]:

(1) *When $k = 1$ in* (2.25), *it reduces to generalized left fractional conformable integral* (2.23).

(2) *For $a = 0$, $k = 1$, and $s = 1$ in* (2.25), *it coincides with Riemann–Liouville fractional integral* (2.3).

(3) *When $a = 0$ and $\alpha \to 0$, it becomes left Hadamard fractional integral* (2.11).

(4) *It gives generalized Katugampola fractional integral* (2.13) *for* $a = 0$.

Theorem 8 *Let* $f \in L_1[a,b]$, $s \in \mathbb{R} \setminus \{0\}$, *and* $k > 0$. *Then* $\mathfrak{F}^{\alpha,s}_{a^+,k} f(x) (\mathfrak{F}^{\alpha,s}_{b^-,k} f(x))$ *exists for any* $x \in [a,b]$, $\operatorname{Re}(\alpha) > 0$.

Proof Let $\triangle' := [a,b] \times [a,b]$ and $P' : \triangle' \to \mathbb{R}$ such that

$$P'(x,t) = \bigl((x-a)^\alpha - (t-a)^\alpha\bigr)^{\frac{\beta}{k}-1} (t-a)^{\alpha-1}.$$

Clearly, it can be seen that

$$P' = P'_+ + P'_-,$$

where

$$P'_+(x,t) := \begin{cases} ((x-a)^\alpha - (t-a)^\alpha)^{\frac{\beta}{k}-1} (t-a)^{\alpha-1}, & a \le t \le x \le b, \\ 0, & a \le x \le t \le b, \end{cases}$$

and

$$P'_-(x,t) := \begin{cases} ((t-a)^\alpha - (x-a)^\alpha)^{\frac{\beta}{k}-1} (x-a)^{\alpha-1}, & a \le t \le x \le b, \\ 0, & a \le x \le t \le b. \end{cases}$$

Since P' is measurable on $\triangle'$, then it can be written as

$$\begin{aligned} \int_a^b P'(x,t)\,dt = \int_a^x P'(x,t)\,dt &= \int_a^x \bigl((x-a)^\alpha - (t-a)^\alpha\bigr)^{\frac{\beta}{k}-1} (t-a)^{\alpha-1}\,dt \\ &= \frac{\alpha k}{\beta} (x-a)^{\frac{\alpha\beta}{k}}. \end{aligned}$$

By using the double integral, we get

$$\begin{aligned} \int_a^b \left(\int_a^b P'(x,t) \bigl|f(x)\bigr|\,dt \right) dx &= \int_a^b \bigl|f(x)\bigr| \left(\int_a^b P'(x,t)\,dt \right) dx \\ &= \frac{\alpha k}{\beta} \int_a^b (x-a)^{\frac{\alpha\beta}{k}} \bigl|f(x)\bigr|\,dx \\ &\le \frac{\alpha k}{\beta} (b-a)^{\frac{\alpha\beta}{k}} \int_a^b \bigl|f(x)\bigr|\,dx, \end{aligned}$$

i.e.,

$$\begin{aligned} \int_a^b \left(\int_a^b P'(x,t) \bigl|f(x)\bigr|\,dt \right) dx &= \int_a^b \bigl|f(x)\bigr| \left(\int_a^b P'(x,t)\,dt \right) dx \\ &\le \frac{\alpha k}{\beta} (b-a)^{\frac{\alpha\beta}{k}} \bigl\|f(x)\bigr\|_{L_1[a,b]} < \infty. \end{aligned}$$

So, the function $Q' : \triangle' \to \mathbb{R}$ such that $Q'(x,t) := P'(x,t)f(x)$ is integrable over $\triangle'$ by Tonelli's theorem. Hence, by Fubini's theorem $\int_a^b P'(x,t)f(x)\,dx$ is an integrable function over $[a,b]$ as a function of $t \in [a,b]$, i.e., $\mathfrak{F}^{\alpha,s}_{a^+,k}f(x)$ exists.

The existence of the right k-fractional conformable integral $\mathfrak{F}^{\alpha,s}_{b^-,k}f(x)$ can be proved in a similar manner. □

3 Reverse Minkowski inequality involving generalized k-fractional conformable integrals

This section contains our main contribution of establishing the proof of the reverse Minkowski inequality via our newly defined generalized k-fractional conformable integral (2.25) and a related theorem referred to as the reverse Minkowski inequality.

Theorem 9 *For $k > 0$, $s \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$ and $p \geq 1$. Let $f_1, f_2 \in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t > a$, $\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t) < \infty$ and $\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t) < \infty$. If $0 < m \leq \frac{f_1(x)}{f_2(x)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $x \in [a,t]$, then*

$$\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\bigr)^{\frac{1}{p}} + \bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\bigr)^{\frac{1}{p}} \leq c_1\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^p(t)\bigr)^{\frac{1}{p}}, \tag{3.1}$$

with $c_1 = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof Under the given conditions $\frac{f_1(x)}{f_2(x)} \leq M$, $a \leq x \leq t$, it can be written

$$f_1(x) \leq M\bigl(f_1(x)+f_2(x)\bigr) - Mf_1(x),$$

which implies that

$$(M+1)^p f_1^p(x) \leq M^p\bigl(f_1(x)+f_2(x)\bigr)^p. \tag{3.2}$$

By multiplying both sides of (3.2) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\begin{aligned}&\frac{(M+1)^p s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1} f_1^p(x)(x-a)^{s-1}\,dx \\ &\quad\leq \frac{M^p s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1}(f_1+f_2)^p(x)(x-a)^{s-1}\,dx.\end{aligned} \tag{3.3}$$

Accordingly, it can be written as

$$\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\bigr)^{\frac{1}{p}} \leq \frac{M}{M+1}\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^p(t)\bigr)^{\frac{1}{p}}. \tag{3.4}$$

In contrast, as $mf_2(x) \leq f_1(x)$, it follows

$$\left(1+\frac{1}{m}\right)^p f_2^p(x) \leq \left(\frac{1}{m}\right)^p \bigl(f_1(x)+f_2(x)\bigr)^p. \tag{3.5}$$

Further, by multiplying both sides of (3.5) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{1}{p}} \leq \frac{1}{m+1}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^p(t)\right)^{\frac{1}{p}}. \tag{3.6}$$

The required result (3.1) follows from (3.4) and (3.6). □

Inequality (3.1) is known as the reverse Minkowski inequality involving generalized k-fractional conformable integral.

Theorem 10 *For $k>0$, $s\in\mathbb{R}\setminus\{0\}$, $\alpha>0$ and $p\geq 1$. Let $f_2,f_2\in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t>a$, $\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)<\infty$ and $\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)<\infty$. If $0<m\leq\frac{f_1(x)}{f_2(x)}\leq M$ for $m,M\in\mathbb{R}^+$ and for all $x\in[a,t]$, then*

$$\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{2}{p}}+\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{2}{p}} \geq c_2\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{1}{p}}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{1}{p}}, \tag{3.7}$$

with $c_2=\frac{(m+1)(M+1)}{M}-2$.

Proof Taking the product between (3.4) and (3.6) results in

$$\frac{(m+1)(M+1)}{M}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{1}{p}}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{1}{p}} \leq \left(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^p(t)\right)^{\frac{2}{p}}. \tag{3.8}$$

Involving the Minkowski inequality, on the right side of (3.8), we get

$$\begin{aligned}&\frac{(m+1)(M+1)}{M}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{1}{p}}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{1}{p}}\\ &\quad\leq\left(\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{1}{p}}+\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{1}{p}}\right)^2.\end{aligned} \tag{3.9}$$

From (3.9), it can be concluded that

$$\begin{aligned}&\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{2}{p}}+\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{2}{p}}\\ &\quad\geq\left(\frac{(m+1)(M+1)}{M}-2\right)\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{1}{p}}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{1}{p}}.\end{aligned}$$

□

4 Related fractional integral inequalities

This section contains the generalization of the results presented by Chinchane, Sulaiman, and Sroysang related to the reverse Minkowski inequality via Riemann integral operator, involving the proposed generalized k-fractional conformable integral (2.25).

Theorem 11 *For $k>0$, $s\in\mathbb{R}\setminus\{0\}$, $\alpha>0$, $p,q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $f_1,f_2\in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t>a$, $\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)<\infty$ and $\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)<\infty$. If $0<m\leq\frac{f_1(x)}{f_2(x)}\leq M$ for $m,M\in\mathbb{R}^+$ and for all $x\in[a,t]$, then*

$$\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\right)^{\frac{1}{p}}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\right)^{\frac{1}{p}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}}\left(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^{\frac{1}{p}}(t)f_2^{\frac{1}{q}}(t)\right). \tag{4.1}$$

Proof Under the given conditions $\frac{f_1(x)}{f_2(x)} \leq M$, $a \leq x \leq t$, it can be written

$$f_1(x) \leq Mf_2(x) \quad \Rightarrow \quad f_2^{\frac{1}{q}}(x) \geq M^{-\frac{1}{q}} f_1^{\frac{1}{q}}(x). \tag{4.2}$$

Multiplying both sides of (4.2) by $f^{\frac{1}{p}}(x)$, we can rewrite it as follows:

$$f_1^{\frac{1}{p}}(x) f_2^{\frac{1}{q}}(x) \geq M^{-\frac{1}{q}} f_1(x). \tag{4.3}$$

Multiplying both sides of (4.3) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\begin{aligned} &\frac{M^{-\frac{1}{q}} s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \bigl((t-a)^s - (x-a)^s\bigr)^{\frac{\alpha}{k}-1} f_1(x)(x-a)^{s-1}\,dx \\ &\quad \leq \frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \bigl((t-a)^s - (x-a)^s\bigr)^{\frac{\alpha}{k}-1} f_1^{\frac{1}{p}}(x) f_2^{\frac{1}{q}}(x)(x-a)^{s-1}\,dx. \end{aligned} \tag{4.4}$$

Accordingly, it can be written as

$$M^{-\frac{1}{pq}} \bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_1(t)\bigr)^{\frac{1}{p}} \leq \bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_1^{\frac{1}{p}}(t) f_2^{\frac{1}{q}}(t)\bigr)^{\frac{1}{p}}. \tag{4.5}$$

In contrast, as $mf_2(x) \leq f_1(x)$, it follows

$$m^{\frac{1}{p}} f_2^{\frac{1}{p}}(x) \leq f_1^{\frac{1}{p}}(x). \tag{4.6}$$

Further, by multiplying both sides of (4.6) by $f_2^{\frac{1}{q}}(x)$ and invoking the relation $\frac{1}{p} + \frac{1}{q} = 1$, it yields

$$m^{\frac{1}{p}} f_2(x) \leq f_1^{\frac{1}{p}}(x) f_2^{\frac{1}{q}}(x). \tag{4.7}$$

Multiplying both sides of (4.7) by $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$m^{\frac{1}{pq}} \bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_2(t)\bigr)^{\frac{1}{q}} \leq \bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_1^{\frac{1}{p}}(t) f_2^{\frac{1}{q}}(t)\bigr)^{\frac{1}{q}}. \tag{4.8}$$

Conducting the product between (4.5) and (4.8) and using the relation $\frac{1}{p} + \frac{1}{q} = 1$, the required inequality (4.1) can be concluded. □

Theorem 12 *For $k > 0$, $s \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$, $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f_1, f_2 \in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t > a$, $\mathfrak{F}_{a^+,k}^{\alpha,s} f_1^p(t) < \infty$ and $\mathfrak{F}_{a^+,k}^{\alpha,s} f_2^p(t) < \infty$. If $0 < m \leq \frac{f_1(x)}{f_2(x)} \leq M$ for $m, M \in \mathbb{R}^+$ and for all $x \in [a,t]$, then*

$$\mathfrak{F}_{a^+,k}^{\alpha,s} f_1(t) f_2(t) \leq c_3 \bigl(\mathfrak{F}_{a^+,k}^{\alpha,s}\bigl(f_1^p + f_2^p\bigr)(t)\bigr) + c_4 \bigl(\mathfrak{F}_{a^+,k}^{\alpha,s}\bigl(f_1^q + f_2^q\bigr)(t)\bigr), \tag{4.9}$$

with $c_3 = \frac{2^{p-1}M^p}{p(M+1)^p}$ and $c_4 = \frac{2^{q-1}}{q(m+1)^q}$.

Proof Using the hypothesis, we get the following identity:

$$(M+1)^p f_1^p(x) \leq M^p (f_1+f_2)^p(x). \tag{4.10}$$

Multiplying both sides of (4.10) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\begin{aligned} &\frac{(M+1)^p s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1} f_1^p(x)(x-a)^{s-1}\,dx \\ &\quad \leq \frac{M^p s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1} (f_1+f_2)^p(x)(x-a)^{s-1}\,dx. \end{aligned} \tag{4.11}$$

Accordingly, it can be written as

$$\mathfrak{F}_{a^+,k}^{\alpha,s} f_1^p(t) \leq \frac{M^p}{(M+1)^p} \mathfrak{F}_{a^+,k}^{\alpha,s} (f_1+f_2)^p(t). \tag{4.12}$$

In contrast, as $0 < m < \frac{f_1(x)}{f_2(x)}$, $a < x < t$, it follows

$$(m+1)^q f_2^q(x) \leq (f_1+f_2)^q(x). \tag{4.13}$$

Further, by multiplying both sides of (4.13) by $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\mathfrak{F}_{a^+,k}^{\alpha,s} f_2^q(t) \leq \frac{1}{(m+1)^q} \mathfrak{F}_{a^+,k}^{\alpha,s} (f_1+f_2)^q(t). \tag{4.14}$$

Taking into account Young's inequality,

$$f_1(x) f_2(x) \leq \frac{f_1^p(x)}{p} + \frac{f_2^q(x)}{q}. \tag{4.15}$$

Now multiplying both sides of (4.15) by $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\mathfrak{F}_{a^+,k}^{\alpha,s} (f_1 f_2)(t) \leq \frac{1}{p}\bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_1^p(t)\bigr) + \frac{1}{q}\bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_2^q(t)\bigr). \tag{4.16}$$

Invoking (4.12) and (4.14) into (4.16), we obtain

$$\begin{aligned} \mathfrak{F}_{a^+,k}^{\alpha,s} (f_1 f_2)(t) &\leq \frac{1}{p}\bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_1^p(t)\bigr) + \frac{1}{q}\bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} f_2^q(t)\bigr) \\ &\leq \frac{M^p}{p(M+1)^p}\bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} (f_1+f_2)^p(t)\bigr) + \frac{1}{q(m+1)^q}\bigl(\mathfrak{F}_{a^+,k}^{\alpha,s} (f_1+f_2)^q(t)\bigr). \end{aligned} \tag{4.17}$$

Using the inequality $(x+y)^s \leq 2^{s-1}(x^s+y^s)$, $s > 1$, $x, y > 0$, one obtains

$$\mathfrak{F}_{a^+,k}^{\alpha,s} (f_1+f_2)^p(t) \leq 2^{p-1} \mathfrak{F}_{a^+,k}^{\alpha,s} \bigl(f_1^p + f_2^p\bigr)(t) \tag{4.18}$$

and

$$\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^q(t) \leq 2^{q-1}\mathfrak{F}^{\alpha,s}_{a^+,k}\big(f_1^q+f_2^q\big)(t). \tag{4.19}$$

The proof of (4.9) can be concluded from (4.17), (4.18), and (4.19) collectively. □

Theorem 13 *For $k>0$, $s\in\mathbb{R}\setminus\{0\}$, $\alpha>0$, $p\geq 1$. Let $f_1,f_2\in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t>a$, $\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)<\infty$ and $\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)<\infty$. If $0<c<m\leq\frac{f_1(x)}{f_2(x)}\leq M$ for $m,M\in\mathbb{R}^+$ and for all $x\in[a,t]$, then*

$$\begin{aligned}\frac{M+1}{M-c}\big(\mathfrak{F}^{\alpha,s}_{a^+,k}\big(f_1(t)-cf_2(t)\big)\big) &\leq \big(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\big)^{\frac{1}{p}}+\big(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\big)^{\frac{1}{p}}\\ &\leq \frac{m+1}{m-c}\big(\mathfrak{F}^{\alpha,s}_{a^+,k}\big(f_1(t)-cf_2(t)\big)\big)^{\frac{1}{p}}.\end{aligned} \tag{4.20}$$

Proof Using the hypothesis $0<c<m\leq M$, we get

$$\begin{aligned} mc\leq Mc \quad &\Rightarrow \quad mc+m\leq mc+M\leq Mc+M\\ &\Rightarrow \quad (M+1)(m-c)\leq (m+1)(M-c).\end{aligned}$$

It can be concluded that

$$\frac{(M+1)}{(M-c)}\leq\frac{(m+1)}{(m-c)}.$$

Further, we have that

$$m-c\leq\frac{f_1(x)-cf_2(x)}{f_2(x)}\leq M-c$$

implies

$$\frac{(f_1(x)-cf_2(x))^p}{(M-c)^p}\leq f_2^p(x)\leq\frac{(f_1(x)-cf_2(x))^p}{(m-c)^p}. \tag{4.21}$$

Again, we have that

$$\frac{1}{M}\leq\frac{f_2(x)}{f_1(x)}\leq\frac{1}{m} \quad\Rightarrow\quad \frac{m-c}{cm}\leq\frac{f_1(x)-cf_2(x)}{cf_1(x)}\leq\frac{M-c}{cM}$$

implies

$$\left(\frac{M}{M-c}\right)^p\big(f_1(x)-cf_2(x)\big)^p\leq f_1^p(x)\leq\left(\frac{m}{m-c}\right)^p\big(f_1(x)-cf_2(x)\big)^p. \tag{4.22}$$

Multiplying both sides of (4.21) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\begin{aligned}
&\frac{s^{1-\frac{\alpha}{k}}}{(M-c)^p k\Gamma_k(\alpha)}\int_a^t \big((t-a)^s-(x-a)^s\big)^{\frac{\alpha}{k}-1}\big(f_1(x)-cf_2(x)\big)^p(x-a)^{s-1}\,dx\\
&\quad\le \frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \big((t-a)^s-(x-a)^s\big)^{\frac{\alpha}{k}-1}f_2^p(x)(x-a)^{s-1}\,dx.\\
&\quad\le \frac{s^{1-\frac{\alpha}{k}}}{(m-c)^p k\Gamma_k(\alpha)}\int_a^t \big((t-a)^s-(x-a)^s\big)^{\frac{\alpha}{k}-1}\big(f_1(x)-cf_2(x)\big)^p(x-a)^{s-1}\,dx.
\end{aligned}$$

Accordingly, it can be written as

$$\begin{aligned}
\frac{1}{M-c}\big(\mathfrak{F}^{\alpha,s}_{a^+,k}\big(f_1(t)-cf_2(t)\big)^p\big)^{\frac{1}{p}} &\le \big(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2(t)^p\big)^{\frac{1}{p}}\\
&\le \frac{1}{m-c}\big(\mathfrak{F}^{\alpha,s}_{a^+,k}\big(f_1(t)-cf_2(t)\big)^p\big)^{\frac{1}{p}}.
\end{aligned}\tag{4.23}$$

Repeating the same steps with (4.22), we obtain

$$\begin{aligned}
\frac{M}{M-c}\big(\mathfrak{F}^{\alpha,s}_{a^+,k}\big(f_1(t)-cf_2(t)\big)^p\big)^{\frac{1}{p}} &\le \big(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1(t)^p\big)^{\frac{1}{p}}\\
&\le \frac{m}{m-c}\big(\mathfrak{F}^{\alpha,s}_{a^+,k}\big(f_1(t)-cf_2(t)\big)^p\big)^{\frac{1}{p}}.
\end{aligned}\tag{4.24}$$

The proof of (4.20) can be concluded by adding (4.23) and (4.24). □

Theorem 14 *For $k>0$, $s\in\mathbb{R}\setminus\{0\}$, $\alpha>0$ and $p\ge 1$. Let $f_1,f_2\in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t>a$, $\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)<\infty$ and $\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)<\infty$. If $0\le a\le f_1(x)\le A$ and $0\le b\le f_2(x)\le B$ for $m,M\in\mathbb{R}^+$ and for all $x\in[a,t]$, then*

$$\big(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\big)^{\frac{1}{p}}+\big(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\big)^{\frac{1}{p}}\le c_5\big(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^p(t)\big)^{\frac{1}{p}},\tag{4.25}$$

with $c_5=\frac{A(a+B)+B(A+b)}{(A+b)(a+B)}$.

Proof Under the given conditions, it follows that

$$\frac{1}{B}\le\frac{1}{f_2(t)}\le\frac{1}{b}.\tag{4.26}$$

Conducting the product between (4.26) and $0<a\le f_1(x)\le A$, we have

$$\frac{a}{B}\le\frac{f_1(t)}{f_2(t)}\le\frac{A}{b}.\tag{4.27}$$

From (4.27), we get

$$f_2^p(x)\le\left(\frac{B}{a+B}\right)^p\big(f_1(x)+f_2(x)\big)^p\tag{4.28}$$

and

$$f_1^p(x) \le \left(\frac{A}{b+A}\right)^p \bigl(f_1(x)+f_2(x)\bigr)^p. \tag{4.29}$$

By multiplying both sides of (4.28) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\begin{aligned}&\frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1} f_2^p(x)(x-a)^{s-1}\,dx\\ &\quad\le \frac{B^p s^{1-\frac{\alpha}{k}}}{(a+B)^p k\Gamma_k(\alpha)}\int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1}\bigl(f_1(x)+f_2(x)\bigr)^p (x-a)^{s-1}\,dx.\end{aligned}$$

Accordingly, it can be written as

$$\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} f_2^p(t)\bigr)^{\frac{1}{p}} \le \frac{B}{a+B}\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^p(t)\bigr)^{\frac{1}{p}}. \tag{4.30}$$

Repeating the same steps with (4.29), we obtain

$$\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} f_1^p(t)\bigr)^{\frac{1}{p}} \le \frac{A}{b+A}\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^p(t)\bigr)^{\frac{1}{p}}. \tag{4.31}$$

The proof of (4.25) can be concluded by adding (4.30) and (4.31). □

Theorem 15 *For $k>0$, $s\in\mathbb{R}\setminus\{0\}$, $\alpha>0$. Let $f_1,f_2\in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t>a$, $\mathfrak{F}^{\alpha,s}_{a^+,k} f_1^p(t)<\infty$ and $\mathfrak{F}^{\alpha,s}_{a^+,k} f_2^p(t)<\infty$. If $0<m\le\frac{f_1(x)}{f_2(x)}\le M$ for $m,M\in\mathbb{R}^+$ and for all $x\in[a,t]$, then*

$$\frac{1}{M}\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} f_1(t)f_2(t)\bigr) \le \frac{1}{(m+1)(M+1)}\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}(f_1+f_2)^2(t)\bigr) \le \frac{1}{m}\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} f_1(t)f_2(t)\bigr). \tag{4.32}$$

Proof Using $0<m\le\frac{f_1(x)}{f_2(x)}\le M$, it follows that

$$f_2(x)(m+1) \le f_2(x)+f_1(x) \le f_2(t)(M+1). \tag{4.33}$$

Also, it follows that $\frac{1}{M}\le\frac{f_2(x)}{f_1(x)}\le\frac{1}{m}$, which yields

$$f_1(x)\left(\frac{M+1}{M}\right) \le f_2(x)+f_1(x) \le f_1(x)\left(\frac{m+1}{m}\right). \tag{4.34}$$

Conducting the product between (4.33) and (4.34), we have

$$\frac{f_1(x)f_2(x)}{M} \le \frac{(f_2(x)+f_1(x))^2}{(m+1)(M+1)} \le \frac{f_1(x)f_2(x)}{m}. \tag{4.35}$$

By multiplying both sides of (4.35) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\begin{aligned}
&\frac{s^{1-\frac{\alpha}{k}}}{Mk\Gamma_k(\alpha)}\int_a^t\bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1}f_1(x)f_2(x)(x-a)^{s-1}\,dx\\
&\quad\le c_6\frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1}\bigl(f_2(x)+f_1(x)\bigr)^2(x-a)^{s-1}\,dx\\
&\quad\le\frac{s^{1-\frac{\alpha}{k}}}{mk\Gamma_k(\alpha)}\int_a^t\bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1}f_1(x)f_2(x)(x-a)^{s-1}\,dx
\end{aligned}$$

with $c_6=\frac{1}{(m+1)(M+1)}$. Accordingly, the required result (4.32) can be concluded. □

Theorem 16 *For $k>0$, $s\in\mathbb{R}\setminus\{0\}$, $\alpha>0$ and $p\ge 1$. Let $f_1,f_2\in L_{1,s}[a,t]$ be two positive functions in $[0,\infty)$ such that, for all $t>a$, $\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)<\infty$ and $\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)<\infty$. If $0<m\le\frac{f_1(x)}{f_2(x)}\le M$ for $m,M\in\mathbb{R}^+$ and for all $x\in[a,t]$, then*

$$\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}f_1^p(t)\bigr)^{\frac{1}{p}}+\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}f_2^p(t)\bigr)^{\frac{1}{p}}\le 2\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k}h^p\bigl(f_1(t),f_2(t)\bigr)\bigr)^{\frac{1}{p}},\tag{4.36}$$

where $h(f_1(x),f_2(x))=\max\{M[(\frac{M}{m}+1)f_1(t)-Mf_2(t)],\frac{(m+M)f_2(t)-f_1(t)}{m}\}$.

Proof Under the given conditions $0<m\le\frac{f_1(x)}{f_2(x)}\le M$, $a\le x\le t$, it can be written

$$0<m\le M+m-\frac{f_1(x)}{f_2(x)},\tag{4.37}$$

and

$$M+m-\frac{f_1(x)}{f_2(x)}\le M.\tag{4.38}$$

From (4.35) and (4.38), we obtain

$$f_2(x)<\frac{(M+m)f_2(x)-f_1(x)}{m}\le h\bigl(f_1(x),f_2(x)\bigr),\tag{4.39}$$

where $h(f_1(x),f_2(x))=\max\{M[(\frac{M}{m}+1)f_1(t)-Mf_2(t)],\frac{(m+M)f_2(t)-f_1(t)}{m}\}$.

From hypothesis, it also follows that $0<\frac{1}{M}\le\frac{f_2(x)}{f_1(x)}\le\frac{1}{m}$ implies that

$$\frac{1}{M}\le\frac{1}{M}+\frac{1}{m}-\frac{f_2(x)}{f_1(x)}\tag{4.40}$$

and

$$\frac{1}{M}+\frac{1}{m}-\frac{f_2(x)}{f_1(x)}\le\frac{1}{m}.\tag{4.41}$$

From (4.40) and (4.41), we get

$$\frac{1}{M}\le\frac{(\frac{1}{M}+\frac{1}{m})f_1(x)-f_2(x)}{f_1(x)}\le\frac{1}{m},\tag{4.42}$$

which can be rewritten as

$$\begin{aligned} f_1(x) &\le M\left(\frac{1}{M}+\frac{1}{m}\right)f_1(x) - Mf_2(x) \\ &= \frac{M(M+m)f_1(x) - M^2 mf_2(x)}{mM} \\ &= \left(\frac{M}{m}+1\right)f_1(x) - Mf_2(x) \\ &\le M\left[\left(\frac{M}{m}+1\right)f_1(x) - Mf_2(x)\right] \\ &\le h\bigl(f_1(x),f_2(x)\bigr). \end{aligned} \tag{4.43}$$

We can write from (4.39) and (4.43)

$$f_1^p(x) \le h^p\bigl(f_1(x),f_2(x)\bigr), \tag{4.44}$$

$$f_2^p(x) \le h^p\bigl(f_1(x),f_2(x)\bigr). \tag{4.45}$$

By multiplying both sides of (4.44) with $\frac{s^{1-\frac{\alpha}{k}}((t-a)^s-(x-a)^s)^{\frac{\alpha}{k}-1}(x-a)^{s-1}}{k\Gamma_k(\alpha)}$ and then integrating with respect to the variable x from a to t, we obtain

$$\begin{aligned} &\frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1} f_1^p(x)(x-a)^{s-1}\,dx \\ &\quad \le \frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \bigl((t-a)^s-(x-a)^s\bigr)^{\frac{\alpha}{k}-1} h^p\bigl(f_1(x),f_2(x)\bigr)(x-a)^{s-1}\,dx. \end{aligned}$$

Accordingly, it can be written as

$$\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} f_1^p(t)\bigr)^{\frac{1}{p}} \le \bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} h^p\bigl(f_1(t),f_2(t)\bigr)\bigr)^{\frac{1}{p}}. \tag{4.46}$$

Repeating the same procedure as above, for (4.45), we have

$$\bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} f_2^p(t)\bigr)^{\frac{1}{p}} \le \bigl(\mathfrak{F}^{\alpha,s}_{a^+,k} h^p\bigl(f_1(t),f_2(t)\bigr)\bigr)^{\frac{1}{p}}. \tag{4.47}$$

The required result (4.36) follows from (4.46) and (4.47). □

By (2.25) and Theorem (7), under the appropriate values of parameters for each individual fractional integral, the preceding introduced and proved theorems (Theorem 9 to Theorem 16) can be deduced as particular cases, each result involving the following fractional integrals: Riemann–Liouville, Hadamard, Liouville, Katugampola, and generalized fractional conformable integrals.

5 Conclusion

This manuscript starts with a concise overview of fractional integrals in the sense of Riemann–Liouville, Hadamard, and Katugampola as well as a new fractional integral operator according to Jarad et al. [1]. We define the formulation of k-fractional conformable integral operators and their existence. We generalize the reverse Minkowski inequality using k-fractional conformable integrals; as a particular case, the inequality involving fractional integrals in the Riemann–Liouville, Hadamard, and Katugampola sense is given [15]. The related important inequalities involving k-fractional conformable integral are also illustrated. Several inequalities can be generalized for the application of these newly introduced fractional integral operators. Amongst them, we cite the Chebyshev inequality, Grüss-type inequality, and Chebyshev–Grüss type inequality recently introduced and proved in [35–37].

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Author details
[1]Department of Mathematics, University of Sargodha, Sargodha, Pakistan. [2]Department of Mathematics, G.C. University Faisalabad, Faisalabad, Pakistan.

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