## RESEARCH

# V-singular values of rectangular tensors and their applications

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## Abstract

The positive definiteness of rectangular tensors has wide applications in solid mechanics and quantum physics. By modifying the existing definition of singular value for rectangular tensors, some V-singular value inclusion sets for rectangular tensors with positive diagonal entries are established to provide verifiable sufficient conditions for the positive definiteness of rectangular tensors. In addition, an upper bound for the largest V-singular value of nonnegative rectangular tensors is provided.

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**Keywords:** Rectangular tensors; V-singular value; Inclusion set; Positive definiteness

## 1 Introduction

Let $\mathbb{R}(\mathbb{C})$ be the real (complex) field, $p$, $q$, $m$, $n$ be positive integers, $m, n \geq 2$, $[n] = \{1, 2, \ldots, n\}$. A $(p,q)$th order $(m \times n)$-dimensional real rectangular tensor, denoted by $\mathcal{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p;q;m;n]}$, is defined as follows:

$$a_{i_1 \ldots i_p j_1 \ldots j_q} \in \mathbb{R}, \quad i_1, \ldots, i_p \in [m], j_1, \ldots, j_q \in [n].$$

When $p = q = 1$, $\mathcal{A}$ is simply a real $m \times n$ rectangular matrix.

$\mathcal{A}$ is called nonnegative if $a_{i_1 \ldots i_p j_1 \ldots j_q} \geq 0$, denoted by $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$. We say that $\mathcal{A}$ is a real partially symmetric rectangular tensor if $a_{i_1 \ldots i_p j_1 \ldots j_q}$ is invariant under any permutation of indices among $i_1, \ldots, i_p$ and any permutation of indices among $j_1, \ldots, j_q$, i.e.,

$$a_{\pi(i_1 \ldots i_p)\sigma(j_1 \ldots j_q)} = a_{i_1 \ldots i_p j_1 \ldots j_q}, \quad \pi \in S_p, \sigma \in S_q,$$

where $S_r$ is the permutation group of $r$ indices.

For any vectors $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$, let $\mathcal{A}x^{p-1}y^q$ be a vector in $\mathbb{C}^m$ such that

$$\left(\mathcal{A}x^{p-1}y^q\right)_i = \sum_{i_2, \ldots, i_p=1}^{m} \sum_{j_1, \ldots, j_q=1}^{n} a_{ii_2 \ldots i_p j_1 \ldots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q},$$

where $i \in [m]$. Let $\mathcal{A}x^p y^{q-1}$ be a vector in $\mathbb{C}^n$ such that

$$\left(\mathcal{A}x^p y^{q-1}\right)_j = \sum_{i_1, \ldots, i_p=1}^{m} \sum_{j_2, \ldots, j_q=1}^{n} a_{i_1 i_2 \ldots i_p j j_2 \ldots j_q} x_{i_1} \cdots x_{i_p} y_{j_2} \cdots y_{j_q},$$

where $j \in [n]$. If there exist a number $\lambda \in \mathbb{C}$, vectors $x \in \mathbb{C}^m \setminus \{0\}$, and $y \in \mathbb{C}^n \setminus \{0\}$ such that

$$
\begin{cases}
\mathcal{A} x^{p-1} y^q = \lambda x^{[l-1]}, \\
\mathcal{A} x^p y^{q-1} = \lambda y^{[l-1]},
\end{cases}
\tag{1}
$$

where $x^{[\alpha]} = [x_1^\alpha, \ldots, x_n^\alpha]^T$ and $l = p + q$, then $\lambda$ is called the *singular value* of $\mathcal{A}$, and $(x, y)$ is the *left and right eigenvector pair* of $\mathcal{A}$, associated with $\lambda$. If $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$, then we say that $\lambda$ is an *H-singular value* of $\mathcal{A}$, and $(x, y)$ is the *left and right H-eigenvector pair* associated with $\lambda$; see [1–4].

We call $\mathcal{A}$ an *elasticity tensor*, if $p = q = 2$, $m = n = 2$ or 3, and $\mathcal{A}$ is a real partially symmetric rectangular tensor. When $\mathcal{A}$ is an elasticity tensor, the strong ellipticity condition holds if and only if $\mathcal{A}$ is positive definite, the strong ellipticity condition plays an important role in the theory of elasticity [5, 6]. By introducing the H-singular values of a partially symmetric tensor, Chang et al. [1] provided a necessary and sufficient condition for the positive definiteness of a partially symmetric rectangular tensor as follows.

**Theorem 1** ([1]) *Suppose that $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular tensor, $p$ and $q$ are even. Then $\mathcal{A}$ is positive definite, i.e.,*

$$
f(x, y) = \mathcal{A} x^p y^q = \sum_{i_1, \ldots, i_p = 1}^{m} \sum_{j_1, \ldots, j_q = 1}^{n} a_{i_1 \ldots i_p j_1 \ldots j_q} x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q} > 0
\tag{2}
$$

*for all nonzero vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ if and only if all of its H-singular values are positive.*

In 2005, Qi obtained a Geršgorin-type H-eigenvalue localization set for real symmetric tensors, which is useful in verifying the positive definiteness of square tensor $\mathcal{A}$ [7, 8]. And then some improved results are given in [9–12]. Recently, the problem of singular value inclusion sets for rectangular tensors became an important topic [13–15]. Unfortunately, these singular value inclusion sets of rectangular tensors cannot be used to verify the positive definiteness of a rectangular tensor.

Inspired by the above, in Sect. 2, we intend to introduce the definition of V-singular value for a rectangular tensor and study its properties. In Sect. 3, we give some V-singular value inclusion sets for rectangular tensors with positive diagonal entries. As applications, we give an upper bound for the largest V-singular value of nonnegative rectangular tensors in Sect. 4, and sufficient conditions of the positive definiteness for rectangular tensors are also provided in Sect. 5.

## 2  V-singular values of rectangular tensors

First, we introduce the definition of a V-singular value for a rectangular tensor, which can be viewed as a variant of the definition of H-singular value [1, 2].

**Definition 1** Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$, $p, q \geq 2$. If there exist a number $\lambda \in \mathbb{R}$, vectors $x \in \mathbb{R}^m \backslash \{0\}$, and $y \in \mathbb{R}^n \backslash \{0\}$ such that

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[p-1]}, \\ \mathcal{A}x^p y^{q-1} = \lambda y^{[q-1]}, \\ \sum_{i=1}^m x_i^p = 1, \\ \sum_{j=1}^n y_j^q = 1, \end{cases} \tag{3}$$

then $\lambda$ is called the V-singular value of $\mathcal{A}$, and $(x, y)$ is the left and right eigenvector pair of $\mathcal{A}$ associated with $\lambda$.

*Remark* 1 When $p = 2$, $q = 2$, our definition is the same as the M-eigenvalue which is introduced in [16].

*Remark* 2 When $p$ is even and $q$ is even, our definition can be viewed as a special case of the $l^{k,s}$-singular value which is introduced in [4]. However, when one of $p$, $q$ is odd, the definition of V-singular value is different from the $l^{k,s}$-singular value.

**Theorem 2** *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$. We have the following conclusions*:

(a) *If $x$ and $y$ are left and right eigenvectors of $\mathcal{A}$, associated with a V-singular value $\lambda$ of $\mathcal{A}$, then*

$$\lambda = f(x, y) = \mathcal{A}x^p y^q.$$

(b) *When $p$ is odd and $q$ is odd, then $(\lambda, -x, -y)$, $(-\lambda, -x, y)$, and $(-\lambda, x, -y)$ are also V-singular values and their associated eigenvectors of $\mathcal{A}$.*

(c) *When $p$ is odd and $q$ is even, then $(\lambda, x, -y)$, $(-\lambda, -x, y)$, and $(-\lambda, -x, -y)$ are also V-singular values and their associated eigenvectors of $\mathcal{A}$.*

(d) *When $p$ is even and $q$ is even, then $(\lambda, -x, -y)$, $(\lambda, -x, y)$, and $(\lambda, x, -y)$ are also V-singular values and their associated eigenvectors of $\mathcal{A}$.*

*Proof* By the first equation of (3), we have

$$f(x, y) = \mathcal{A}x^p y^q = \lambda \left(x^{p-1}\right)^T x = \lambda \sum_{i=1}^m x_i^p = \lambda.$$

By the second equation of (3), we have

$$f(x, y) = \mathcal{A}x^p y^q = \lambda \left(y^{q-1}\right)^T y = \lambda \sum_{j=1}^n y_j^q = \lambda.$$

We thus have conclusion (a).

It is straightforward to verify the assertions (b), (c) and (d).  □

### 3  V-singular value inclusion sets for rectangular tensors with positive diagonal entries

Let $\sigma(\mathcal{A})$ be the set containing all V-singular values of $\mathcal{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p;q;m;n]}$. We call the entries $a_{i \ldots ij \ldots j}$ ($i \in [m]$, $j \in [n]$) diagonal, and other entries are called off-diagonal. In this section, some new V-singular value inclusion sets are given to locate all V-singular values of rectangular tensors with positive diagonal entries.

**Theorem 3** *Consider* $\mathcal{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}^{[p;q;m;n]}$ *with positive diagonal entries, $p$ and $q$ even, and* $\Delta = \{(j_1, \ldots, j_q) : (1, \ldots, 1), \ldots, (n, \ldots, n)\}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma_1(\mathcal{A}) = \bigcup_{i \in [m]} \{z \in \mathbb{R} : |z - \underline{\alpha_i}| \le R_i(\mathcal{A}) + \overline{\alpha_i} - \underline{\alpha_i}\},$$

*where* $\min_{i \in [m]}\{a_{i \ldots i1 \ldots 1}, \ldots, a_{i \ldots in \ldots n}\} = \underline{\alpha_i}$, $\max_{i \in [m]}\{a_{i \ldots i1 \ldots 1}, \ldots, a_{i \ldots in \ldots n}\} = \overline{\alpha_i}$,

$$R_i(\mathcal{A}) = \sum_{\substack{i_2, \ldots, i_p = 1}}^{m} \sum_{\substack{j_1, \ldots, j_q = 1, \\ (j_1, \ldots, j_q) \notin \Delta}}^{n} |a_{ii_2 \ldots i_p j_1 \ldots j_q}| + \gamma_i,$$

$$\gamma_i = \max_{k \in [n]} \left\{ \sum_{\substack{i_2, \ldots, i_p = 1, \\ (i_2, \ldots, i_p) \ne (i, \ldots, i)}}^{m} |a_{ii_2 \ldots i_p k \ldots k}| \right\}.$$

*Proof* Assume that $\lambda$ is a V-singular value of $\mathcal{A}$, $x = (x_i)_{i=1}^{m} \in \mathbb{R}^m \backslash \{0\}$ and $y = (y_j)_{j=1}^{n} \in \mathbb{R}^n \backslash \{0\}$ are the corresponding left and right eigenvectors. Let $|x_s| = \max_{i \in [m]}\{|x_i|\}$. Then $|x_s| \ne 0$. The $s$th equation of $\mathcal{A}x^{p-1}y^q = \lambda x^{[p-1]}$ is

$$\lambda x_s^{p-1} = \sum_{i_2, \ldots, i_p = 1}^{m} \sum_{j_1, \ldots, j_q = 1}^{n} a_{si_2 \ldots i_p j_1 \ldots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$= \sum_{\substack{i_2, \ldots, i_p = 1, \\ (i_2, \ldots, i_p) = (s, \ldots, s)}}^{m} \sum_{\substack{j_1, \ldots, j_q = 1, \\ (j_1, \ldots, j_q) \in \Delta}}^{n} a_{si_2 \ldots i_p j_1 \ldots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$+ \sum_{\substack{i_2, \ldots, i_p = 1, \\ (i_2, \ldots, i_p) = (s, \ldots, s)}}^{m} \sum_{\substack{j_1, \ldots, j_q = 1, \\ (j_1, \ldots, j_q) \notin \Delta}}^{n} a_{si_2 \ldots i_p j_1 \ldots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$+ \sum_{\substack{i_2, \ldots, i_p = 1, \\ (i_2, \ldots, i_p) \ne (s, \ldots, s)}}^{m} \sum_{\substack{j_1, \ldots, j_q = 1, \\ (j_1, \ldots, j_q) \notin \Delta}}^{n} a_{si_2 \ldots i_p j_1 \ldots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$+ \sum_{\substack{i_2, \ldots, i_p = 1, \\ i_2, \ldots, i_p \ne s, \ldots, s}}^{m} \sum_{\substack{j_1, \ldots, j_q = 1, \\ (j_1, \ldots, j_q) \in \Delta}}^{n} a_{si_2 \ldots i_p j_1 \ldots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q},$$

which can be rewritten as

$$\left(\lambda - \left(a_{s \ldots s1 \ldots 1} y_1^q + \cdots + a_{s \ldots sn \ldots n} y_n^q\right)\right) x_s^{p-1}$$

$$
= \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)=(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} a_{si_2\ldots i_p j_1\ldots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q}
$$

$$
+ \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)\neq(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} a_{si_2\ldots i_p j_1\ldots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q}
$$

$$
+ \sum_{\substack{i_2,\ldots,i_p=1, \\ i_2,\ldots,i_p\neq s,\ldots,s}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\in\Delta}}^{n} a_{si_2\ldots i_p j_1\ldots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q}.
$$

Therefore,

$$
\bigl(\lambda - \underline{\alpha_s}\bigl(y_1^q + \cdots + y_n^q\bigr)\bigr)x_s^{p-1}
$$

$$
= \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)=(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} a_{si_2\ldots i_p j_1\ldots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q}
$$

$$
+ \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)\neq(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} a_{si_2\ldots i_p j_1\ldots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q}
$$

$$
+ \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)\neq(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\in\Delta}}^{n} a_{si_2\ldots i_p j_1\ldots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q}
$$

$$
+ (a_{s\ldots s1\ldots1} - \underline{\alpha_s})x_s^{p-1}y_1^q + \cdots + (a_{s\ldots sn\ldots n} - \underline{\alpha_s})x_s^{p-1}y_n^q.
$$

By taking absolute values on both sides and the fact that $\sum_{j=1}^{n} y_j^q = 1$, $q$ is even, we have

$$
|\lambda - \underline{\alpha_s}|\bigl|x_s^{p-1}\bigr| \leq \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)=(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} |a_{si_2\ldots i_p j_1\ldots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|
$$

$$
+ \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)\neq(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} |a_{si_2\ldots i_p j_1\ldots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|
$$

$$
+ \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)\neq(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\in\Delta}}^{n} |a_{si_2\ldots i_p j_1\ldots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|
$$

$$
+ |a_{s\ldots s1\ldots1} - \underline{\alpha_s}|\bigl|x_s^{p-1}\bigr|\bigl|y_1^q\bigr| + \cdots + |a_{s\ldots sn\ldots n} - \underline{\alpha_s}|\bigl|x_s^{p-1}\bigr|\bigl|y_n^q\bigr|
$$

$$
\leq \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)=(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} |a_{si_2\ldots i_p j_1\ldots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|
$$

$$
+ \sum_{\substack{i_2,\ldots,i_p=1, \\ (i_2,\ldots,i_p)\neq(s,\ldots,s)}}^{m} \sum_{\substack{j_1,\ldots,j_q=1, \\ (j_1,\ldots,j_q)\notin\Delta}}^{n} |a_{si_2\ldots i_p j_1\ldots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|
$$

$$+ \sum_{\substack{i_2,\dots,i_p=1, \\ (i_2,\dots,i_p) \neq (s,\dots,s)}} \sum_{\substack{j_1,\dots,j_q=1, \\ (j_1,\dots,j_q)\in\Delta}} |a_{si_2\dots i_p j_1\dots j_q}| |x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|$$

$$+ (\overline{\alpha_s} - \underline{\alpha_s})(y_1^q + \cdots + y_n^q)|x_s^{p-1}|.$$

Note that

$$\sum_{\substack{i_2,\dots,i_p=1, \\ (i_2,\dots,i_p) \neq (s,\dots,s)}} \sum_{\substack{j_1,\dots,j_q=1, \\ (j_1,\dots,j_q)\in\Delta}} |a_{si_2\dots i_p j_1\dots j_q}| |x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}|$$

$$\leq \left( \sum_{\substack{i_2,\dots,i_p=1, \\ (i_2,\dots,i_p) \neq (s,\dots,s)}} |a_{si_2\dots i_p 1\dots 1}||y_1^q| + \cdots + \sum_{\substack{i_2,\dots,i_p=1, \\ (i_2,\dots,i_p) \neq (s,\dots,s)}} |a_{si_2\dots i_p n\dots n}||y_n^q| \right) |x_s^{p-1}|$$

$$\leq \max_{k\in[n]} \left\{ \sum_{\substack{i_2,\dots,i_p=1, \\ (i_2,\dots,i_p) \neq (s,\dots,s)}} |a_{si_2\dots i_p k\dots k}| \right\} (y_1^q + \cdots + y_n^q)|x_s^{p-1}|$$

$$= \max_{k\in[n]} \left\{ \sum_{\substack{i_2,\dots,i_p=1, \\ (i_2,\dots,i_p) \neq (s,\dots,s)}} |a_{si_2\dots i_p k\dots k}| \right\} |x_s^{p-1}|.$$

Then,

$$|\lambda - \underline{\alpha_s}||x_s^{p-1}| \leq R_s(\mathcal{A})|x_s^{p-1}| + (\overline{\alpha_s} - \underline{\alpha_s})|x_s^{p-1}|. \tag{4}$$

Dividing $|x_s|$ on both sides of (4) yields

$$|\lambda - \underline{\alpha_s}| \leq R_s(\mathcal{A}) + (\overline{\alpha_s} - \underline{\alpha_s}). \tag{5}$$

Thus $\sigma(\mathcal{A}) \subseteq \Gamma_1(\mathcal{A})$. This completes the proof.                                  □
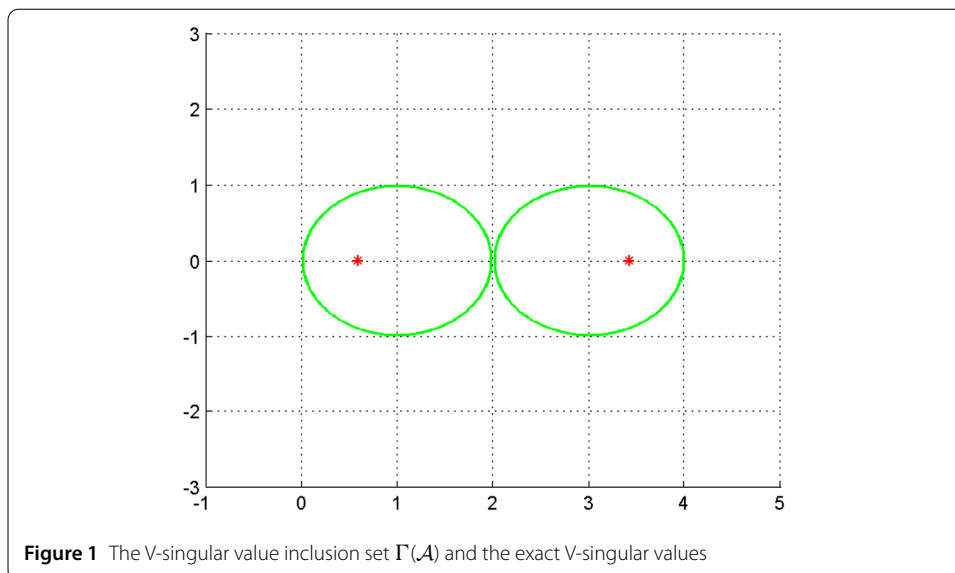
Similarly, using the forms $\mathcal{A}x^p y^{q-1} = \lambda y^{[q-1]}$ and $\sum_{i=1}^m x_i^p = 1$, we can get the following V-singular value interval.

**Theorem 4** *Consider* $\mathcal{A} = (a_{i_1\dots i_p j_1\dots j_q}) \in \mathbb{R}^{[p;q;m;n]}$ *with positive diagonal entries, $p$ and $q$ even, and* $\Omega = \{(i_1,\dots,i_p) : (1,\dots,1),\dots,(m,\dots,m)\}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma_2(\mathcal{A}) = \bigcup_{j\in[n]} \{z \in \mathbb{R} : |z - \underline{\beta_j}| \leq C_j(\mathcal{A}) + \overline{\beta_j} - \underline{\beta_j}\},$$

*where* $\min_{j\in[n]}\{a_{1\dots 1j\dots j},\dots,a_{m\dots mj\dots j}\} = \underline{\beta_j}$, $\max_{j\in[n]}\{a_{1\dots 1j\dots j},\dots,a_{m\dots mj\dots j}\} = \overline{\beta_j}$,

$$C_j(\mathcal{A}) = \sum_{\substack{i_1,\dots,i_p=1, \\ (i_1,\dots,i_p)\notin\Omega}} \sum_{j_2,\dots,j_q=1} |a_{i_1\dots i_p j j_2\dots j_q}| + \delta_j,$$

$$\delta_j = \max_{k\in[m]} \left\{ \sum_{\substack{j_2,\dots,j_q=1, \\ (j_2,\dots,j_q) \neq (j,\dots,j)}} |a_{k\dots kj j_2\dots j_q}| \right\}.$$

**Figure 1** The V-singular value inclusion set $\Gamma(\mathcal{A})$ and the exact V-singular values

By Theorems 3 and 4, we can get an inclusion set to locate all V-singular value of a rectangular tensor.

**Corollary 1** *Consider* $\mathcal{A} = (a_{i_1 \dots i_p j_1 \dots j_q}) \in \mathbb{R}^{[p;q;m;n]}$ *with positive diagonal entries, as well as* $p$ *and* $q$ *even. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \Gamma_1(\mathcal{A}) \cap \Gamma_2(\mathcal{A}).$$

*Example* 1 Let us consider the rectangular tensor $\mathcal{A} \in \mathbb{R}^{[2;4;2;2]}$ with

$$a_{111111} = a_{112222} = 1, \qquad a_{221111} = a_{222222} = 3,$$

$$a_{121111} = a_{122222} = a_{211111} = a_{212222} = -1,$$

and other $a_{i_1 i_2 j_1 \dots j_4} = 0$.

The V-singular value inclusion set $\Gamma(\mathcal{A})$ and the exact V-singular values are drawn in Fig. 1, where $\Gamma(\mathcal{A})$ is represented by solid boundary, and the exact V-singular values are plotted by "*". It is easy to see that $\sigma(\mathcal{A}) = \{2 \pm \sqrt{2}\} \subseteq \Gamma(\mathcal{A})$, that is, $\Gamma(\mathcal{A})$ can capture all V-singular values of $\mathcal{A}$.

## 4 Nonnegative rectangular tensors

In this section, we give some results for the V-singular value of nonnegative rectangular tensors. The following definition for irreducibility has been introduced in [17].

**Definition 2** The square tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \dots, n\}$ such that $a_{i_1, i_2, \dots, i_m} = 0$, $\forall i_1 \in \mathbb{J}$, $\forall i_2, \dots, i_m \notin \mathbb{J}$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

The Perron–Frobenius theorem for nonnegative square tensors is introduced in [18], which states that the spectral radius of any nonnegative square tensor is an eigenvalue with a nonnegative eigenvector, and the eigenvector is positive and unique if the square

tensor is irreducible. For any $j \in [n]$, let $\mathcal{A}(\cdot, j) = (a_{i_1 \ldots i_p j \ldots j})$ be a $p$th order $m$-dimensional square tensor. For any $i \in [m]$, let $\mathcal{A}(\cdot, i) = (a_{i \ldots i j_1 \ldots j_q})$ be a $q$th order $n$-dimensional square tensor.

**Definition 3** ([1]) A nonnegative rectangular tensor $\mathcal{A}$ is called irreducible if all the square tensors $\mathcal{A}(\cdot, j)$, $j \in [n]$, and $\mathcal{A}(\cdot, i)$, $i \in [m]$, are irreducible.

Denote by $\lambda_{\max}(\mathcal{A})$ the maximal V-singular value of $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$. Then

$$\lambda_{\max}(\mathcal{A}) = \max\left\{ \mathcal{A}x^p y^q, \sum_{i=1}^{m} x_i^p = 1, \sum_{j=1}^{n} y_j^q = 1 \right\}.$$

And we call

$$\rho_V(\mathcal{A}) = \max\left\{ |\lambda| : \lambda \in \sigma(\mathcal{A}) \right\}$$

the largest V-singular value of $\mathcal{A}$. We have the following result.

**Theorem 5** *Let* $\mathcal{A} = (a_{i_1 \ldots i_p j_1 \ldots j_q}) \in \mathbb{R}_+^{[p;q;m;n]}$. *Then,* $\rho_V(\mathcal{A}) = \lambda_{\max}(\mathcal{A})$, *and there is a pair of nonnegative eigenvectors corresponding to* $\rho_V(\mathcal{A})$. *Furthermore, if* $\mathcal{A}$ *is irreducible, then* $\rho_V(\mathcal{A}) = \lambda_{\max}(\mathcal{A})$, *and there is a pair of positive eigenvectors corresponding to* $\rho_V(\mathcal{A})$.

*Proof* Assume that $\lambda$ is a V-singular value of $\mathcal{A}$, $x = (x_i)_{i=1}^{m} \in \mathbb{R}^m$ and $y = (y_j)_{j=1}^{n} \in \mathbb{R}^n$ are the corresponding left and right eigenvectors. Then $\lambda = \mathcal{A}x^p y^q$ and, by the nonnegativity of the entries of $\mathcal{A}$, we have

$$|\lambda| \leq \lambda_{\max}(\mathcal{A}).$$

Next, we consider the eigenvectors of $\rho_V(\mathcal{A})$. Assume that $\lambda_{\max}(\mathcal{A})$ is the maximal V-singular value of $\mathcal{A}$, $\mu$ and $\nu$ are the corresponding left and right eigenvectors, and one of $\mu$ or $\nu$ is not a nonnegative vector. Then we have

$$\lambda_{\max}(\mathcal{A}) = \mathcal{A}\mu^p \nu^q \leq \mathcal{A}|\mu^p||\nu^q| \leq \lambda_{\max}(\mathcal{A}),$$

which implies $\lambda_{\max}(\mathcal{A}) = \mathcal{A}|\mu^p||\nu^q|$. Therefore, $|\mu|$ and $|\nu|$ are the left and right nonnegative eigenvectors corresponding to $\lambda_{\max}(\mathcal{A})$.

If $\mathcal{A}$ is irreducible, by Lemma 2 in [1], $\mathcal{A}|\mu^{p-1}||\nu^q| \neq 0$, $\mathcal{A}|\mu^p||\nu^{q-1}| \neq 0$, that is to say, $|\mu| > 0$ and $|\nu| > 0$.

Hence, the proof is completed. □

From Theorem 5, we obtain the the following corollary.

**Corollary 2** *Let* $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$. *Then,*

$$\rho_V(\mathcal{A}) = \max\left\{ \mathcal{A}x^p y^q, \sum_{i=1}^{m} x_i^p = 1, \sum_{j=1}^{n} y_j^q = 1 \right\}.$$

We have the following lemma.

**Lemma 1** *Let $\mathcal{A} \in \mathbb{R}_+^{[p;q;m;n]}$. Then,*

$$\rho_V(\mathcal{A}) \geq \max_{i \in [m], j \in [n]} \{a_{i...ij...j}\}.$$

*Proof* Let $e_i$ be an $m$-dimensional real vector, whose $i$th entry is 1, and others are 0, also let $e_j$ be an $n$-dimensional real vector, whose $j$th entry is 1 and others are 0. Hence, from Corollary 2,

$$\rho(\mathcal{A}) \geq \mathcal{A}e_i \cdots e_i e_j \cdots e_j = a_{i...ij...j},$$

for any $i \in [m], j \in [n]$. This means $\rho(\mathcal{A}) \geq \max_{i \in [m], j \in [n]} \{a_{i...ij...j}\}$.                                    $\square$

Based on Theorem 5, an upper bound for the largest V-singular value of nonnegative rectangular tensors is given.

**Theorem 6** *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be nonnegative. Then,*

$$\rho_V(\mathcal{A}) \leq \min\left\{ \max_{i \in [m]}\left\{R_i(\mathcal{A}) + \overline{\alpha_i}\right\}, \max_{j \in [n]}\left\{C_j(\mathcal{A}) + \overline{\beta_j}\right\} \right\}.$$

*Proof* Because $\rho_V(\mathcal{A})$ is a V-singular value of $\mathcal{A}$, $x = (x_i)_{i=1}^m \in \mathbb{R}^m\setminus\{0\}$ and $y = (y_j)_{j=1}^n \in \mathbb{R}^n\setminus\{0\}$ are the corresponding left and right eigenvectors. From Theorem 5, we have $x, y \geq 0$. Let $x_s = \max_{i \in [m]}\{x_i\}$. Then $x_s > 0$. The $s$th equation of $\mathcal{A}x^{p-1}y^q = \lambda x^{[p-1]}$ is

$$\rho_V(\mathcal{A})x_s^{p-1} = \sum_{i_2,...,i_p=1}^m \sum_{j_1,...,j_q=1}^n a_{si_2...i_pj_1...j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$= \sum_{\substack{i_2,...,i_p=1, \\ (i_2,...,i_p)=(s,...,s)}}^m \sum_{\substack{j_1,...,j_q=1, \\ (j_1,...,j_q)\in\Delta}}^n a_{si_2...i_pj_1...j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$+ \sum_{\substack{i_2,...,i_p=1, \\ (i_2,...,i_p)\neq(s,...,s)}}^m \sum_{\substack{j_1,...,j_q=1, \\ (j_1,...,j_q)\in\Delta}}^n a_{si_2...i_pj_1...j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$+ \sum_{i_2,...,i_p=1}^m \sum_{\substack{j_1,...,j_q=1, \\ (j_1,...,j_q)\notin\Delta}}^n a_{si_2...i_pj_1...j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$$

$$\leq R_s(\mathcal{A})x_s^{p-1} + \overline{\alpha_s}x_s^{p-1}.$$

Then,

$$\rho_V(\mathcal{A}) \leq R_s(\mathcal{A}) + \overline{\alpha_s}.$$

Similarly, let $y_t = \max_{j \in [n]}\{y_j\}$. Then $y_t > 0$. Considering the $t$th equation of $\mathcal{A}x^p y^{q-1} = \lambda y^{[q-1]}$, we can get

$$\rho_V(\mathcal{A}) \leq C_t(\mathcal{A}) + \overline{\beta_t}.$$

The proof is completed.                                    $\square$

*Example* 2  Let us consider the rectangular tensor $\mathcal{A} \in \mathbb{R}^{[2;4;2;2]}$ with

$$a_{111111} = a_{112222} = 3, \qquad a_{221111} = a_{222222} = 3,$$

$$a_{121111} = a_{122222} = a_{211111} = a_{212222} = 1,$$

and other $a_{i_1 i_2 j_1 \ldots j_4} = 0$.

By Theorem 6, we have

$$\rho_V(\mathcal{A}) \leq 4.$$

In fact, $\rho_V(\mathcal{A}) = 4$.

*Example* 3  Let us consider the rectangular tensor $\mathcal{A} \in \mathbb{R}^{[3;2;2;2]}$ with

$$a_{11111} = a_{11122} = 3, \qquad a_{22211} = a_{22222} = 3,$$

$$a_{11211} = a_{11222} = 1, \qquad a_{21111} = a_{21122} = 1,$$

and other $a_{i_1 i_2 i_3 j_1 j_2} = 0$.

By Theorem 6, we have

$$\rho_V(\mathcal{A}) \leq 4.$$

In fact, $\rho_V(\mathcal{A}) = 4$.

## 5  Positive definiteness of rectangular tensors

In this section, we provide some verifiable sufficient conditions for the positive definiteness of rectangular tensors. We have the following theorem.

**Theorem 7**  *Suppose that $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular tensor, $p$ and $q$ are even. Then,*

  (a)  *There exist V-singular value of $\mathcal{A}$ and associated left and right eigenvectors.*
  (b)  *$\mathcal{A}$ is positive definite if and only if all of its V-singular values are positive.*

*Proof*  Consider the following optimization problem:

$$\min \left\{ f(x,y) = \mathcal{A}x^p y^q : \sum_{i=1}^{m} x_i^p = 1, \sum_{j=1}^{n} y_j^q = 1 \right\}. \tag{6}$$

The objective function $f(x,y)$ is continuous in variables $x$ and $y$. When $p$ and $q$ are even, the feasible set $\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} x_i^p = 1, \sum_{j=1}^{n} y_j^q = 1\}$ is compact. Then, there exist vectors $x_*$ and $y_*$ that solve the optimization problem with the minimal objective value $\lambda_* = \mathcal{A}x_*^p y_*^q$.

By the Lagrangian multiplier method, for the optimal solution $(x_*, y_*)$, there exist multipliers $\lambda$ and $\mu$ such that

$$\begin{cases} \mathcal{A}x_*^{p-1}y_*^q = \lambda x_*^{[p-1]}, \\ \mathcal{A}x_*^p y_*^{q-1} = \mu y_*^{[q-1]}, \\ \sum_{i=1}^m (x_*)_i^p = 1, \\ \sum_{j=1}^n (y_*)_j^q = 1, \end{cases}$$

and then we have $\lambda = \mu = \lambda_* = \mathcal{A}x_*^p y_*^q$. Therefore, $\lambda_*$, $x_*$ and $y_*$ satisfy the conditions of Definition 1. This proves existence.

When $p$ and $q$ are even, $\mathcal{A}$ is positive definite if and only if the optimal objective function value of (6) is positive. Suppose that every V-singular value of $\mathcal{A}$ is positive. By the proof for (a), $\lambda_*$, $x_*$ and $y_*$ are the optimal solution of (6), and $\lambda_*$ is a V-singular value of $\mathcal{A}$. That is to say, $\lambda_* > 0$. Hence, $\mathcal{A}$ is positive definite.

On the other hand, suppose that $\mathcal{A}$ is positive definite. Let $\lambda$ be a V-singular value of $\mathcal{A}$, $x$ and $y$ be a left and right eigenvectors of $\mathcal{A}$ associated with $\lambda$. If $f(x, y) > 0$, by Theorem 2, we have $\lambda > 0$.

The proof is completed. $\qquad\square$

Two sufficient conditions for the positive definiteness of partially symmetric rectangular tensors are obtained. First, we need the following lemma.

**Lemma 2** *Suppose that $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ is a partially symmetric rectangular tensor, and $p$ and $q$ are even. If $\mathcal{A}$ is positive definite, then*

$$a_{i\ldots ij\ldots j} > 0, \quad i \in [m], j \in [n].$$

*Proof* Let $e_i$ be an $m$-dimensional real vector, whose $i$th entry is 1 and others are 0, also let $e_j$ be an $n$-dimensional real vector, whose $j$th entry is 1 and others are 0. Since the partially symmetric rectangular tensor $\mathcal{A}$ is positive definite,

$$\mathcal{A}e_i \cdots e_i e_j \cdots e_j = a_{i\ldots ij\ldots j} > 0,$$

for any $i \in [m], j \in [n]$. $\qquad\square$

**Theorem 8** *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor, $p$ and $q$ even, and $a_{i\ldots i1\ldots 1} = \cdots = a_{i\ldots in\ldots n} = \alpha_i > 0$, $i \in [m]$. If for all $i \in [m]$,*

$$\alpha_i > R_i(\mathcal{A}),$$

*then $\mathcal{A}$ is positive definite.*

*Proof* Assume that $\lambda \leq 0$ is a V-singular value of $\mathcal{A}$. From Theorem 3, we have $\lambda \in \Gamma_1(\mathcal{A})$, hence, there is $i_0 \in [m]$ such that

$$|\lambda - \alpha_{i_0}| \leq R_{i_0}(\mathcal{A}).$$

From $\alpha_{i_0} > 0$ for all $i_0 \in [m]$, we get

$$|\lambda - \alpha_{i_0}| \geq \alpha_{i_0} > R_{i_0}(\mathcal{A}).$$

This is a contradiction. Hence, $\lambda > 0$. Then, the partially symmetric rectangular tensor $\mathcal{A}$ is positive definite.                                                                                                 □

Similarly to the proof of Theorem 8, we can get the following result.

**Theorem 9** *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor, $p$ and $q$ even, and $a_{1...1j...j} = \cdots = a_{m...mj...j} = \beta_j > 0$, $j \in [n]$. If for all $j \in [n]$,*

$$\beta_j > C_j(\mathcal{A}),$$

*then $\mathcal{A}$ is positive definite.*

**Theorem 10** *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor with positive diagonal entries, $p$ and $q$ even, and*

$$\min_{i\in[m]}\{a_{i...i1...1},\ldots,a_{i...in...n}\} = \alpha_i, \qquad \min_{j\in[n]}\{a_{1...1j...j},\ldots,a_{m...mj...j}\} = \beta_j.$$

*Assume $\mathcal{B}_1$ and $\mathcal{B}_2$ are partially symmetric rectangular tensors whose $(i_1 \ldots i_p j_1 \ldots j_q)$th entries are respectively defined as follows:*

$$(\mathcal{B}_1)_{i_1...i_p j_1...j_q} = \begin{cases} b_{i...i1...1} = \cdots = b_{i...in...n} = \alpha_i, & i \in [m], \\ a_{i_1...i_p j_1...j_q}, & otherwise, \end{cases} \tag{7}$$

*and*

$$(\mathcal{B}_2)_{i_1...i_p j_1...j_q} = \begin{cases} b_{1...1j...j} = \cdots = b_{m...mj...j} = \beta_j, & j \in [n], \\ a_{i_1...i_p j_1...j_q}, & otherwise. \end{cases} \tag{8}$$

*If $\mathcal{B}_1$ (or $\mathcal{B}_2$) is positive definite, then $\mathcal{A}$ is positive definite.*

*Proof* Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ be two nonzero vectors. Since $\mathcal{B}_1$ is positive definite, from the definition of positive definite partially symmetric rectangular tensor, we have

$$\mathcal{B}_1 x^p y^q > 0.$$

Then, we have

$$\mathcal{A} x^p y^q = \mathcal{B}_1 x^p y^q + \sum_{i\in[m],j\in[n]} (a_{i...ij...j} - b_{i...ij...j})x_i^p y_j^q > 0.$$

Thus $\mathcal{A}$ is positive definite.

If $\mathcal{B}_2$ is positive definite, then

$$\mathcal{B}_2 x^p y^q > 0.$$

Then, we have

$$\mathcal{A}x^p y^q = \mathcal{B}_2 x^p y^q + \sum_{i \in [m], j \in [n]} (a_{i...ij...j} - b_{i...ij...j}) x_i^p y_j^q > 0.$$

Thus $\mathcal{A}$ is positive definite.                                                                □

By Theorem 8, 9 and 10, we have the following sufficient conditions for the positive definiteness of partially symmetric rectangular tensors.

**Theorem 11** *Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor with positive diagonal entries, $p$ and $q$ even, and*

$$\min_{i \in [m]}\{a_{i...i1...1},\ldots,a_{i...in...n}\} = \alpha_i, \qquad \min_{j \in [n]}\{a_{1...1j...j},\ldots,a_{m...mj...j}\} = \beta_j.$$

*If for all $i \in [m], j \in [n]$,*

$$\alpha_i > R_i(\mathcal{A}),$$

*or*

$$\beta_j > C_j(\mathcal{A}),$$

*then $\mathcal{A}$ is positive definite.*

Based on the above theorem, we introduce the definition of strictly diagonally dominated (SDD) partially symmetric rectangular tensors.

**Definition 4** Let $\mathcal{A} \in \mathbb{R}^{[p;q;m;n]}$ be a partially symmetric rectangular tensor with positive diagonal entries, $p$ and $q$ even, and

$$\min_{i \in [m]}\{a_{i...i1...1},\ldots,a_{i...in...n}\} = \alpha_i, \qquad \min_{j \in [n]}\{a_{1...1j...j},\ldots,a_{m...mj...j}\} = \beta_j.$$

Then, the partially symmetric rectangular tensor $\mathcal{A}$ is called strictly diagonally dominated (SDD), if for all $i \in [m], j \in [n]$,

$$\alpha_i > R_i(\mathcal{A}),$$

or

$$\beta_j > C_j(\mathcal{A}).$$

*Example* 4 Let us consider the rectangular tensor $\mathcal{A} \in \mathbb{R}^{[2;4;2;2]}$ with

$$a_{111111} = a_{112222} = a_{221111} = a_{222222} = 1.1,$$

$$a_{121111} = a_{122222} = a_{211111} = a_{212222} = -1,$$

and other $a_{i_1 i_2 j_1 \ldots j_4} = 0$.

It can be verified that $\mathcal{A}$ satisfies all the conditions of Theorem 11, i.e., $\mathcal{A}$ is an SDD partially symmetric rectangular tensor. Hence, from Theorem 11, $\mathcal{A}$ is a positive definite rectangular tensor. In fact, the V-singular values of $\mathcal{A}$ are 2.1 and 0.1. From the definition of positive definite partially symmetric rectangular tensors, $\mathcal{A}$ is positive definite.

*Example* 5 Let us consider the rectangular tensor $\mathcal{A} \in \mathbb{R}^{[2;6;2;2]}$ with

$$a_{11111111} = a_{11222222} = a_{22111111} = a_{22222222} = 2,$$

$$a_{12111111} = a_{12222222} = a_{21111111} = a_{21222222} = 1.9,$$

and other $a_{i_1 i_2 j_1 \ldots j_6} = 0$.

It can be verified that $\mathcal{A}$ satisfies all the conditions of Theorem 11, i.e., $\mathcal{A}$ is an SDD partially symmetric rectangular tensor. Hence, from Theorem 11, $\mathcal{A}$ is a positive definite rectangular tensor. In fact, the V-singular values of $\mathcal{A}$ are 3.9 and 0.1. From the definition of positive definite partially symmetric rectangular tensors, $\mathcal{A}$ is positive definite.

## 6 Conclusions

In this paper, we introduce the definition of V-singular value for rectangular tensors and obtain some V-singular value inclusion sets for rectangular tensors with positive diagonal entries, an upper bound for the largest V-singular value of nonnegative rectangular tensors is also given. As applications, some sufficient conditions for the positive definiteness of partially symmetric rectangular tensors are obtained. These conditions only depend on the elements of partially symmetric rectangular tensors and thus can be easily verified. At last, we introduce the definition of SDD partially symmetric rectangular tensor and show that the SDD partially symmetric rectangular tensor is positive definite.

**Availability of data and materials**
The datasets generated during and/or analyzed during the current study are available from the corresponding author upon request.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
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