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Identities, inequalities for Boole-type polynomials: approach to generating functions and infinite series

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Abstract
The main purpose and motivation of this work is to investigate and provide some new identities, inequalities and relations for combinatorial numbers and polynomials, and for Peters type polynomials with the help of their generating functions. The results of this paper involve some special numbers and polynomials such as Stirling numbers, the Apostol–Euler numbers and polynomials, Peters polynomials, Boole polynomials, Changhee numbers and the other well-known combinatorial numbers and polynomials. Finally, in the light of Boole's inequality (Bonferroni's inequalities) and bounds of the Stirling numbers of the second kind, some inequalities for a combinatorial finite sum are derived. We mention an open problem including bounds for our numbers. Some remarks and observations are presented.

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1 Introduction

Some well-known notations and definitions are given first: $\mathbb{N} = \{1,2,3,\dots\}$, $\mathbb{N}_0 = \{0,1,2,3,\dots\}$. $\mathbb{Z},\mathbb{R},\mathbb{C}$ and $\mathbb{Z}_p$ denote the set of integers, the set of real numbers, the set of complex numbers, and the set of p-adic integers, respectively. $0^n = 1$ if $n = 0$ and $0^n = 0$ if $n \in \mathbb{N}$. For $v \in \mathbb{N}_0$ we have

$$(x)_v = x(x-1)\cdots(x-v+1),$$

$(x)_0 = 1$ and

$$\binom{x}{v} = \frac{x(x-1)\cdots(x-v+1)}{v!} = \frac{(x)_v}{v!}$$

(*cf.* [1–22] and the references cited therein).

The Apostol–Euler polynomials $\mathcal{E}_n(x,\lambda)$ are defined by

$$F_{p1}(t,x;\lambda) = \frac{2e^{tx}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x,\lambda)\frac{t^n}{n!}.$$

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When $x = 0$, the above equation reduces to the following Apostol–Euler numbers:

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n(0, \lambda).$$

Similarly when $\lambda = 1$, the above equation reduces to the following Euler numbers:

$$E_n = \mathcal{E}_n^{(1)}(1)$$

(*cf.* [1–20] and the references cited therein).

Let $k \in \mathbb{N}_0$. The Stirling numbers of the first kind, $S_1(n,k)$, are by

$$F_{S_1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{1}$$

and

$$(x)_k = \sum_{j=0}^{k} x^j S_1(k,j). \tag{2}$$

By the above generating function, we have

$$S_1(0,0) = 1.$$

The other properties are given as follows:

$S_1(0,k) = 0$ if $k > 0$. $S_1(n,0) = 0$ if $n > 0$ and $S_1(n,k) = 0$ if $k > n$ (*cf.* [1–23] and the references therein).

The Stirling numbers of the second kind, $S_2(n,k)$, are defined as follows:

$$F_S(t,k) = \frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{3}$$

By using (1), an explicit formula for the numbers $S_2(n,k)$ is given by

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

From the above equation, we also have

$$S_2(0,0) = 1.$$

If $k > n$, then

$$S_2(n,k) = 0.$$

$S_2(n,0) = 0$ if $n > 0$ and also

$$S_2(n+1,k) = S_2(n,k-1) + kS_2(n,k)$$

(*cf.* [1–23] and the references cited therein).

The Peters polynomials are defined by

$$F_P(t,x;\lambda,\mu)=\frac{(1+t)^x}{(1+(1+t)^\lambda)^\mu}=\sum_{n=0}^{\infty}s_n(x;\lambda,\mu)\frac{t^n}{n!}, \tag{4}$$

where $x,y\in\mathbb{C}$ (*cf.* [6, 13]).

We now give some special values of this polynomials as follows:

When $x=0$, we have the Peters numbers [15, 20]:

$$s_n(\lambda,\mu)=s_n(0;\lambda,\mu).$$

When $\mu=1$, we have the Boole polynomials [6, 13]:

$$\xi(x,\lambda)=s_n(x;\lambda,1).$$

If $\lambda=\mu=1$, we get the Changhee polynomials [7, 9]:

$$Ch_n(x)=2s_n(x;1,1).$$

The combinatorial numbers $Y_n(\lambda)$ and the combinatorial polynomials $Y_n(x;\lambda)$ are defined, respectively, by [16]

$$F(t,\lambda)=\frac{2}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}Y_n(\lambda)\frac{t^n}{n!}$$

and

$$F(t,x,\lambda)=\frac{2(1+\lambda t)^x}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}.$$

The motivation of this paper as regards generating functions for combinatorial numbers is related to the work of Simsek [16]. Let d be an odd integer. If χ is the Dirichlet character with conductor d, then we have the following equation [16]:

$$\int_{\mathbb{X}}\lambda^x(1+\lambda t)^x\chi(x)\,d\mu_{-q}(x)=\frac{[2]}{(\lambda q)^d(1+\lambda t)^d+1}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j(1+\lambda t)^j.$$

By using the above integral equation, the first author gave the generalized Apostol–Changhee numbers and polynomials by means of the following generating functions, respectively:

$$F_{\mathfrak{C}}(t;\lambda,q,\chi)=\frac{[2]_q\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j(1+\lambda t)^j}{(\lambda q)^d(1+\lambda t)^d+1}=\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(\lambda,q)\frac{t^n}{n!}, \tag{5}$$

where $[x]_q=\frac{1-q^x}{1-q}$, $\lim_{q\to 1}[x]_q=x$ and $\chi(x+d)=\chi(x)$.

Also

$$F_{\mathfrak{C}}(t,z;\lambda,q,\chi) = F_{\mathbb{C}}(t;\lambda,q,\chi)(1+\lambda t)^z = \sum_{n=0}^{\infty} \mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}.$$

We summarize our paper as follows:

In Sect. 2, we give generating functions for Peters type combinatorial numbers and polynomials. We investigate and provide some fundamental properties of these numbers and polynomials.

In Sect. 3, we derive some inequalities including Stirling's numbers of the second kind and finite combinatorial sums. We also mention an open problem for a Peters type combinatorial number.

2 Generating functions for Peters type combinatorial numbers and polynomials

In this section, by (5), we give a generating function for combinatorial numbers and polynomials and investigate properties for the function. By using these functions, we derive identities and relations.

From the above equation, we derive the following generating function:

$$K_d(t,x;\lambda,q) = \frac{[2]_q(1+\lambda t)^x}{(\lambda q)^d(1+\lambda t)^d+1} = \sum_{n=0}^{\infty} y_{7,n}(x;\lambda,q,d)\frac{t^n}{n!}. \tag{6}$$

If $x = 0$, then we get

$$K_d(t,0;\lambda,q) = \frac{[2]_q}{(\lambda q)^d(1+\lambda t)^d+1} = \sum_{n=0}^{\infty} y_{7,n}(\lambda,q,d)\frac{t^n}{n!}. \tag{7}$$

Remark 1 Substituting $d = 1$ into (6), we have

$$y_{7,n}(x;\lambda,q) = y_{7,n}(x;\lambda,q,1),$$

which was defined by the first author (*cf.* [17]).

Substituting $\mu = 1$ into (4) after combining with (6), we have

$$F_P(\lambda t,x;d,1) = \frac{(1+t)^x}{1+(1+t)^d} = \sum_{n=0}^{\infty} s_n(x;d,1)\frac{t^n}{n!}$$

and

$$K_d(t,x;\lambda,q) = \frac{[2]_q(1+\lambda t)^x}{(\lambda q)^d(1+\lambda t)^d+1} = \sum_{n=0}^{\infty} y_{7,n}(x;\lambda,q,d)\frac{t^n}{n!}.$$

Having $\lambda = \frac{1}{q}$ in (6), we have the following functional equation:

$$K_d\left(t,x;\frac{1}{q},q\right) = [2]_q F_P\left(\frac{1}{q}t,x;d,1\right).$$

By using the above equation, we have

$$y_{7,n}\left(x;\frac{1}{q},q,d\right)=[2]_q q^{-n}s_n(x;d,1).$$

Setting $d=1$ and $\lambda=\frac{1}{q}$ in (6), we have

$$\begin{aligned}K_1\left(t,x;\frac{1}{q},q\right)&=\frac{[2]_q(1+\frac{1}{q}t)^x}{(1+\frac{1}{q}t)+1}\\&=\frac{[2]_q(1+\frac{1}{q}t)^x}{\frac{1}{q}t+2}\\&=\sum_{n=0}^{\infty}\frac{[2]_q}{2}q^{-n}Ch_n(x)\frac{t^n}{n!}.\end{aligned}$$

Thus we have

$$y_{7,n}\left(x;\frac{1}{q},q,1\right)=\frac{[2]_q}{2}q^{-n}Ch_n(x).$$

2.1 Peters type combinatorial numbers $y_{7,n}(\lambda,q,d)$

Here, by (7), we derive some formulas, indentities and relations.

We assume $|1+\lambda q|<\frac{1}{|\lambda q|}$, by (7), we get

$$K_d(t,0,\lambda,q)=[2]_q\sum_{n=0}^{\infty}(-1)^n(\lambda q)^{nd}(1+\lambda t)^{nd}.$$

By using the above equation, we also assume that $|\lambda t|<1$, after that we have

$$\sum_{v=0}^{\infty}y_{7,v}(\lambda,q,d)\frac{t^v}{v!}=[2]_q\sum_{n=0}^{\infty}(-1)^n(\lambda q)^{nd}\sum_{v=0}^{\infty}\binom{nd}{v}\lambda^v t^v.$$

Comparing the coefficients of t^v on both sides of the above equation, we get the following theorem.

Theorem 1 *Let $|\lambda q|<1$. Then we have*

$$y_{7,v}(\lambda,q,d)=[2]_q\sum_{n=0}^{\infty}(-1)^n\binom{nd}{v}v!q^{nd}\lambda^{v+nd}.$$

By Remark 4 in [21], we have the following formula:

$$\binom{xy}{k}=\sum_{l,m=1}^{k}\binom{x}{l}\binom{y}{m}\left(\frac{l!m!}{k!}\right)\sum_{j=1}^{k}(-1)^{k-j}S_1(k,j)S_2(j,l)S_2(j,m).\tag{8}$$

Combining (8) with Theorem 1, we can obtain another version of the recurrence formula for the numbers $y_{7,n}(\lambda,q,d)$.

Theorem 2 *Let*

$$y_{7,0}(\lambda, q, d) = \frac{[2]_q}{1 + (\lambda q)^d}.$$

For $\nu \geq 1$, *we have*

$$y_{7,\nu}(\lambda, q, d) = [2]_q \lambda^\nu \sum_{l=1}^{\nu} \sum_{m=1}^{\nu} (-1)^m \binom{d}{l} \frac{l!m!(\lambda q)^{dm}}{(1 + \lambda^d q^d)^{m+1}} \times \sum_{j=0}^{\nu} (-1)^{\nu-j} S_1(\nu, j) S_2(j, l) S_2(j, m).$$

Proof Combining Theorem 1 and (8), we obtain

$$\begin{aligned} y_{7,\nu}(\lambda, q, d) &= [2]_q \lambda^\nu \nu! \sum_{n=0}^{\infty} (-1)^n (\lambda q)^{nd} \sum_{l=1}^{\nu} \sum_{m=1}^{\nu} \binom{n}{m} \binom{d}{l} \frac{l!m!}{\nu!} \\ &\quad \times \sum_{j=0}^{\nu} (-1)^{\nu-j} S_1(\nu, j) S_2(j, l) S_2(j, m) \\ &= [2]_q \lambda^\nu \nu! \sum_{l=1}^{\nu} \sum_{m=1}^{\nu} \binom{d}{l} \frac{l!m!}{\nu!} \sum_{j=0}^{\nu} (-1)^{\nu-j} S_1(\nu, j) S_2(j, l) S_2(j, m) \\ &\quad \times \sum_{n=0}^{\infty} (-1)^n \binom{n}{m} (\lambda^d q^d)^n \\ &= [2]_q \lambda^\nu \nu! \sum_{l=1}^{\nu} \sum_{m=1}^{\nu} \binom{d}{l} \frac{l!m!}{\nu!} \sum_{j=0}^{\nu} (-1)^{\nu-j} S_1(\nu, j) S_2(j, l) S_2(j, m) \\ &\quad \times (-1)^m \frac{(\lambda^d q^d)^m}{(1 + \lambda^d q^d)^{m+1}}. \end{aligned}$$

□

By using (7), we have

$$\sum_{n=0}^{\infty} y_{7,n}(\lambda, q, d) \frac{t^n}{n!} = \frac{[2]_q}{2} \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda^d q^d) \frac{d^n \ln(1 + \lambda t)^n}{n!}.$$

Combining the above equation with (1), we get

$$\sum_{n=0}^{\infty} y_{7,n}(\lambda, q, d) \frac{t^n}{n!} = \frac{[2]_q}{2} \sum_{m=0}^{\infty} \sum_{j=0}^{m} d^n \lambda^m \mathcal{E}_n(\lambda^d q^d) S_1(m, n) \frac{t^m}{m!}.$$

Since $S_1(m, n) = 0$ if $m > n$, by some calculation we derive the following theorem.

Theorem 3 *Let* $m \in \mathbb{N}_0$. *Then we have*

$$y_{7,m}(\lambda, q, d) = \frac{1 + q}{2} \sum_{j=0}^{m} d^n \lambda^m \mathcal{E}_n(\lambda^d q^d) S_1(m, n).$$

By (7), we obtain

$$[2]_q = \left((\lambda q)^d (1+\lambda t)^d + 1\right) \sum_{n=0}^{\infty} y_{7,n}(\lambda, q, d) \frac{t^n}{n!}.$$

From the above equation, we get

$$[2]_q = (\lambda q)^d \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (d)_j \lambda^j y_{7,n-j}(\lambda, q, d) \frac{t^n}{n!} + \sum_{n=0}^{\infty} y_{7,n}(\lambda, q, d) \frac{t^n}{n!}.$$

After some elementary calculations and comparing coefficients $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following recurrence formula:

Theorem 4 *Let*

$$y_{7,0}(\lambda, q, d) = \frac{1+q}{(\lambda q)^d + 1}.$$

For $n \geq 1$, we have

$$y_{7,n}(\lambda, q, d) = -(\lambda q)^d \sum_{j=0}^{n} \binom{n}{j} (d)_j \lambda^j y_{7,n-j}(\lambda, q, d). \tag{9}$$

By substituting (2) into (1), we derive the following corollary.

Corollary 1

$$y_{7,n}(\lambda, q, d) = -(\lambda q)^d \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} d^k \lambda^j S_1(j,k) y_{7,n-j}(\lambda, q, d).$$

2.2 Peters type combinatorial polynomials $y_{7,n}(x; \lambda, q, d)$

Here, by (6), we derive some formulas, identities and relations for the polynomials $y_{7,n}(x; \lambda, q, d)$ and the numbers $y_{7,n}(\lambda, q, d)$.

Theorem 5

$$y_{7,n}(x, \lambda, q, d) = \sum_{j=0}^{n} \binom{n}{j} (x)_j \lambda^j y_{7,n-j}(\lambda, q, d).$$

Proof Substituting (7) into (6) and assume $|\lambda t| < 1$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} y_{7,n}(x, \lambda, q, d) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \binom{x}{n} (\lambda t)^n \sum_{n=0}^{\infty} y_{7,n}(\lambda, q, d) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (x)_j \lambda^j y_{7,n-j}(\lambda, q, d) \frac{t^n}{n!}. \end{aligned}$$

□

Theorem 6 *Let χ be the Dirichlet character with conductor d. Then we have*

$$\mathfrak{Ch}_{n,\chi}(\lambda,q)=\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j y_{7,n}(j;\lambda,q,d).$$

Proof By combining (5) with (6), we get the following functional equation:

$$F_{\mathfrak{C}}(t;\lambda,q,\chi)=\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j K_d(t,j;\lambda,q).$$

From the above equation, we get

$$\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j y_{7,n}(j;\lambda,q,d)\frac{t^n}{n!}.$$

Comparing coefficients $\frac{t^n}{n!}$ on both sides of the above equation, we get the derived result. □

In [11], Kucukoglu and Simsek defined

$$\begin{aligned} F_d(t;\lambda,q)&=\frac{\ln(1+\lambda t)}{(\lambda q)^d(1+\lambda t)^d-1}=\sum_{n=0}^{\infty}I_{n,d}(\lambda,q)\frac{t^n}{n!},\\ G_d(t,x;\lambda,q)&=(1+\lambda t)^x F_d(t;\lambda,q)=\sum_{n=0}^{\infty}I_{n,d}(x;\lambda,q)\frac{t^n}{n!}. \end{aligned} \tag{10}$$

By combining (6) and the above equation, we get the following functional equation:

$$G_{2d}(t,2x;\lambda,q)=\frac{1}{[2]_q}G_d(t,x;\lambda,q)K_d(t,x;\lambda,q).$$

Using the above equation, we derive

$$\sum_{n=0}^{\infty}I_{n,2d}(2x;\lambda,q)\frac{t^n}{n!}=\frac{1}{[2]_q}\sum_{n=0}^{\infty}y_{7,n}(x;\lambda,q,d)\frac{t^n}{n!}\sum_{n=0}^{\infty}I_{n,d}(x;\lambda,q)\frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}I_{n,2d}(2x;\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{[2]_q}\sum_{j=0}^{n}\binom{n}{j}y_{7,j}(x;\lambda,q,d)I_{n-j,d}(x;\lambda,q)\frac{t^n}{n!}.$$

Comparing coefficients $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem.

Theorem 7

$$I_{n,2d}(2x;\lambda,q)=\frac{1}{[2]_q}\sum_{j=0}^{n}\binom{n}{j}y_{7,j}(x;\lambda,q,d)I_{n-j,d}(x;\lambda,q).$$

Theorem 8

$$(x)_n = \frac{(n)_d}{[2]_q} \sum_{j=0}^{n-d} \binom{n-d}{j} (d)_j \lambda^{d+j} y_{7,n+d-j}(x;\lambda,q,d) + y_{7,n}(x;\lambda,q,d),$$

where $(d)_j = d(d-1)\cdots(d-j+1)$.

Proof By definition of $y_{7,n}(x;\lambda,q,d)$, we have

$$[2]_q(1+\lambda t)^x = t^d \sum_{n=0}^{\infty} \frac{(d)_n}{n!} \lambda^{d+n} t^n \sum_{n=o}^{\infty} y_{7,n}(x;\lambda,q,d) \frac{t^n}{n!}.$$

After some calculation in the above equation, we get

$$\begin{aligned}
[2]_q \sum_{n=0}^{\infty} \lambda^n (x)_n \frac{t^n}{n!} &= t^d \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (d)_j \lambda^{d+j} y_{7,n-j}(x;\lambda,q,d) \frac{t^n}{n!} \\
&\quad + \sum_{n=o}^{\infty} y_{7,n}(x;\lambda,q,d) \frac{t^n}{n!} \\
&= \sum_{n=o}^{\infty} (n)_d \sum_{j=0}^{n-d} \binom{n-d}{j} (d)_j \lambda^{d+j} y_{7,n+d-j}(x;\lambda,q,d) \frac{t^n}{n!} \\
&\quad + \sum_{n=o}^{\infty} y_{7,n}(x;\lambda,q,d) \frac{t^n}{n!}.
\end{aligned}$$

□

3 Inequalities for Stirling numbers of the second kind and finite combinatorial sums

Bonferroni's inequalities, also known as Boole's inequality, are dependent on probability and also associated with the principle of inclusion and exclusion; for detailed information as regards these inequalities see the work of Comtet and Wagner [4, 23].

We mention an open question for bounds for $y_{7,n}(\lambda,q,d)$.

By using (3), we also have the following well-known explicit Stirling numbers of the second kind:

$$S^*(n,k;m) = \sum_{j=1}^{m} (-1)^{j-1} \binom{k}{j} (k-j)^n \tag{11}$$

for $n,m,k \in \mathbb{N}$.

By using Bonferroni's inequalities the above finite sum satisfies the following well-known inequalities:

If $m = k-1$, $S^*(n,k;m) = k^n - k!S_2(n,k)$.

If m is an odd integer and $m < k-1$ we have $S^*(n,k;m) > k^n - k!S_2(n,k)$.

If m is an even integer and $m < k-1$ we have $S^*(n,k;m) < k^n - k!S_2(n,k)$.

Wagner [23] gave proofs of the above results.

Lemma 1 *If* $m = k-1$, *we have*

$$S^*(n,k;k-1) \geq k^n \left(1 - \frac{k!}{n} \binom{n}{k} k^{1-k}\right). \tag{12}$$

Proof Upper bound for the $S_2(n,k)$ is given as follows [4]:

$$S_2(n,k) \le \binom{n-1}{k-1} k^{n-k}.$$

Since

$$S^*(n,k;k-1) = k^n - k! S_2(n,k)$$

if $m = k-1$, we obtain

$$S^*(n,k;k-1) \ge k^n k! \binom{n-1}{k-1} k^{n-k}.$$

After some elementary calculation, we complete proof of Lemma 1. □

Lemma 2 *If $m = k-1$, we have*

$$S^*(n,k;k-1) \le k^n \left(1 - \frac{k!}{k^k}\right). \tag{13}$$

Proof The lower bound for the $S_2(n,k)$ is given as follows:

$$S_2(n,k) \ge k^{n-k}$$

(*cf.* [4]).

Since

$$S^*(n,k;k-1) = k^n - k! S_2(n,k)$$

if $m = k-1$, we obtain

$$S^*(n,k;k-1) \le k^n - k! k^{n-k}.$$

After some elementary calculation, we complete proof of Lemma 2. □

Combining (12) and (13), we get bounds for $S^*(n,k;m)$ by the following theorem.

Theorem 9

$$k^n \left(1 - \frac{k!}{n} \binom{n}{k} k^{1-k}\right) \le S^*(n,k;k-1) \le k^n \left(1 - \frac{k!}{k^k}\right). \tag{14}$$

Open Problem. By using (12) and (13), is it possible to find bounds for $y_{7,n}(\lambda,q,d)$ with the help of Theorem 2?

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