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Optimality conditions for interval-valued univex programming

Lifeng Li[1,2*], Jianke Zhang[1,2] and Chang Zhou[1]

*Correspondence:
lilifeng80@163.com
[1] School of Science, Xi'an University of Posts and Telecommunications, Xi'an, China
[2] Shaanxi Key Laboratory of Network Data Analysis and Intelligent Processing, Xi'an University of Posts and Telecommunications, Xi'an, China

Abstract

We introduce concepts of interval-valued univex mappings, consider optimization conditions for interval-valued univex functions for the constrained interval-valued minimization problem, and show examples for the illustration purposes.

Keywords: Interval-valued univex mappings; Interval-valued univex programming; Optimality

1 Introduction

Convexity and generalized convexity are important in mathematical programming. Invex functions, introduced by Hanson [17], are important generalized convex functions and are successfully used in optimization and equilibrium problems. For example, necessary and sufficient conditions are obtained for K-invex functions in [14]. The concept of G-invex functions is introduced by Antczak [3]. Optimality and duality for differentiable G-multiobjective problems are considered in [4, 5]. Noor [26] considered invex equilibrium problems in the context of invexity. As an extension and refinement of Noor [26], Farajzadeh [15] gave some results for invex Ky Fan inequalities in topological vector spaces.

Another important type of generalized convex functions, called univex functions and preunivex functions, is introduced in [8]. Suppose $\emptyset \neq X \subseteq R^n$, $\eta : X \times X \to R^n$, $\Phi : R \to R$, and $b = b(x,y) : X \times X \to R^+$. A differentiable function $F : X \to R$ is said to be univex at $y \in X$ with respect to η, Φ, b if, for all $x \in X$,

$$b(x,y)\Phi\left[F(x) - F(y)\right] \geq \eta^t(x,y)\nabla F(y). \tag{1}$$

Later, some generalized optimality conditions of primal and dual problems were considered by Hanson and Mond [18]. Combing with generalized type I and univex functions, optimality conditions and duality for several mathematical programming problems were considered by many researchers [1, 16, 29], and more and more scholars pay attention to type I and univex functions [24, 25, 34, 35].

The authors of [2, 6, 9, 12, 27, 30, 33, 36–39] have studied generalized convex interval-valued mappings and their connection with interval-valued optimization. For example, Steuer [33] proposed three algorithms, called the F-cone algorithm, E-cone algorithm, and emanating algorithms, to solve the linear programming problems with interval-valued objective functions. To prove strong duality theorems, Wu [37] derived KKT optimality conditions in the interval-valued problems under convexity hypotheses. Wu [36] also obtained KKT conditions in an optimization problem with an interval-valued objective function using H-derivatives and the concept of weakly differentiable functions. Since the H-derivative suffers certain disadvantages, Chalco-Cano et al. [10] gave KKT-type optimality conditions, which were obtained using the gH-derivatives of interval-valued functions. Also, they studied the relationship between the approach presented with other known approaches given by Wu [36]. However, these methods cannot solve a kind of optimization problems with interval-valued objective functions that are not LU-convex but univex. Antczak [6] used the classical exact l_1 penalty function method for solving nondifferentiable interval-valued optimization problems under convexity hypotheses. Optimality conditions in invex optimization problems with an interval-valued objective function were discussed by Zhang et al. [39]. Using gH-differentiability, Li et al. [21] introduced interval-valued invex mappings and gave the optimality conditions for interval-valued objective functions under invexity. By using the weak derivative of fuzzy functions, Li et al. [22] defined fuzzy weakly univex functions and considered optimization conditions for fuzzy minimization problem.

Followed by [21] and [22], in this paper, we introduce the concept of interval-valued univex mappings, consider optimization conditions for interval-valued univex functions for the constrained interval-valued minimization problem, and show examples for illustration purposes. The present paper can be seen as promotion and expansion of [20]. The method presented in this paper is different from that in [6]. Our method cannot solve Example 3.1 of [6] because the objective function is not gH-differentiable. Example 4.1 shows that the methods given by [6, 33, 36, 37] cannot solve a kind of optimization problems for interval-valued univex mappings. Example 4.2 shows that the methods given by Li et al. [22] cannot solve a kind of fuzzy optimization problems for interval-valued univex mappings. Finally, Example 4.3 shows that the method given in [10] cannot solve a kind of optimization problems for interval-valued univex mappings. In Sect. 3, we introduce the concept of interval-valued univex mappings and discuss some their properties. Section 4 deals with optimality conditions for the constrained interval-valued minimization problem under the assumption of interval-valued univexity.

2 Preliminaries

In this paper, a closed interval in R is denoted by $A = [a^L, a^U]$. Every $a \in R$ is considered as a particular closed interval $a = [a, a]$. The set of closed intervals is denoted by $\mathcal{I}$.

Given $A = [a^L, a^U]$ and $B = [b^L, b^U] \in \mathcal{I}$, the arithmetic operations and order are defined in [32] as follows:

(1) $A + B = [a^L + b^L, a^U + b^U]$ and $-A = \{-a : a \in A\} = [-a^U, -a^L]$;

(2) $A \ominus_{gH} B = [\min(a^L - b^L, a^U - b^U), \max(a^L - b^L, a^U - b^U)]$;

(3) $A \preceq B \Leftrightarrow a^L \leq b^L$ and $a^U \leq b^U$; $A \prec B \Leftrightarrow A \preceq B$ and $A \neq B$.

For $X \subseteq R^n$, a mapping $F : X \to \mathcal{I}$ is called an interval-valued function. Then $F(x) = [F^L(x), F^U(x)]$, where $F^L(x)$ and $F^U(x)$ are two real-valued functions defined on R^n and satisfying $F^L(x) \leq F^U(x)$ for every $x \in X$. If $F^L(x)$ and $F^U(x)$ are continuous, then $F(x)$ is said to be continuous.

It is well known that the derivative and subderivative of a function is important in the study of generalized convexity and mathematical programming. For example, a classic subdifferential is introduced by Azimov and Gasimov [7]. Some theorems connecting operations on the weak subdifferential in the nonsmooth and nonconvex analysis are provided in [13]. The derivative and subderivative of interval-valued functions are extensions of real-valued functions. Due to different arithmetics of intervals, several definitions about derivatives of interval-valued functions are introduced by the authors, such as weakly differentiable functions [36], H-differentiable functions (based on the Hukuhara difference of two closed intervals [36]), gH-differentiable functions (based on the operation $\ominus_{gH}$ of two closed intervals [11, 31]), and subdifferentiable functions (based on the difference $A - B = [a^L - b^U, a^U - b^L]$ of two closed intervals [6]). In this paper, we always use weakly differentiable and gH-differentiable functions, which are defined as follows.

Let X be an open set in R^n, and let $F(x) = [F^L(x), F^U(x)]$. Then $F(x)$ is called weakly differentiable at x_0 if $F^L(x)$ and $F^U(x)$ are differentiable at x_0.

Let $x_0 \in (a,b)$ and h be such that $x_0 + h \in (a,b)$. Then

$$F'(x_0) = \lim_{x\to 0}\bigl[F(x_0 + h) \ominus_{gH} F(x_0)\bigr]. \tag{2}$$

If $F'(x_0) \in \mathcal{I}$ exists, then F is gH- differentiable at x_0.

If $F^L(x)$ and $F^U(x)$ are differentiable functions at $x \in (a,b)$, then $F(x)$ is gH-differentiable at x, and

$$F'(x) = \bigl[\min\bigl\{\bigl(F^L\bigr)'(x), \bigl(F^U\bigr)'(x)\bigr\}, \max\bigl\{\bigl(F^L\bigr)'(x), \bigl(F^U\bigr)'(x)\bigr\}\bigr]. \tag{3}$$

We say that an interval-valued function F is gH-differentiable at $x = (x_1, \ldots, x_n) \in X$ if all the partial gH-derivatives $(\frac{\partial F}{\partial x_1})(x), \ldots, (\frac{\partial F}{\partial x_n})(x)$ exist on some neighborhood of x and are continuous at x. We write

$$\nabla F(x) = \left(\left(\frac{\partial F}{\partial x_1}\right)(x), \left(\frac{\partial F}{\partial x_2}\right)(x), \ldots, \left(\frac{\partial F}{\partial x_n}\right)(x)\right)^t,$$

and we call $\nabla F(x)$ the gradient of a gH-differentiable interval-valued function F at x.

Let $\mathbb{H}(R^n)$ denote the family of nonempty compact subsets of R^n. For $A, B \in \mathbb{H}(R^n)$, the Hausdorff metric $h(A,B)$ on $\mathbb{H}(R^n)$ is defined by

$$h(A,B) = \inf\bigl\{\varepsilon \mid A \subseteq N(B,\varepsilon), B \subseteq N(A,\varepsilon)\bigr\},$$

where

$$N(A,\varepsilon) = \bigl\{x \in R^n \mid d(x,A) < \varepsilon\bigr\}, \qquad d(x,A) = \inf_{a\in A} \|x - a\|.$$

The following basic result (which can be found in Lemma 3.1. of [19]) of the mathematical analysis is well known:

Suppose that $\Phi : R^n \to R^n$ is continuous and let $X \in \mathbb{H}(R^n)$. Then the mapping

$$\Psi : \mathbb{H}\bigl(X^n\bigr) \to \mathbb{H}\bigl(R^n\bigr), \qquad \Psi(A) = \bigl\{\phi(a) \mid a \in A\bigr\}$$

is uniformly continuous in h-metric.

We say that $\Psi : \mathcal{I} \to \mathcal{I}$ is increasing if $A \preceq B$ implies $\Psi(A) \preceq \Psi(B)$. From the above result we can prove the following result:

If function $\Phi : R \to R$ is increasing, then $\Psi : \mathcal{I} \to \mathcal{I}$ is increasing. Moreover, $\Psi([a^L, a^U]) = [\Phi(a^L), \Phi(a^U)]$.

3 Interval-valued univex functions

In this section, we define interval-valued univex functions as a generalization of univex functions [8] and discuss some their properties.

Let X be an invex set in R^n (the concept of an invex set can be found in [8]), and let F be an interval-valued function. The following definition is a particular case of fuzzy weakly univex functions, which has been introduced in [22].

Suppose F is a weakly differentiable interval-valued function. Then F is weakly univex at $y \in X$ with respect to η, Φ, b if and only if both $F^L(x)$ and $F^U(x)$ are univex at $y \in X$, that is, for all $x \in X$,

$$b(x,y)\Phi\bigl[F^L(x) - F^L(y)\bigr] \geq \eta^t(x,y)\nabla F^L(y), \tag{4}$$

$$b(x,y)\Phi\bigl[F^U(x) - F^U(y)\bigr] \geq \eta^t(x,y)\nabla F^U(y), \tag{5}$$

where $\eta = \eta(x,y) : X \times X \to R^n$, $\Phi : R \to R$, and $b = b(x,y) : X \times X \times [0,1] \to R^+$.

Remark 3.1 The concept of LU-invexity for interval-valued functions is introduced in [39], since it considers the endpoint functions; in this paper, we call them weakly invex. Every interval-valued weakly invex function is interval-valued weakly univex with respect to η, b, Φ, where

$$\Phi(x) = x, \qquad b = 1,$$

but the converse is not true.

Example 3.1 Consider the function $F : (-\infty, 0) \to \mathcal{I}$ defined by

$$F(x) = [1,2]x^3,$$

$$\eta(x,y) = \begin{cases} x^2 + xy + y^2, & x > y, \\ x - y, & x \leq y, \end{cases}$$

$$b(x,y) = \begin{cases} \frac{y^2}{x-y}, & x > y, \\ 0, & x \leq y. \end{cases}$$

Let $\Phi : R \to R$ be defined by $\Phi(V) = 3V$, $F^L(x) = 2x^3$, and $F^U(x) = x^3$; then $\nabla F^L(x) = 6x^2$ and $\nabla F^U(x) = 3x^2$. Then F is interval-valued weakly univex but not interval-valued weakly invex, since for $x = -2$ and $y = -1$, $F^U(x) - F^U(y) < \eta^t(x,y)\nabla F^U(y)$.

Let X be a nonempty open set in R^n, $\eta : X \times X \to R^n$, $\Psi : \mathcal{I} \to \mathcal{I}$, and $b = b(x,y) : X \times X \to R^+$.

Definition 3.1 Suppose F is a gH-differentiable interval-valued function. Then F is univex at $y \in X$ with respect to η, Ψ, b if for all $x \in X$,

$$b(x,y)\Psi\bigl[F(x) \ominus_{gH} F(y)\bigr] \succeq \eta^t(x,y)\nabla F(y). \tag{6}$$

The following example shows that an interval-valued univex function may not be an interval-valued weakly univex function.

Example 3.2 Suppose $F(x) = [-|x|, |x|]$, $x \in R$, $b = 1$, and $\Phi(a) = a$. Then $\Psi[a,b] = [a,b]$ is induced by $\Phi(a) = a$, and

$$\eta(x,y) = \begin{cases} x - y, & xy \geq 0, \\ x + y, & xy < 0. \end{cases}$$

Then $F(x)$ is gH-differentiable on R, and $F'(y) = [-1, 1]$. We can prove that

$$b\Psi\bigl[F(x) \ominus_{gH} F(y)\bigr] \succeq \eta^t(x,y)\nabla F(y).$$

Therefore $F(x)$ is univex with respect to η, b, Ψ, but $F(x)$ is not weakly univex since $F^L(x)$ is not univex with respect to η, b, Φ.

Theorem 3.1 *Suppose $F(x)$ is gH-differentiable. If $F(x)$ is an interval-valued weakly univex function with respect to η, b, Φ and Φ is increasing, then $F(x)$ is an interval-valued univex function with respect to the same η, b, and Ψ, where Ψ is an extension of Φ.*

Proof Since $F(x)$ is weakly univex at y, then real-valued functions F^L and F^U are univex at y, that is,

$$b(x,y)\Phi\bigl[F^L(x) - F^L(y)\bigr] \geq \eta^t(x,y)\nabla F^L(y) \quad \text{and}$$
$$b(x,y)\Phi\bigl[F^U(x) - F^U(y)\bigr] \geq \eta^t(x,y)\nabla F^U(y)$$

for all $x \in X$.

(i) Under the condition $\eta^t(x,y)\nabla F^L(y) \leq \eta^t(x,y)\nabla F^U(y)$, we have

$$\eta^t(x,y)\nabla F(y) = \bigl[\eta^t(x,y)\nabla F^L(y), \eta^t(x,y)\nabla F^U(y)\bigr].$$

If $F(x) \ominus_{gH} F(y) = [F^L(x) - F^L(y), F^U(x) - F^U(y)]$, then since Φ is increasing, we have

$$\begin{aligned} &b(x,y)\Psi\bigl[F(x) \ominus_{gH} F(y)\bigr] \\ &\quad = \bigl[b(x,y)\Phi\bigl(F^L(x) - F^L(y)\bigr), b(x,y)\Phi\bigl(F^U(x) - F^U(y)\bigr)\bigr] \\ &\quad \succeq \bigl[\eta^t(x,y)\nabla F^L(y), \eta^t(x,y)\nabla F^U(y)\bigr] \\ &\quad = \eta^t(x,y)\nabla F(y). \end{aligned}$$

If $F(x) \ominus_{gH} F(y) = [F^U(x) - F^U(y), F^L(x) - F^L(y)]$, then

$$\begin{aligned} & b(x,y)\Phi\bigl(F^L(x) - F^L(y)\bigr) \\ & \quad \succeq b(x,y)\Phi\bigl(F^U(x) - F^U(y)\bigr) \\ & \quad \succeq \eta^t(x,y)\nabla F^U(y) \\ & \quad \succeq \eta^t(x,y)\nabla F^L(y), \end{aligned}$$

and since Φ is increasing, we have

$$\begin{aligned} & b(x,y)\Psi\bigl[F(x) \ominus_{gH} F(y)\bigr] \\ & \quad = [b(x,y)\Psi\bigl[F^U(x) - F^U(y), F^L(x) - F^L(y)\bigr] \\ & \quad = \bigl[b(x,y)\Phi\bigl(F^U(x) - F^U(y)\bigr), b(x,y)\Phi\bigl(F^L(x) - F^L(y)\bigr)\bigr] \\ & \quad \succeq \bigl[\eta^t(x,y)\nabla F^L(y), \eta^t(x,y)\nabla F^U(y)\bigr] \\ & \quad = \eta^t(x,y)\nabla F(y). \end{aligned}$$

(ii) Under the condition $\eta^t(x,y)\nabla F^L(y) > \eta^t(x,y)\nabla F^U(y)$, we have

$$\eta^t(x,y)\nabla F(y) = \bigl[\eta^t(x,y)\nabla F^U(y), \eta^t(x,y)\nabla F^L(y)\bigr].$$

If $F(x) \ominus_{gH} F(y) = [F^U(x) - F^U(y), F^L(x) - F^L(y)]$, then since Φ is increasing, we have

$$\begin{aligned} & b(x,y)\Psi\bigl[F(x) \ominus_{gH} F(y)\bigr] \\ & \quad = \bigl[b(x,y)\Phi\bigl(F^U(x) - F^U(y)\bigr), b(x,y)\Phi\bigl(F^L(x) - F^L(y)\bigr)\bigr] \\ & \quad \succeq \bigl[\eta^t(x,y)\nabla F^U(y), \eta^t(x,y)\nabla F^L(y)\bigr] \\ & \quad = \eta^t(x,y)\nabla F(y). \end{aligned}$$

If $F(x) \ominus_{gH} F(y) = [F^L(x) - F^L(y), F^U(x) - F^U(y)]$, then

$$\begin{aligned} & b(x,y)\Phi\bigl(F^U(x) - F^U(y)\bigr) \\ & \quad \succeq b(x,y)\Phi\bigl(F^L(x) - F^L(y)\bigr) \\ & \quad \succeq \eta^t(x,y)\nabla F^L(y) \\ & \quad \succeq \eta^t(x,y)\nabla F^U(y). \end{aligned}$$

Since Φ is increasing, we have

$$\begin{aligned} & b(x,y)\Psi\bigl[F(x) \ominus_{gH} F(y)\bigr] \\ & \quad = b(x,y)\Psi\bigl[F^L(x) - F^L(y), F^U(x) - F^U(y)\bigr] \\ & \quad = \bigl[b(x,y)\Phi\bigl(F^L(x) - F^L(y)\bigr), b(x,y)\Phi\bigl(F^U(x) - F^U(y)\bigr)\bigr] \\ & \quad \succeq \bigl[\eta^t(x,y)\nabla F^U(y), \eta^t(x,y)\nabla F^L(y)\bigr] \\ & \quad = \eta^t(x,y)\nabla F(y). \end{aligned}$$

□

Remark 3.2 If Φ is nonincreasing, then Theorem 3.1 may not be true (as shown in the following Example 3.3).

Example 3.3 Suppose $F(x) = [-2,1]x^2$, $x < 0$. Then $F(x)$ is gH-differentiable and weakly differentiable. It is easy to check that $F(x)$ is weakly univex with respect to $\eta(x,y) = x - y$,

$$b(x,y) = \begin{cases} 1, & x \leq y < 0, \\ \frac{-2y(x-y)}{-x^2+y^2}, & y < x < 0, \end{cases}$$

and $\Phi(a) = |a|$. However, $F(x)$ is not univex with respect to the same $\eta(x,y)$, b, and Ψ, where Ψ is defined by the extension of $\Phi(a) = |a|$.

4 Optimality criteria for interval-valued univex mappings

In this section, for gH-differentiable interval-valued univex functions, we establish sufficient optimality conditions for a feasible solution x^* to be an optimal solution or a nondominated solution for (P).

Suppose $F(x), g_1(x), \ldots, g_m(x)$ are gH-differentiable interval-valued mappings defined on a nonempty open set $X \subseteq R^n$. Then, we consider the primal problem:

$$(P) \quad \min F(x)$$
$$\text{s.t.} \quad g(x) \preceq 0.$$

Let $P := \{x \in X : g(x) \preceq 0\}$ denote the feasible set of (P).

Since $\preceq$ is a partial order, the optimal solution may not exist for some interval-valued optimization problems. Therefore, authors always consider the concept of a nondominated solution in this situation. We reconsider an optimal solution and nondominated solution as follows.

Definition 4.1

(i) $x^* \in P$ is an optimal solution of (P) $\Leftrightarrow$ $F(x^*) \preceq F(x)$ for all $x \in P$. In this case, $F(x^*)$ is called the optimal objective value of F.

(ii) $x^* \in P$ is a nondominated solution of (P) $\Leftrightarrow$ there exists no $x_0 \in P$ such that $F(x_0) \prec F(x^*)$. In this case, $F(x^*)$ is called the nondominated objective value of F.

Theorem 4.1 *Let x^* be P-feasible. Suppose that*:

(i) *there exist η, Ψ_0, b_0, Ψ_i, b_i, $i = 1,2,\ldots,m$, such that*

$$b_0(x,y)\Psi_0\bigl[F(x) \ominus_{gH} F\bigl(x^*\bigr)\bigr] \succeq \eta^t\bigl(x,x^*\bigr)\nabla F\bigl(x^*\bigr) \tag{7}$$

and

$$-b_i\bigl(x,x^*\bigr)\Psi_i\bigl[g_i\bigl(x^*\bigr)\bigr] \succeq \eta^t\bigl(x,x^*\bigr)\nabla g_i\bigl(x^*\bigr) \tag{8}$$

for all feasible x;

(ii) *there exists* $y^* \in R^m$ *such that*

$$\nabla F(x^*) = -y^{*t} \nabla g(x^*), \tag{9}$$

$$y^* \succeq 0. \tag{10}$$

Further suppose that

$$\Psi_0(\mu) \succeq 0 \quad \Rightarrow \quad \mu \succeq 0, \tag{11}$$

$$\mu \preceq 0 \quad \Rightarrow \quad \Psi_i(\mu) \succeq 0, \tag{12}$$

and

$$b_0(x, x^*) > 0, \qquad b_i(x, x^*) \geq 0, \tag{13}$$

for all feasible x. Then x^* *is an optimal solution of* (P).

Proof Let x be P-feasible. Then

$$g(x) \preceq 0.$$

This, along with (12), yields

$$\Psi_i\big[g_i(x)\big] \succeq 0.$$

From (7)–(13) it follows that

$$\begin{aligned} b_0(x, x^*)\Psi_0\big[F(x) \ominus_{gH} F(x^*)\big] &\succeq \eta^t(x, x^*) \nabla F(x^*) \\ &= -\eta^t(x, x^*) \sum_{i=1}^m y_i \nabla g_i(x^*) \\ &\succeq \sum_{i=1}^m b_i(x, x^*) y_i \Psi_i\big[g_i(x^*)\big] \\ &\succeq 0. \end{aligned}$$

From (13) it follows that

$$\Psi_0\big[F(x) \ominus_{gH} F(x^*)\big] \succeq 0.$$

By (11) we have

$$F(x) \ominus_{gH} F(x^*) \succeq 0.$$

Thus

$$F(x) \succeq F(x^*).$$

Therefore x^* is an optimal solution of (P). □

Remark 4.1 If we change the condition

$$\Psi_0(\mu) \succeq 0 \quad \Rightarrow \quad \mu \succeq 0$$

of Theorem 4.1 by

$$\Psi_0(\mu) \not\prec 0 \quad \Rightarrow \quad \mu \not\prec 0, \tag{14}$$

then x^* is a nondominated solution of (P).

In Theorem 18 of [20], the authors also gave a sufficient optimality condition for a feasible solution x^* to be an optimal solution. In this theorem, the equation

$$\nabla F\bigl(x^*\bigr) + y^{*t}\nabla g\bigl(x^*\bigr) = 0$$

was used, substituted for (9) of Theorem 4.1. We can prove that the previous equation is very restrictive. In fact, in case $F(x)$ is a unary function, suppose $\nabla F(x^*) = [a,b]$ and $y^{*t}\nabla g(x^*) = [yc, yd]$. Then we have $[a,b] + [yc,yd] = [a+yc, b+yd] = 0$, where $a \leq b$ and $yc \leq yd$. Therefore we have $a = b$ and $c = d$ since $y \geq 0$. That is to say, $\nabla F(x^*)$ is a real number instead of an interval. In the following example, we can observe that x^* is an optimal solution of (P), but x^* do not satisfies the previous equation. The following example also shows the advantages of our method over [6, 33, 36, 37].

Example 4.1

$$\begin{aligned}
&\min F(x) = \left[\frac{1}{2}, \frac{3}{2}\right]\sin^2 x_1 + \left[\frac{1}{2}, \frac{3}{2}\right]\sin^2 x_2 \\
&\textit{s.t.}\quad g(x) = \left[\frac{1}{2}, \frac{3}{2}\right](\sin x_1 - 1)^2 + \left[\frac{1}{2}, \frac{3}{2}\right](\sin x_2 - 1)^2 \preceq \frac{1}{4}\left[\frac{1}{2}, \frac{3}{2}\right], \\
&\qquad x_1, x_2 \in \left(0, \frac{\pi}{2}\right).
\end{aligned}$$

We can observe that $F(x)$ is weakly differentiable, H-differentiable, and gH-differentiable. Since the interval-valued function $F(x)$ is not convex, the method in [6, 33, 36, 37] cannot be used.

The function $F(x)$ is interval-valued univex with respect to

$$\eta(x,y) = \begin{cases} (\frac{\sin x_1 - \sin y_1}{\cos y_1}, \frac{\sin x_2 - \sin y_2}{\cos y_2})^t, & (x_1,x_2) \geq (y_1,y_2), \\ 0 & \text{otherwise}, \end{cases}$$

$$b_0(x,y) = \begin{cases} 1, & (x_1,x_2) \geq (y_1,y_2), \\ 0 & \text{otherwise}, \end{cases}$$

and Ψ is induced by $\Phi(a) = 2a$, $b_1(x,y) = b_0(x,y)$, and Ψ_1 is induced by $\Phi_1(a) = |a|$, where $x = (x_1,x_2)^t$ and $y = (y_1,y_2)^t$. The point $x^* = (\sin^{-1}(1 - \frac{1}{2\sqrt{2}}), \sin^{-1}(1 - \frac{1}{2\sqrt{2}}))^t$ is a feasible solution. We can also see that (F,g) satisfies the hypotheses of Theorem 4.1. Therefore $x^* = (\sin^{-1}(1 - \frac{1}{2\sqrt{2}}), \sin^{-1}(1 - \frac{1}{2\sqrt{2}}))^t$ is an optimal solution.

Theorem 4.2 *Let x^* be P-feasible. Suppose that:*

(i) *there exist η, Ψ_0, b_0, Ψ_i, b_i, $i = 1, 2, \dots, m$, such that*

$$b_0(x,y)\Psi_0\big[F(x) \ominus_{gH} F\big(x^*\big)\big] \succeq \eta^t\big(x,x^*\big)\nabla F\big(x^*\big) \tag{15}$$

and

$$-b_i\big(x,x^*\big)\Psi_i\big[g_i\big(x^*\big)\big] \succeq \eta^t\big(x,x^*\big)\nabla g_i\big(x^*\big) \tag{16}$$

for all feasible x;

(ii) *there exists $y^* \in R^m$ such that*

$$\big\{\nabla F\big(x^*\big)\big\}^L = \big\{-y^{*t}\nabla g\big(x^*\big)\big\}^L, \tag{17}$$

$$y^* \geq 0. \tag{18}$$

Further, suppose that

$$\Psi_0(\mu) \not\prec 0 \quad \Rightarrow \quad \mu \not\prec 0, \tag{19}$$

$$\mu \preceq 0 \quad \Rightarrow \quad \Psi_i(\mu) \succeq 0, \tag{20}$$

and

$$b_0\big(x,x^*\big) > 0, \qquad b_i\big(x,x^*\big) \geq 0 \tag{21}$$

for all feasible x. Then x^* is a nondominated solution of (P).

Proof Let x be P-feasible. Then

$$\widetilde{g}(x) \preceq 0.$$

From (20) we conclude

$$\Psi_i\big[g_i(x)\big] \succeq 0.$$

From (15), (16) it follows that

$$b_0(x,y)\big\{\Psi_0\big[F(x) \ominus_{gH} F\big(x^*\big)\big]\big\}^L \geq \big\{\eta^t\big(x,x^*\big)\nabla F\big(x^*\big)\big\}^L,$$
$$b_0(x,y)\big\{\Psi_0\big[F(x) \ominus_{g} F\big(x^*\big)\big]\big\}^U \geq \big\{\eta^t\big(x,x^*\big)\nabla F\big(x^*\big)\big\}^U,$$

and

$$b_i\big(x,x^*\big)\big\{\Psi_i\big[g_i\big(x^*\big)\big]\big\}^L \leq \big\{-\eta^t\big(x,x^*\big)\nabla g_i\big(x^*\big)\big\}^L,$$
$$b_i\big(x,x^*\big)\big\{\Psi_i\big[g_i\big(x^*\big)\big]\big\}^U \leq \big\{-\eta^t\big(x,x^*\big)\nabla g_i\big(x^*\big)\big\}^U.$$

Since

$$\eta^t(x,x^*)\nabla F(x^*) = \eta^t(x,x^*)\big[\{\nabla F(x^*)\}^L, \{\nabla F(x^*)\}^U\big]$$
$$= \begin{cases} [\eta^t(x,x^*)\{\nabla F(x^*)\}^L, \eta^t(x,x^*)\{\nabla F(x^*)\}^U], & \eta^t(x,x^*) \geq 0, \\ [\eta^t(x,x^*)\{\nabla F(x^*)\}^U, \eta^t(x,x^*)\{\nabla F(x^*)\}^L], & \eta^t(x,x^*) < 0, \end{cases}$$

and

$$-\eta^t(x,x^*)\nabla g_i(x^*) = -\eta^t(x,x^*)\big[\{\nabla g_i(x^*)\}^L, \{\nabla g_i(x^*)\}^U\big]$$
$$= \begin{cases} [\eta^t(x,x^*)\{-\nabla g_i(x^*)\}^U, \eta^t(x,x^*)\{-\nabla g_i(x^*)\}^L], & \eta^t(x,x^*) \geq 0, \\ [\eta^t(x,x^*)\{-\nabla g_i(x^*)\}^L, \eta^t(x,x^*)\{-\nabla g_i(x^*)\}^U], & \eta^t(x,x^*) < 0, \end{cases}$$

we consider the following two cases.

Case (i)

$$\{\eta^t(x,x^*)\nabla F(x^*)\}^L = \eta^t(x,x^*)\{\nabla F(x^*)\}^L$$

and

$$\{-\eta^t(x,x^*)\nabla g_i(x^*)\}^L = \eta^t(x,x^*)\{-\nabla g_i(x^*)\}^U$$

yield

$$\{\eta^t(x,x^*)\nabla F(x^*)\}^U = \eta^t(x,x^*)\{\nabla F(x^*)\}^U$$

and

$$\{-\eta^t(x,x^*)\nabla g_i(x^*)\}^U = \eta^t(x,x^*)\{-\nabla g_i(x^*)\}^L.$$

Thus

$$\begin{aligned} b_0(x,y)\{\Psi_0[F(x) \ominus_{gH} F(x^*)]\}^L &\geq \{\eta^t(x,x^*)\nabla F(x^*)\}^L \\ &= \eta^t(x,x^*)\{\nabla F(x^*)\}^L \\ &= \eta^t(x,x^*)\{-y^{*t}\nabla g(x^*)\}^L \\ &\geq \sum_{i=1}^m b_i(x,x^*)y_i\{\Psi_i[g_i(x^*)]\}^L \\ &\geq 0. \end{aligned}$$

From (21) it follows that

$$\Psi_0[F(x) \ominus_{gH} F(x^*)] \succeq 0.$$

Then

$$F(x) \ominus_{gH} F(x^*) \nprec 0,$$

and thus

$$F(x) \nprec F(x^*).$$

Therefore x^* is a nondominated solution of (P).

Case (ii)

$$\{\eta^t(x,x^*)\nabla F(x^*)\}^L = \eta^t(x,x^*)\{\nabla F(x^*)\}^U$$

and

$$\{-\eta^t(x,x^*)\nabla g_i(x^*)\}^L = \eta^t(x,x^*)\{-\nabla g_i(x^*)\}^L$$

yield

$$\{\eta^t(x,x^*)\nabla F(x^*)\}^U = \eta^t(x,x^*)\{\nabla F(x^*)\}^L$$

and

$$\{-\eta^t(x,x^*)\nabla g_i(x^*)\}^U = \eta^t(x,x^*)\{-\nabla g_i(x^*)\}^U.$$

Thus

$$\begin{aligned} b_0(x,y)\{\Psi_0[F(x) \ominus_{gH} F(x^*)]\}^U &\geq \{\eta^t(x,x^*)\nabla F(x^*)\}^U \\ &= \eta^t(x,x^*)\{\nabla F(x^*)\}^L \\ &= \eta^t(x,x^*)\{-y^{*t}\nabla g(x^*)\}^L \\ &\geq \sum_{i=1}^m b_i(x,x^*)y_i\{\Psi_i[g_i(x^*)]\}^L \\ &\geq 0, \end{aligned}$$

From (21) it follows that

$$\Psi_0[F(x) \ominus_{gH} F(x^*)] \nprec 0.$$

Then

$$F(x) \ominus_{gH} F(x^*) \nprec 0,$$

and thus

$$F(x) \nprec F(x^*).$$

Therefore x^* is a nondominated solution of (P). □

Theorem 4.3 *Let x^* be P-feasible. Suppose that*:

(i) *there exist η, Ψ_0, b_0, Ψ_i, b_i, $i = 1, 2, \ldots, m$, such that*

$$b_0(x,y)\Psi_0\bigl[F(x) \ominus_{gH} F(x^*)\bigr] \succeq \eta^t(x,x^*)\nabla F(x^*) \tag{22}$$

and

$$-b_i(x,x^*)\Psi_i\bigl[g_i(x^*)\bigr] \succeq \eta^t(x,x^*)\nabla g_i(x^*) \tag{23}$$

for all feasible x;

(ii) *there exists $y^* \in R^m$ such that*

$$\bigl\{\nabla F(x^*)\bigr\}^U = \bigl\{-y^{*t}\nabla g(x^*)\bigr\}^U, \tag{24}$$

$$y^* \geq 0. \tag{25}$$

Further, suppose that

$$\Psi_0(\mu) \not\prec 0 \quad \Rightarrow \quad \mu \not\prec 0, \tag{26}$$

$$\mu \preceq 0 \quad \Rightarrow \quad \Psi_i(\mu) \succeq 0, \tag{27}$$

and

$$b_0(x,x^*) > 0, \qquad b_i(x,x^*) \geq 0 \tag{28}$$

for all feasible x. Then x^* is a nondominated solution of (P).

The following example shows the advantages of our method over [22].

Example 4.2

$$\min F(x) = [-1,1]|x|$$

$$s.t. \quad g(x) = x - 1 \leq 0.$$

Since $F^L(x) = -|x|$ and $F^U(x) = |x|$ is not differentiable at $x = 0$, $F(x)$ is not weakly differentiable at $x = 0$. Therefore the method in [22] cannot be used.

Note that the objective function $F(x)$ is gH-differentiable on R and that $F'(y) = [-1,1]$. Let

$$b_0(x,y) = \begin{cases} 1, & x < y < 0 \text{ or } 0 < x < y, \\ 0 & \text{otherwise.} \end{cases}$$

the function $\Psi_0[a,b] = [a,b]$ is induced by $\Phi_0(a) = a$, and

$$\eta(x,y) = \begin{cases} x - y, & x < y < 0 \text{ or } 0 < x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $b_1 = 1$, and let Ψ_1 be induced by $\Phi_1(a) = |a|$. The point $x^* = 1$ is a feasible solution. We can see that (F, g) satisfies the hypotheses of Theorem 4.2. Therefore $x^* = 1$ is a nondominated solution.

The following example also shows the advantages of our method over [10] and [23, 28].

Example 4.3

$$\min F(x) = [-2, 1]x^2, \quad x < 0,$$
$$s.t. \quad g(x) = x + 1 \leq 0.$$

Then $F(x)$ is gH-differentiable and weakly differentiable. Since $F(x)$ is not LU-convex, the methods of [10] cannot be used, and since $F^L(x) + F^U(x) = -x^2$ is not convex, the methods of [23, 28] cannot be used.

Let

$$b_0(x,y) = \begin{cases} 1, & x \leq y < 0, \\ \frac{-2y(x-y)}{-x^2+y^2}, & y < x < 0, \end{cases} \quad \text{and} \quad \Psi_0[a,b] = \begin{cases} [a,b], & [a,b] \preceq 0, \\ \Psi([a,b]), & [a,b] \npreceq 0, \end{cases}$$

where $\Psi([a,b])$ induced by $\Phi(a) = |a|$, and

$$\eta(x,y) = \begin{cases} x - y, & x < y < 0 \text{ or } 0 < x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $b_1(x,y) = 1$ and $\Phi_1(a) = \Phi(a) = |a|$. The point $x^* = -1$ is a feasible solution. We can see that (F, g) satisfies the hypotheses of Theorem 4.3, and therefore $x^* = -1$ is a nondominated solution.

5 Conclusion

The objective of this paper is to introduce the concept of gH-differentiable interval-valued univex mappings and discuss the relationship between interval-valued univex mappings and interval-valued weakly univex mappings. We derive sufficient optimality conditions for constrained interval-valued minimization problem under interval-valued univex mappings. In future work, we hope to give sufficient optimality conditions for a nondifferentiable interval-valued optimization problem under univexity hypotheses.

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Authors' contributions
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