Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On complete moment convergence for arrays of rowwise pairwise negatively quadrant dependent random variables

Meimei Ge[1], Zexing Dai[1] and Yongfeng Wu[1,2*]

*Correspondence: wyfwyf@126.com
[1]School of Mathematics and Finance, Chuzhou University, Chuzhou, China
[2]College of Mathematics and Computer Science, Tongling University, Tongling, China

Abstract

In this paper, we establish some results on the complete moment convergence for weighted sums of pairwise negatively quadrant dependent (PNQD) random variables. The obtained results improve the corresponding ones of Ko (Stoch. Int. J. Probab. Stoch. Process. 85:172–180, 2013).

MSC: 60F15

Keywords: Pairwise negatively quadrant dependent; Complete moment convergence; Weighted sums

1 Introduction

The following concept of PNQD random variables was introduced by Lehmann [2].

Definition 1.1 A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise negatively quadrant dependent (PNQD) if for any r_i, r_j and $i \neq j$,

$$P(X_i > r_i, X_j > r_j) \leq P(X_i > r_i)P(X_j > r_j).$$

Negative quadrant dependence is shown to be a stronger notion of dependence than negative correlation but weaker than negative association. The convergence properties of NQD random sequences have been studied in many papers. We refer to Wu [3] for Kolmogorov-type three-series theorem, Matula [4] for the Kolmogorov-type strong law of large numbers, Jabbari [5] for the almost sure limit theorems for weighted sums of pairwise NQD random variables under some fragile conditions, Li and Yang [6], Wu [7], and Xu and Tang [8] for strong convergence, Gan and Chen [9] for complete convergence and complete moment convergence, Wu and Guan [10] for a mean convergence theorem and weak laws of large numbers for dependent random variables, and so on.

The concept of complete convergence of a sequence of random variables was first given by Hsu and Robbins [11].

Definition 1.2 A sequence of random variables $\{U_n, n \in N\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\bigl(|U_n - a| > \varepsilon\bigr) < \infty.$$

By the Borel–Cantelli lemma this result implies that $U_n \to a$ almost surely. Therefore, the complete convergence is a very important tool in establishing the almost sure convergence of sums of random variables and weighted sums of random variables.

Recently, Ko [1] proved the following complete convergence theorem for arrays of PNQD random variables.

Theorem A *Let* $\{X_{nj}, 1 \le j \le b_n, n \ge 1\}$ *be an array of rowwise and PNQD random variables with mean zero, and let* $\{a_{nj}, j \ge 1, n \ge 1\}$ *be an array of positive numbers. Let* $\{b_n, n \ge 1\}$ *be a nondecreasing sequence of positive numbers. Assume that, for some* $0 < t < 2$ *and all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} P\bigl(|a_{nj} X_{nj}| \ge \varepsilon b_n^{\frac{1}{t}}\bigr) < \infty \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} a_{nj}^2 E(X_{nj})^2 I\bigl(|a_{nj} X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\bigr) < \infty. \tag{1.2}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left\{ \max_{1 \le k \le b_n} \left| \sum_{j=1}^{k} \bigl(a_{nj} X_{nj} - a_{nj} E X_{nj} I\bigl[|a_{nj} X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\bigr]\bigr) \right| \ge \varepsilon b_n^{\frac{1}{t}} \right\} < \infty. \tag{1.3}$$

Chow [12] was the first who showed the complete moment convergence for a sequence of independent and identically distributed random variables by generalizing the result of Baum and Katz [13]. The concept of complete moment convergence is as follows.

Definition 1.3 Let $\{Z_n, n \ge 1\}$ be a sequence of random variables, and let $a_n > 0$, $b_n > 0$, and $q > 0$. If for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n E\bigl\{b_n^{-1} |Z_n| - \varepsilon\bigr\}_+^q < \infty,$$

then this is called the complete moment convergence.

It is easily seen that complete moment convergence is stronger than complete convergence. There are many papers on complete moment convergence; see, for example, Sung [14] for independent random variables, Wang and Hu [15] for the maximal partial sums of a martingale difference sequence, Shen et al. [16] for arrays of rowwise negatively superadditive dependent (NSD) random variables. Wu et al. [17] for arrays of rowwise END random variables, Wu [7] for negatively associated random variables, Wu et al. [18] for weighted sums of weakly dependent random variables, Wang et al. [19] for double indexed randomly weighted sums and its applications, Wu and Wang [20] for a class of dependent random variables, and so forth.

In this work, we improve Theorem A from complete convergence to complete moment convergence for PNQD random variables under some stronger conditions. In addition, we obtain some much stronger conclusions under the same conditions of the corresponding theorems in Ko [1].

Throughout this paper, the symbol C always stands for a generic positive constant which may differ from one place to another. By $I(A)$ we denote the indicator function of a set A. We also denote $x_+ = xI(x \geq 0)$.

2 Main results

Now we state the main results of this paper. The proofs are given in next section.

Theorem 2.1 *Let $\{X_{nj}, 1 \leq j \leq b_n, n \geq 1\}$ be an array of rowwise and PNQD random variables with mean zero, and let $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive numbers. Let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Assume that, for some $0 < t < 2$ and all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} |a_{nj}| E|X_{nj}| I\big(|a_{nj}X_{nj}| \geq \varepsilon b_n^{\frac{1}{t}}\big) < \infty \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} a_{nj}^2 E(X_{nj})^2 I\big(|a_{nj}X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\big) < \infty. \tag{2.2}$$

Then, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E\left\{ b_n^{-\frac{1}{t}} \max_{1 \leq k \leq b_n} \left| \sum_{j=1}^{k} \big(a_{nj}X_{nj} - a_{nj}EX_{nj}I\big[|a_{nj}X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\big]\big) \right| - \varepsilon \right\}_+ < \infty. \tag{2.3}$$

Remark 2.1 Let $H_{nj} = \sum_{j=1}^{k}(a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \varepsilon b_n^{\frac{1}{t}}])$. Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n E\left\{ b_n^{-\frac{1}{t}} \max_{1 \leq k \leq b_n} |H_{nj}| - \varepsilon \right\}_+ \\
&\quad = \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\Big(b_n^{-\frac{1}{t}} \max_{1 \leq j \leq b_n} |H_{nj}| > \varepsilon + u \Big)\, du \\
&\quad \geq \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon} P\Big(\max_{1 \leq j \leq b_n} |H_{nj}| > (\varepsilon + u) b_n^{-\frac{1}{t}} \Big)\, du \\
&\quad \geq \sum_{n=1}^{\infty} c_n P\Big(\max_{1 \leq j \leq b_n} |H_{nj}| > 2\varepsilon b_n^{-\frac{1}{t}} \Big).
\end{aligned}$$

Thus (2.3) is much stronger than (1.3).

Theorem 2.2 *Let $\{X_{nj}, 1 \le j \le b_n, n \ge 1\}$ be an array of rowwise and PNQD random variables with mean zero, and let $\{a_{nj}, j \ge 1, n \ge 1\}$ be an array of positive numbers. Let $\{b_n, n \ge 1\}$ be a nondecreasing sequence of positive numbers. Assume that, for some sequence $\{\lambda_n, n \ge 1\}$ with $0 < \lambda_n \le 1$, we have $E|X_{nj}|^{1+\lambda_n} < \infty$ for $1 \le j \le b_n$, $n \ge 1$. If for some sequence $\{c_n, n \ge 1\}$ of positive real numbers and $0 < t < 2$,*

$$\sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \bigl(b_n^{\frac{1}{t}}\bigr)^{-1-\lambda_n} \sum_{j=1}^{b_n} E|a_{nj}X_{nj}|^{1+\lambda_n} < \infty, \tag{2.4}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E\left\{ b_n^{-\frac{1}{t}} \max_{1\le k\le b_n} \left| \sum_{j=1}^{k} a_{nj}X_{nj} \right| - \varepsilon \right\}_+ < \infty. \tag{2.5}$$

Remark 2.2 Noting that the conditions of Theorem 2.2 are the same as in Theorem 3.2 in Ko [1], we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n E\left\{ b_n^{-\frac{1}{t}} \max_{1\le k\le b_n} \left| \sum_{j=1}^{k} a_{nj}X_{nj} \right| - \varepsilon \right\}_+ \\
&\quad = \int_0^{\infty} P\left(b_n^{-\frac{1}{t}} \max_{1\le k\le b_n} \left| \sum_{j=1}^{k} a_{nj}X_{nj} \right| > \varepsilon + u \right) du \\
&\quad \ge \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon} P\left(\max_{1\le k\le b_n} \left| \sum_{j=1}^{k} a_{nj}X_{nj} \right| > (\varepsilon + u) b_n^{-\frac{1}{t}} \right) du \\
&\quad \ge \sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le b_n} \left| \sum_{j=1}^{k} a_{nj}X_{nj} \right| > 2\varepsilon b_n^{-\frac{1}{t}} \right).
\end{aligned}$$

Therefore (2.5) is much stronger than (3.8) of Theorem 3.2 in Ko [1]. To sum up, Theorem 2.2 improves Theorem 3.2 in Ko [1].

Corollary 2.3 *Let $\{X_{nj}, 1 \le j \le b_n, n \ge 1\}$ be an array of rowwise PNQD random variables, and let $\{a_{nj}, j \ge 1, n \ge 1\}$ be an array of positive numbers. Let $h(x) > 0$ be a slowly varying function as $x \to \infty$, and let $\alpha > \frac{1}{2}$ and $\alpha r \ge 1$. Suppose that, for $0 < t < 2$, the following conditions hold for any $\varepsilon > 0$:*

$$\sum_{n=1}^{\infty} n^{\alpha r - 2 - \frac{1}{t}} (\log_2 n)^2 h(n) \sum_{j=1}^{n} |a_{nj}| E|X_{nj}| I\bigl(|a_{nj}X_{nj}| \ge \varepsilon n^{\frac{1}{t}}\bigr) < \infty \tag{2.6}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha r - 2 - \frac{2}{t}} (\log_2 n)^2 h(n) \sum_{j=1}^{n} a_{nj}^2 E(X_{nj})^2 I\bigl[|a_{nj}X_{nj}| < \varepsilon n^{\frac{1}{t}}\bigr] < \infty. \tag{2.7}$$

Then, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha r-2}h(n)E\left\{n^{-\frac{1}{t}}\max_{1\le k\le n}\left|\sum_{j=1}^{k}\bigl(a_{nj}X_{nj}-a_{nj}EX_{nj}I\bigl[|a_{nj}X_{nj}|<\varepsilon n^{\frac{1}{t}}\bigr]\bigr)\right|-\varepsilon\right\}_{+}<\infty. \tag{2.8}$$

Theorem 2.4 *Let* $\{X_{nj}, j \ge 1, n \ge 1\}$ *be an array of rowwise identically distributed PNQD random variables with* $EX_{11} = 0$, *and let* $h(x) > 0$ *be a slowly varying function as* $x \to \infty$. *If* $E|X_{11}|^{(\alpha r+2)t}h(|X_{11}|^t) < \infty$ *for* $\alpha > \frac{1}{2}$, $\alpha r \ge 1$, *and* $0 < t < 2$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha r-2}h(n)E\left\{n^{-\frac{1}{t}}\max_{1\le k\le b_n}\left|\sum_{j=1}^{k}X_{nj}\right|-\varepsilon\right\}_{+}<\infty. \tag{2.9}$$

Remark 2.3 Noting that the conditions of Theorem 2.2 are the same as in Theorem 3.4 in Ko [1], we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha r-2}h(n)E\left\{n^{-\frac{1}{t}}\max_{1\le k\le b_n}\left|\sum_{j=1}^{k}X_{nj}\right|-\varepsilon\right\}_{+}\\
&\quad=\sum_{n=1}^{\infty} n^{\alpha r-2}h(n)\int_0^{\infty}P\left(n^{-\frac{1}{t}}\max_{1\le k\le b_n}\left|\sum_{j=1}^{k}X_{nj}\right|>\varepsilon+u\right)du\\
&\quad\ge\sum_{n=1}^{\infty} n^{\alpha r-2}h(n)\int_0^{\varepsilon}P\left(\max_{1\le k\le b_n}\left|\sum_{j=1}^{k}X_{nj}\right|>(\varepsilon+u)n^{\frac{1}{t}}\right)du\\
&\quad\ge\sum_{n=1}^{\infty} n^{\alpha r-2}h(n)P\left(\max_{1\le k\le b_n}\left|\sum_{j=1}^{k}X_{nj}\right|>2\varepsilon n^{\frac{1}{t}}\right).
\end{aligned}$$

Therefore (2.9) is much stronger than (3.13) of Theorem 3.4 in Ko [1]. Theorem 2.4 improves Theorem 3.4 in Ko [1].

Corollary 2.5 *Let* $\{X_{nj}, 1 \le j \le b_n, n \ge 1\}$ *be an array of rowwise and PNQD random variables with mean zero, and let* $\{a_{nj}, j \ge 1, n \ge 1\}$ *be an array of positive numbers. Let* $\{b_n, n \ge 1\}$ *be a nondecreasing sequence of positive numbers, and let* $\{c_n, n \ge 1\}$ *be a sequence of positive numbers. Assume that, for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n(\log_2 b_n)^2\sum_{j=1}^{b_n}|a_{nj}|E|X_{nj}|I\bigl(|a_{nj}X_{nj}|\ge\varepsilon\log_2 b_n\bigr)<\infty \tag{2.10}$$

and

$$\sum_{n=1}^{\infty} c_n\sum_{j=1}^{b_n}a_{nj}^2E(X_{nj})^2I\bigl[|a_{nj}X_{nj}|<\varepsilon\log_2 b_n\bigr]<\infty. \tag{2.11}$$

Then, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_nE\left\{(\log_2 b_n)^{-1}\max_{1\le k\le b_n}\left|\sum_{j=1}^{k}\bigl(a_{nj}X_{nj}-a_{nj}EX_{nj}I\bigl[|a_{nj}X_{nj}|<\varepsilon\log_2 b_n\bigr]\bigr)\right|-\varepsilon\right\}_{+}$$

$$< \infty. \tag{2.12}$$

Corollary 2.6 *Let $\{X_{nj}, 1 \le j \le b_n, n \ge 1\}$ be an array of rowwise and PNQD random variables with mean zero and finite variances. Let $\{a_{nj}, j \ge 1, n \ge 1\}$ be an array of positive numbers satisfying*

$$\sum_{j=1}^{n} a_{nj}^2 E(X_{nj})^2 = O\bigl(n^\delta\bigr) \quad \textit{as } n \to \infty \tag{2.13}$$

for some $0 < \delta < 1$. Then, for all $\varepsilon > 0$ and $\alpha > 0$,

$$\sum_{n=1}^{\infty} n^{2(\alpha-1)} E\left\{ n^{-\alpha} (\log_2 n)^{-1} \max_{1 \le k \le b_n} \left| \sum_{j=1}^{k} a_{nj} X_{nj} \right| - \varepsilon \right\}_+ < \infty. \tag{2.14}$$

Remark 2.4 Note that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{2(\alpha-1)} E\left\{ n^{-\alpha} (\log_2 n)^{-1} \max_{1 \le k \le b_n} \left| \sum_{j=1}^{k} a_{nj} X_{nj} \right| - \varepsilon \right\}_+ \\ &\quad = \sum_{n=1}^{\infty} n^{2(\alpha-1)} \int_0^{\infty} P\left(n^{-\alpha} (\log_2 n)^{-1} \max_{1 \le k \le b_n} \left| \sum_{j=1}^{k} a_{nj} X_{nj} \right| > \varepsilon + u \right) du \\ &\quad \ge \sum_{n=1}^{\infty} n^{2(\alpha-1)} \int_0^{\varepsilon} P\left(n^{-\alpha} (\log_2 n)^{-1} \max_{1 \le k \le b_n} \left| \sum_{j=1}^{k} a_{nj} X_{nj} \right| > (\varepsilon + u) n^{\alpha} \log_2 n \right) du \\ &\quad \ge \sum_{n=1}^{\infty} n^{2(\alpha-1)} P\left(\max_{1 \le k \le b_n} \left| \sum_{j=1}^{k} a_{nj} X_{nj} \right| > 2\varepsilon n^{\alpha} \log_2 n \right). \end{aligned}$$

Therefore (2.14) is much stronger than (3.18) of Corollary 3.6 in Ko [1].

3 The proofs

To prove our results, we need some lemmas. The first one is the basic property for PNQD random variables, which can be referred to Lehmann [2].

Lemma 3.1 *Let $\{X_n, n \ge 1\}$ be a sequence of PNQD random variables, and let $\{f_n, n \ge 1\}$ be a sequence of nondecreasing functions. Then $\{f_n(X_n), n \ge 1\}$ is still a sequence of PNQD random variables.*

The next lemma comes from Wu [3] and plays an essential role to prove the result of the paper.

Lemma 3.2 *Let $\{X_n, n \ge 1\}$ be a sequence of PNQD random variables with mean zero and finite second moments. Then*

$$E \max_{1 \le k \le n} \left| \sum_{j=1}^{k} X_j \right|^2 \le C(\log_2 n)^2 \sum_{j=1}^{n} EX_j^2. \tag{3.1}$$

A positive measurable function $h(x)$ on $[a,\infty)$ for some $a>0$ is said to be slowly varying as $x\to\infty$ if

$$\lim_{x\to\infty}\frac{h(\lambda x)}{h(x)}=1 \quad \text{for each } \lambda>0. \tag{3.2}$$

The last lemma can be found in Wu [21].

Lemma 3.3 *If $h(x)>0$ is a slowly varying function as $x\to\infty$, then*
(i) $\lim_{x\to\infty}\sup_{2^k\le x<2^{k+1}} h(x)/h(2^k)=1$, *and*
(ii) $c_1 2^k h(\varepsilon 2^k)\le\sum_{j=1}^k 2^j h(\varepsilon 2^j)\le c_2 2^k h(\varepsilon 2^k)$
for all $r>0$, $\varepsilon>0$, and positive integers k and some positive constants c_1 and c_2.

Proof of Theorem 2.1 Let $S_j=\sum_{j=1}^k(a_{nj}X_{nj}-a_{nj}EX_{nj}I[|a_{nj}X_{nj}|<\varepsilon b_n^{\frac{1}{t}}])$. For any fixed $\varepsilon>0$,

$$\begin{aligned}
&\sum_{n=1}^\infty c_n E\Big\{b_n^{-\frac{1}{t}}\max_{1\le j\le b_n}|S_j|-\varepsilon\Big\}_+\\
&\quad=\sum_{n=1}^\infty c_n\int_0^\infty P\Big(b_n^{-\frac{1}{t}}\max_{1\le j\le b_n}|S_j|-\varepsilon>u\Big)\,du\\
&\quad\le\varepsilon\sum_{n=1}^\infty c_n P\Big(\max_{1\le j\le b_n}|S_j|>\varepsilon b_n^{\frac{1}{t}}\Big)+\sum_{n=1}^\infty c_n\int_\varepsilon^\infty P\Big(\max_{1\le j\le b_n}|S_j|>ub_n^{\frac{1}{t}}\Big)\,du\\
&\quad=:I_1+I_2.
\end{aligned}$$

Obviously, we have $I_1<\infty$ by Theorem A. Hence we need only to prove $I_2<\infty$. Clearly,

$$\begin{aligned}
&P\Big(\max_{1\le j\le b_n}|S_j|>ub_n^{\frac{1}{t}}\Big)\\
&\quad=P\Bigg(\max_{1\le j\le b_n}|S_j|>ub_n^{\frac{1}{t}},\bigcup_{j=1}^{b_n}\big\{|a_{nj}X_{nj}|\ge ub_n^{\frac{1}{t}}\big\}\Bigg)\\
&\qquad+P\Bigg(\max_{1\le j\le b_n}|S_j|>ub_n^{\frac{1}{t}},\bigcap_{j=1}^{b_n}\big\{|a_{nj}X_{nj}|<ub_n^{\frac{1}{t}}\big\}\Bigg)\\
&\quad\le\sum_{j=1}^{b_n}P\big(|a_{nj}X_{nj}|\ge ub_n^{\frac{1}{t}}\big)\\
&\qquad+P\Bigg(\max_{1\le k\le b_n}\Bigg|\sum_{j=1}^k\big(a_{nj}X_{nj}I\big(|a_{nj}X_{nj}|<\varepsilon b_n^{\frac{1}{t}}\big)-a_{nj}EX_{nj}I\big(|a_{nj}X_{nj}|<\varepsilon b_n^{\frac{1}{t}}\big)\big)\Bigg|>ub_n^{\frac{1}{t}}\Bigg).
\end{aligned}$$

Then we can get

$$\begin{aligned}
I_2&\le\sum_{n=1}^\infty c_n\sum_{j=1}^{b_n}\int_\varepsilon^\infty P\big(|a_{nj}X_{nj}|>ub_n^{\frac{1}{t}}\big)\,du\\
&\quad+\sum_{n=1}^\infty c_n\int_\varepsilon^\infty P\Bigg(\max_{1\le k\le b_n}\Bigg|\sum_{j=1}^k\big(a_{nj}X_{nj}I\big(|a_{nj}X_{nj}|<ub_n^{\frac{1}{t}}\big)
\end{aligned}$$

$$- a_{nj}EX_{nj}I\bigl(|a_{nj}X_{nj}| < ub_n^{\frac{1}{t}}\bigr)\Big| > ub_n^{\frac{1}{t}}\Big)$$

$$=: I_3 + I_4.$$

Firstly, we will prove that $I_3 < \infty$. Noting that

$$\int_\varepsilon^\infty P\bigl(|a_{nj}X_{nj}| \geq ub_n^{\frac{1}{t}}\bigr)\,du \leq b_n^{-\frac{1}{t}} E|a_{nj}X_{nj}|I\bigl(|a_{nj}X_{nj}| \geq \varepsilon b_n^{\frac{1}{t}}\bigr),$$

by (2.1) we have

$$I_3 \leq \sum_{n=1}^\infty c_n \sum_{j=1}^{b_n} \int_\varepsilon^\infty P\bigl(|a_{nj}X_{nj}| \geq ub_n^{\frac{1}{t}}\bigr)\,du$$

$$\leq \sum_{n=1}^\infty c_n b_n^{-\frac{1}{t}} \sum_{j=1}^{b_n} E|a_{nj}X_{nj}|I\bigl(|a_{nj}X_{nj}| \geq \varepsilon b_n^{\frac{1}{t}}\bigr) < \infty.$$

To prove that $I_4 < \infty$, let

$$Y_{nk} = -ub_n^{\frac{1}{t}} I\bigl(a_{nj}X_{nj} < -ub_n^{\frac{1}{t}}\bigr) + a_{nj}X_{nj}I\bigl(|a_{nj}X_{nj}| \leq ub_n^{\frac{1}{t}}\bigr) + ub_n^{\frac{1}{t}} I\bigl(a_{nj}X_{nj} > ub_n^{\frac{1}{t}}\bigr),$$

$$Z_{nk} = -ub_n^{\frac{1}{t}} I\bigl(a_{nj}X_{nj} < -ub_n^{\frac{1}{\mu}}\bigr) + ub_n^{\frac{1}{t}} I\bigl(a_{nj}X_{nj} > ub_n^{\frac{1}{t}}\bigr).$$

We have

$$P\left(\max_{1\leq k\leq b_n}\left|\sum_{j=1}^k \bigl(a_{nj}X_{nj}I\bigl(|a_{nj}X_{nj}| \leq \varepsilon b_n^{\frac{1}{t}}\bigr) - a_{nj}EX_{nj}I\bigl(|a_{nj}X_{nj}| \leq \varepsilon b_n^{\frac{1}{t}}\bigr)\bigr)\right| > ub_n^{\frac{1}{t}}\right)$$

$$= P\left(\max_{1\leq k\leq b_n}\left|\sum_{j=1}^k (Y_{nk} - EY_{nk} - Z_{nk} + EZ_{nk})\right| > ub_n^{\frac{1}{t}}\right).$$

Then we have

$$I_4 \leq \sum_{n=1}^\infty c_n \int_\varepsilon^\infty P\left(\max_{1\leq k\leq b_n}\left|\sum_{j=1}^k (Z_{nk} - EZ_{nk})\right| > \frac{ub_n^{\frac{1}{t}}}{2}\right)du$$

$$+ \sum_{n=1}^\infty c_n \int_\varepsilon^\infty P\left(\max_{1\leq k\leq b_n}\left|\sum_{j=1}^k (Y_{nk} - EY_{nk})\right| > \frac{ub_n^{\frac{1}{t}}}{2}\right)du$$

$$=: I_5 + I_6.$$

For I_5, by the Markov inequality and (2.1) we have

$$I_5 \leq C\sum_{n=1}^\infty c_n \sum_{j=1}^{b_n} \int_\varepsilon^\infty u^{-1}E|Z_{nk}|\,du$$

$$\leq C\sum_{n=1}^\infty c_n b_n^{\frac{1}{t}} \sum_{j=1}^{b_n} \int_\varepsilon^\infty P\bigl(|a_{nj}X_{nj}| > ub_n^{\frac{1}{t}}\bigr)\,du$$

$$\begin{aligned}
&\leq C\sum_{n=1}^{\infty} c_n b_n^{\frac{1}{t}} \sum_{j=1}^{b_n} b_n^{-\frac{1}{t}} E|a_{nj}X_{nj}| I\big(|a_{nj}X_{nj}| > \varepsilon b_n^{\frac{1}{t}}\big) \\
&\leq C\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} E|a_{nj}X_{nj}| I\big(|a_{nj}X_{nj}| > u b_n^{\frac{1}{t}}\big) < \infty.
\end{aligned}$$

Now consider I_6. By the Markov inequality and Lemma 3.2 we have

$$\begin{aligned}
I_6 &\leq C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} \int_{\varepsilon}^{\infty} u^{-2} E\left(\max_{1\leq k\leq b_n}\left|\sum_{j=1}^{k}(Y_{nk} - EY_{nk})\right|\right)^2 du \\
&\leq C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \int_{\varepsilon}^{\infty} u^{-2} E|Y_{nk}|^2 \, du \\
&= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \int_{\varepsilon}^{\infty} u^{-2} |a_{nj}|^2 E|X_{nj}|^2 I\big(|a_{nj}X_{nj}| \leq u b_n^{\frac{1}{t}}\big) \\
&\quad + C\sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} \int_{\varepsilon}^{\infty} P\big(|a_{nj}X_{nj}| > u b_n^{\frac{1}{t}}\big)\, du \\
&= I_7 + I_8.
\end{aligned}$$

Firstly, we will prove that $I_8 < \infty$. By (2.1) we have

$$\begin{aligned}
I_8 &\leq C\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} |a_{nj}| E|X_{nj}| I\big(|a_{nj}X_{nj}| > \varepsilon b_n^{\frac{1}{t}}\big) \\
&< \infty.
\end{aligned}$$

Next, consider $I_7 < \infty$. We have

$$\begin{aligned}
I_7 &= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \int_{\varepsilon}^{\infty} u^{-2} |a_{nj}|^2 E|X_{nj}|^2 I\big(|a_{nj}X_{nj}| \leq \varepsilon b_n^{\frac{1}{t}}\big) \\
&\quad + C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \int_{\varepsilon}^{\infty} u^{-2} |a_{nj}|^2 E|X_{nj}|^2 I\big(\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \leq u b_n^{\frac{1}{t}}\big) \\
&=: I_7' + I_7''.
\end{aligned}$$

By (2.2) it is easy to see that

$$\begin{aligned}
I_7' &\leq C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} |a_{nj}|^2 E|X_{nj}|^2 I\big(|a_{nj}X_{nj}| \leq \varepsilon b_n^{\frac{1}{t}}\big) \int_{\varepsilon}^{\infty} u^{-2}\, du \\
&< \infty.
\end{aligned}$$

By the Markov inequality and (2.1) we have

$$\begin{aligned}
I_7'' &= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{m=1}^{\infty} \int_{m\varepsilon}^{(m+1)\varepsilon} u^{-2} E|a_{nj}X_{nj}|^2 I\bigl(\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le u b_n^{\frac{1}{t}}\bigr)\, du \\
&\le C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{m=1}^{\infty} m^{-2} E|a_{nj}X_{nj}|^2 I\bigl(\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (m+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \\
&= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{m=1}^{\infty} m^{-2} \sum_{s=1}^{m} E|a_{nj}X_{nj}|^2 I\bigl(s\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (s+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \\
&= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{s=1}^{\infty} E|a_{nj}X_{nj}|^2 I\bigl(s\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (s+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \sum_{m=s}^{\infty} m^{-2} \\
&\le C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{s=1}^{\infty} s^{-1} E|a_{nj}X_{nj}|^2 I\bigl(s\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (s+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \\
&= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{s=1}^{\infty} s^{-1} (s+1)^2 \varepsilon^2 b_n^{\frac{2}{t}} \\
&\quad \times E\left|\frac{a_{nj}X_{nj}}{(s+1)\varepsilon b_n^{\frac{1}{t}}}\right|^2 I\bigl(s\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (s+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \\
&\le C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{s=1}^{\infty} s^{-1} (s+1)^2 b_n^{\frac{2}{t}} \\
&\quad \times E\left|\frac{a_{nj}X_{nj}}{(s+1)\varepsilon b_n^{\frac{1}{t}}}\right| I\bigl(s\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (s+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \\
&= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{s=1}^{\infty} s^{-1} (s+1) b_n^{\frac{1}{t}} E|a_{nj}X_{nj}| I\bigl(s\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (s+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \\
&\le C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} \sum_{s=1}^{\infty} E|a_{nj}X_{nj}| I\bigl(s\varepsilon b_n^{\frac{1}{t}} < |a_{nj}X_{nj}| \le (s+1)\varepsilon b_n^{\frac{1}{t}}\bigr) \\
&= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} b_n^{\frac{1}{t}} E|a_{nj}X_{nj}| I\bigl(|a_{nj}X_{nj}| > \varepsilon b_n^{\frac{1}{t}}\bigr) \\
&= C\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} (\log_2 b_n)^2 \sum_{j=1}^{n} E|a_{nj}X_{nj}| I\bigl(|a_{nj}X_{nj}| > \varepsilon b_n^{\frac{1}{t}}\bigr) < \infty.
\end{aligned}$$

This completes the proof of the theorem. □

Proof of Theorem 2.2 We estimate

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} |a_{nj}| E|X_{nj}| I\bigl(|a_{nj}X_{nj}| \ge \varepsilon b_n^{\frac{1}{t}}\bigr)$$

$$= \varepsilon \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} E\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| I\left(\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| \geq 1\right)$$
$$\leq \varepsilon \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} E\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right|^{1+\lambda_n} I\left(\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| \geq 1\right)$$
$$\leq C \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \left(b_n^{\frac{1}{t}}\right)^{-1-\lambda_n} \sum_{j=1}^{b_n} E|a_{nj}X_{nj}|^{1+\lambda_n} < \infty$$

and

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} a_{nj}^2 E(X_{nj})^2 I\left[|a_{nj}X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\right]$$
$$= \varepsilon^2 \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 E\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right|^2 I\left(\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| < 1\right)$$
$$\leq C \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} E\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right|^{1+\lambda_n} I\left(\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| < 1\right)$$
$$\leq C \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \left(b_n^{\frac{1}{t}}\right)^{-1-\lambda_n} \sum_{j=1}^{b_n} E|a_{nj}X_{nj}|^{1+\lambda_n} < \infty.$$

Hence conditions (2.1) and (2.2) of Theorem 2.1 are satisfied. Since $EX_{nj} = 0$, we get

$$b_n^{-\frac{1}{t}} \max_{1\leq k\leq b_n} \left|\sum_{j=1}^{k} E a_{nj}X_{nj} I\left[|a_{nj}X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\right]\right|$$
$$\leq b_n^{-\frac{1}{t}} \max_{1\leq k\leq b_n} \sum_{j=1}^{k} |a_{nj}| E|X_{nj}| I\left[|a_{nj}X_{nj}| \geq \varepsilon b_n^{\frac{1}{t}}\right]$$
$$\leq \left(b_n^{-\frac{1}{t}}\right)^{-1-\lambda_n} \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} \to 0 \quad \text{as } n \to \infty,$$

and thus (2.5) is completed. □

Proof of Corollary 2.3 Let $c_n = n^{\alpha r-2}h(n)$ and $b_n = n$. Then, by Theorem 2.1, (2.8) is completed. □

Proof of Theorem 2.4 For $a_{nj} = 1, j \geq 1, n \geq 1$, by Lemma 3.3 we have

$$\sum_{n=1}^{\infty} n^{\alpha r-1} (\log_2 n)^2 h(n) E|X_{11}| I\left(|X_{11}| \geq \varepsilon n^{\frac{1}{t}}\right)$$
$$\leq C \sum_{n=1}^{\infty} \left(2^k\right)^{\alpha r} k^2 h\left(2^k\right) E|X_{11}| I\left(|X_{11}| \geq \varepsilon \left(2^k\right)^{\frac{1}{t}}\right)$$
$$\leq C \sum_{k=1}^{\infty} \left(2^k\right)^{\alpha r+2} h\left(2^k\right) E|X_{11}| I\left(|X_{11}| \geq \varepsilon \left(2^k\right)^{\frac{1}{t}}\right)$$

$$
\begin{aligned}
&\le C\sum_{m=1}^{\infty} E|X_{11}|I\bigl[\varepsilon\bigl(2^m\bigr)^{\frac{1}{t}} \le |X_{11}| < \bigl(2^{m+1}\bigr)^{\frac{1}{t}}\bigr]\sum_{j=1}^{m}\bigl(2^j\bigr)^{\alpha r+2}h\bigl(2^j\bigr) \\
&\le C\sum_{m=1}^{\infty}\bigl(2^m\bigr)^{\alpha r+2}h\bigl(2^m\bigr)E|X_{11}|I\bigl[\varepsilon\bigl(2^m\bigr)^{\frac{1}{t}} \le |X_{11}| < \varepsilon\bigl(2^{m+1}\bigr)^{\frac{1}{t}}\bigr] \\
&\le E|X_{11}|^{(\alpha r+2)t}h\bigl(|X_{11}|\bigr) < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha r-1-\frac{2}{t}}(\log_2 n)^2 h(n)\sum_{j=1}^{n} a_{nj}^2 E(X_{11})^2 I\bigl[|X_{11}| < \varepsilon n^{\frac{1}{t}}\bigr] \\
&\quad\le \sum_{n=1}^{\infty}\bigl(2^k\bigr)^{\alpha r-\frac{2}{t}}k^2 h\bigl(2^k\bigr)\int_0^{(2^k)^{\frac{1}{t}}} x^2\,dF(x) \\
&\quad\le \sum_{k=1}^{\infty}\bigl(2^k\bigr)^{\alpha r+2-\frac{2}{t}}h\bigl(2^k\bigr)\int_0^{(2^k)^{\frac{1}{t}}} x^2\,dF(x) \\
&\quad\le \sum_{m=1}^{\infty}\bigl(2^m\bigr)^{\alpha r+2-\frac{2}{t}}h\bigl(2^m\bigr)\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} x^2\,dF(x) \\
&\quad= C\sum_{m=1}^{\infty}\bigl(2^m\bigr)^{\alpha r+2-\frac{2}{t}}\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}}\frac{h(2\times 2^{m-1})}{h(|x|^t)}h\bigl(|x|^t\bigr)x^2\,dF(x) \\
&\quad\le C\sum_{m=1}^{\infty}\bigl(2^m\bigr)^{\alpha r+2-\frac{2}{t}}\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} h\bigl(|x|^t\bigr)x^2\,dF(x) \\
&\quad\le C\sum_{m=1}^{\infty}\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}}\bigl(|x|^t\bigr)^{\alpha r+2}h\bigl(|x|^t\bigr)x^2\,dF(x) \\
&\quad= CE|X_{11}|^{(\alpha r+2)t}h\bigl(|X_{11}|^t\bigr) < \infty,
\end{aligned}
$$

and thus (2.6) and (2.7) are satisfied. Then, to complete the proof, it remains to show that, for $1 \le j \le n$, $n^{-\frac{1}{t}}j|EX_{11}I[|X_{11}| < \varepsilon n^{\frac{1}{t}}]| \to 0$ as $n \to \infty$.

If $(\alpha r+2)t < 1$, then we have, as $n \to \infty$,

$$
n^{-\frac{1}{t}}j\bigl|EX_{11}I\bigl[|X_{11}| < \varepsilon n^{\frac{1}{t}}\bigr]\bigr| \le (\varepsilon)^{1-(\alpha r+2)t}n^{1-(\alpha r+2)t}E|X_{11}|^{(\alpha r+2)t} \to 0,
$$

and if $(\alpha r+2)t \ge 1$, then since $|EX_{11}| = 0$, we have, as $n \to \infty$,

$$
\begin{aligned}
n^{-\frac{1}{t}}j\bigl|EX_{11}I\bigl[|X_{11}| < \varepsilon n^{\frac{1}{t}}\bigr]\bigr| &\le n^{1-\frac{1}{t}}\bigl|-EX_{11}I\bigl[|X_{11}| \ge \varepsilon n^{\frac{1}{t}}\bigr]\bigr| \\
&\le (\varepsilon)^{1-(\alpha r+2)t}n^{1-(\alpha r+2)t}E|X_{11}|^{(\alpha r+2)t} \to 0.
\end{aligned}
$$

Hence the proof of Theorem 2.4 is completed. □

Proof of Corollary 2.5 Taking $\log_2 b_n$ instead of $b_n^{\frac{1}{t}}$ in Theorem 2.1, we get (2.12). □

Proof of Corollary 2.6 In Theorem 2.1, let $c_n = n^{2(\alpha-1)}$ and $b_n^{-\frac{1}{t}} = n^{-\alpha}(\log_2 n)^{-1}$. By (2.13) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} |a_{nj}| E|X_{nj}| I\bigl(|a_{nj}X_{nj}| \geq \varepsilon b_n^{\frac{1}{t}}\bigr) \\
&\quad = \varepsilon \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} E\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| I\left(\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| \geq 1\right) \\
&\quad \leq \varepsilon \sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} E\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right|^2 I\left(\left|\frac{a_{nj}X_{nj}}{\varepsilon b_n^{\frac{1}{t}}}\right| \geq 1\right) \\
&\quad \leq C \sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} E|a_{nj}X_{nj}|^2 \\
&\quad \leq C \sum_{n=1}^{\infty} n^{2(\alpha-1)} n^{-2\alpha} (\log_2 n)^{-2} (\log_2 b_n)^2 n^{\delta} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} a_{nj}^2 E(X_{nj})^2 I\bigl[|a_{nj}X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\bigr] \\
&\quad \leq \sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} E|a_{nj}X_{nj}|^2 \\
&\quad \leq C \sum_{n=1}^{\infty} n^{2(\alpha-1)} n^{-2\alpha} (\log_2 n)^{-2} (\log_2 b_n)^2 n^{\delta} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty.
\end{aligned}$$

Hence conditions of (2.1) and (2.2) of Theorem 2.1 are satisfied. Since $EX_{nj} = 0$, by (2.13) we get

$$\begin{aligned}
&b_n^{-\frac{1}{t}} \max_{1 \leq k \leq b_n} \left| \sum_{j=1}^{k} E a_{nj} X_{nj} I\bigl[|a_{nj}X_{nj}| < \varepsilon b_n^{\frac{1}{t}}\bigr] \right| \\
&\quad \leq b_n^{-\frac{1}{t}} \max_{1 \leq k \leq b_n} \sum_{j=1}^{k} |a_{nj}| E|X_{nj}| I\bigl[|a_{nj}X_{nj}| \geq \varepsilon b_n^{\frac{1}{t}}\bigr] \\
&\quad \leq b_n^{-\frac{2}{t}} \sum_{j=1}^{b_n} |a_{nj}|^2 E|X_{nj}|^2 \\
&\quad \leq C n^{-2\alpha+\delta} (\log_2 n)^{-2} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

and thus (2.14) is completed. □

Acknowledgements
The authors are grateful to the referee for carefully reading the manuscript and for providing some comments and suggestions, which led to improvements in this paper.

Funding
The research of M. Ge is partially supported by the NSF of Anhui Educational Committe (KJ2017B11, KJ2018A0428). The research of Z. Dai is partially supported by the NSF of Anhui Educational Committe (KJ2017B09). The research of Y. Wu is partially supported by the Natural Science Foundation of Anhui Province (1708085MA04), the Key Program in the Young Talent Support Plan in Universities of Anhui Province (gxyqZD2016316), and Chuzhou University scientific research fund (2017qd17).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 April 2018 Accepted: 11 February 2019 Published online: 22 February 2019

References

1. Ko, M.: Complete convergences for arrays of row-wise PNQD random variables. Stoch. Int. J. Probab. Stoch. Process. **85**, 172–180 (2013)
2. Lehmann, E.L.: Some concepts of dependence. Ann. Math. Stat. **37**, 1137–1153 (1996)
3. Wu, Q.Y.: Convergence properties of pairwise NQD random sequence. Acta Math. Sin. **45**, 617–624 (2002) (in Chinese)
4. Matula, P.: A note on the almost sure convergence of sums of negatively dependent random variables. Stat. Probab. Lett. **15**, 209–213 (1992)
5. Jabbari, H.: On almost sure convergence for weighted sums of pairwise negatively quadrant dependent random variables. Stat. Pap. **54**, 765–772 (2013)
6. Li, R., Yang, W.G.: Strong convergence of pairwise NQD random sequences. J. Math. Anal. Appl. **334**, 741–747 (2008)
7. Wu, Y.F.: Strong convergence for weighted sums of arrays of row-wise pairwise NQD random variables. Collect. Math. **65**, 119–130 (2014)
8. Xu, H., Tang, L.: Some convergence properties for weighted sums of pairwise NQD sequences. J. Inequal. Appl. **2012**, 255 (2012)
9. Gan, S.X., Chen, P.Y.: Some limit theorems for sequences of pairwise NQD random variables. Acta Math. Sci. **28**, 269–281 (2008)
10. Wu, Y.F., Guan, M.: Mean convergence theorems and weak laws of large numbers for weighted sums of dependent random variables. J. Math. Anal. Appl. **377**, 613–623 (2011)
11. Hsu, P.L., Robbins, H.: Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. USA **33**, 25–31 (1947)
12. Chow, Y.S.: On the rate of moment convergence of sample sums and extremes. Bull. Inst. Math. Acad. Sin. **16**(3), 177–201 (1988)
13. Baum, L.E., Katz, M.: Convergence rates in the law of large numbers. Trans. Am. Math. Soc. **120**, 108–123 (1965)
14. Sung, S.H.: Moment inequalities and complete moment convergence. J. Inequal. Appl. **2009**, Article ID 271265 (2009)
15. Wang, X.J., Hu, S.H.: Complete convergence and complete moment convergence for martingale difference sequence. Acta Math. Sin. **30**, 119–132 (2014)
16. Shen, A.T., Xue, M.X., Volodin, A.: Complete moment convergence for arrays of rowwise NSD random variables. Stoch. Int. J. Probab. Stoch. Process. **88**(4), 606–621 (2016)
17. Wu, Y.F., Cabrea, M.O., Volodin, A.: Complete convergence and complete moment convergence for arrays of rowwise END random variables. Glas. Mat. **49**(69), 449–468 (2014)
18. Wu, Y., Wang, X.J., Hu, S.H.: Complete moment convergence for weighted sums of weakly dependent random variables and its application in nonparametric regression model. Stat. Probab. Lett. **127**, 56–66 (2017)
19. Wang, X.J., Wu, Y., Hu, S.H.: Complete moment convergence for double indexed randomly weighted sums and its applications. Statistics **52**, 503–518 (2018)
20. Wu, Y., Wang, X.J.: Equivalent conditions of complete moment and integral convergence for a class of dependent random variables. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**, 575–592 (2018)
21. Wu, Q.Y.: Limit Theorems of Probability Theory for Mixing Sequences. Science Press, Beijing (2006)