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Dimension estimates of the attractor for the dissipative quantum Zakharov equations

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Abstract

The finite dimension estimates of Hausdorff dimension and fractal dimension of the attractor of the dissipative quantum Zakharov equations are mainly studied by using the estimates of the Lyapunov exponents. The main results on the attractor are obtained through studying the linear system DS(t), which is the differential system of solution operator $S(t)$ for the dissipative quantum Zakharov equations at the initial value (m_0, n_0, E_0).

MSC: 60H15; 76A05

Keywords: Quantum Zakharov equations; Lyapunov exponents; Hausdorff dimension; Fractal dimension

1 Introduction

It is well known that the classical Zakharov equations are coupled nonlinear wave equations, which describe the interaction between high-frequency Langmuir waves and low-frequency ion-acoustic waves, and had been firstly derived by Zakharov [1] and can be written (in normalized units) as

$$iE_t + \Delta E - nE = 0, \tag{1.1}$$

$$n_{tt} - \Delta n - \Delta |E|^2 = 0, \tag{1.2}$$

where $E : \Omega \times \mathbf{R}^+ \to C$ is the envelope of the high-frequency electric field, and $n : \Omega \times \mathbf{R}^+ \to \mathbf{R}$ is the plasma density measured from its equilibrium value. Recently, a lot of results for (1.1)–(1.2) have been obtained by many authors. The existence of solutions of (1.1)–(1.2) is obtained in [2–4] in the one-dimensional case. The attractors of the dissipative Zakharov equations are investigated in [5]. Some authors have proved that two-dimensional and three-dimensional Zakharov equations develop singular infinite time in numerical and theoretical proofs [6–8].

However, in recent decades, the quantum effects have been considered in ultrasmall electronic devices, in dense astrophysical plasma systems, and in laser plasmas, which play an important role in studying nonlinear phenomena in some problems. There has been an increasing interest in the investigation of the quantum counterpart of some of the classical plasma physics phenomena. By further studying the nonlinear coupling roles

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between the quantum ion-acoustic waves and the quantum Langmuir waves, the modified Zakharov equations for plasmas with a quantum correction, simply called quantum Zakharov equations, are obtained by using of a quantum fluid approach [9]. In fact, to get a better qualitative agreement, it is necessary to include damping effects or effects of the loss of energy in the system. Many authors are interested in the quantum dissipative Zakharov equations as follows:

$$iE_t + \Delta E - H^2\Delta^2 E - nE + i\gamma E = 0, \tag{1.3}$$

$$n_{tt} + \alpha n_t - \Delta n + H^2\Delta^2 n - \Delta|E|^2 = 0, \tag{1.4}$$

where $H > 0$ is the quantum parameter expressing the ratio between the ion plasmon energy and the electron thermal energy, and $\alpha > 0$ and $\gamma > 0$. Until now, the global well-posedness and the limit behaviors of quantum Zakharov equations have been investigated as $H \to 0$ in [10], and the existence of attractor of (1.3)–(1.4) has been studied in [11]. In this paper, on the basis of [11], the finite dimension estimate of the Hausdorff dimension and fractal dimension of the attractor for the dissipative quantum Zakharov equations are further considered.

The paper is organized as follows. In Sect. 2, we give the work spaces and some results known in [11]. Section 3 is devoted to the investigation of the linear system $\mathrm{DS}(t)$, which is the differential system of the solution operator $S(t)$ for the dissipative quantum Zakharov equations at initial value (m_0, n_0, E_0). In Sect. 4, we study the Hausdorff dimension and fractal dimension of attractor using the equivalent form in space V_1. By c or c' we denote a generic positive constant, which may change their values from terms to terms.

2 Preliminary

We consider the bounded set $\Omega = [0, L]^d$ in $\mathbf{R}^d$ ($d = 1, 2, 3$). Assume that the coupled nonlinear dissipative quantum Zakharov equations on Ω satisfy the periodic boundary conditions $E(x + Le_j) = E(x)$ and $n(x + Le_j) = n(x)$ ($j = 1, 2, 3$) and the initial conditions $n(x, 0) = n_0(x)$, $n_t(x, 0) = n_1(x)$, and $E(x, 0) = E_0(x)$, $x \in \Omega$, and assume that $\int_\Omega E(x)\,dx = \int_\Omega n(x)\,dx = 0$.

We are going to work in the L-periodic functions spaces $L^p_{\mathrm{per}}(\Omega)$ and $H^s_{\mathrm{per}}(\Omega)$ for E and n; see [12]. As in [12] for the nonlinear wave equations, we can set $m = n_t + \varepsilon n$, where ε will be chosen later. So we have three unknown variables, which satisfy the following equations:

$$iE_t + \Delta E - H^2\Delta^2 E - nE + i\gamma E = 0, \tag{2.1}$$

$$m = n_t + \varepsilon n, \tag{2.2}$$

$$m_t + (\alpha - \varepsilon)m - \varepsilon(\alpha - \varepsilon)n - \Delta n + H^2\Delta^2 n - \Delta|E|^2 = 0, \tag{2.3}$$

where $\alpha > 0$, $\gamma > 0$, $H > 0$, and $\varepsilon > 0$.

The corresponding initial conditions are

$$m(x,0) = n_1(x) + \varepsilon n_0(x) = m_0(x), \qquad n(x,0) = n_0(x), \qquad E(x,0) = E_0(x).$$

For system (2.1)–(2.3), we also define the product spaces V_0, V_1, and V_2 which are different from the product spaces in [5] due to the higher-order terms and are written as

$$V_0 = H^{-1}_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega) \times H^2_{\mathrm{per}}(\Omega),$$
$$V_1 = H^{1}_{\mathrm{per}}(\Omega) \times H^3_{\mathrm{per}}(\Omega) \times H^4_{\mathrm{per}}(\Omega),$$
$$V_2 = H^{3}_{\mathrm{per}}(\Omega) \times H^5_{\mathrm{per}}(\Omega) \times H^6_{\mathrm{per}}(\Omega).$$

These spaces satisfy $V_2 \subset V_1 \subset V_0$ with compact embeddings. For the convenience, we use $\|\cdot\|$ to represent the norm of $\|\cdot\|_{\dot{L}^2_{\mathrm{per}}}$ on the corresponding space $\dot{L}^2_{\mathrm{per}}$, and $\dot{H}^s_{\mathrm{per}}(\Omega)$ can be regarded as $H^s_{\mathrm{per}}(\Omega)$ for the variables m, n, E because the norm $\dot{H}^s_{\mathrm{per}}(\Omega)$ is equivalent to the standard norm $H^s_{\mathrm{per}}(\Omega)$ if $\int_\Omega u(x)\,dx = 0$.

In this paper, we use some inequalities, such as the Gagliardo–Nirenberg inequality, Agmon inequality, and so on; see [12]. From Agmon's inequality, for $d = 1, 2, 3$, we have $\|u\|_{L^\infty} \le c\|E\|^{1-\frac{d}{4}}_{L^2}\|E\|^{\frac{d}{4}}_{H^2}$.

By the a priori uniform time estimates, using the Galerkin approximation, we obtain the uniqueness and existence of the solution of dissipative quantum Zakharov equations in spaces V_1 and V_2. In addition, the continuous dynamical systems in the spaces V_1 and V_2 have been given. The attractors in V_1 and V_2 have been obtained equipped with weak topology for dissipative quantum Zakharov equations; see [11].

We consider system (2.1)–(2.3). If $(m_0, n_0, E_0) \in V_1$ (resp., $(m_0, n_0, E_0) \in V_2$), then this system admits a unique solution $(m, n, E) \in L^\infty(\mathbf{R}^+; V_1)$ (resp., $(m, n, E) \in L^\infty(\mathbf{R}^+; V_2)$) and $(m, n, E) \in C(\mathbf{R}^+, V_2)$ (resp., $(m, n, E) \in C(\mathbf{R}^+, V_2)$).

We set $S(t)(m_0, n_0, E_0) = (m, n, E)$ for $t \ge 0$. Since the considered system is autonomous, $\{S(t)\}_{t\in\mathbf{R}}$ forms a group; see [12].

3 The linearized system

We prove that the attractor Q is finite by using the estimates of the Lyapunov exponents on the attractor Q and through studying the linear application $\mathrm{DS}(t)$. Now we consider its linearized system

$$iF_t + \Delta F - H^2\Delta^2 F - nF - uE + i\gamma F = 0, \tag{3.1}$$

$$v = u_t + \varepsilon u, \tag{3.2}$$

$$\begin{aligned} &v_t + (\alpha - \varepsilon)v - \varepsilon(\alpha - \varepsilon)u - \Delta u + H^2\Delta^2 u - 2\operatorname{Re}(F\Delta\bar{E}) \\ &\quad - 2\operatorname{Re}(\bar{E}\Delta F) - 4\operatorname{Re}(\nabla\bar{E}\nabla F) = 0, \end{aligned} \tag{3.3}$$

where $(m, n, E) = S(t)(m_0, n_0, E_0)$, $(m_0, n_0, E_0) \in V_2$, and $(v_0, u_0, F_0) \in V_1$.

Since $(m, n, E) \in L^\infty(\mathbf{R}^+; V_1)$, it is easy to show that this system has a unique solution $(v, u, F) \in L^\infty(\mathbf{R}^+; V_1)$.

Let

$$\bigl(v(t), u(t), F(t)\bigr) = \mathrm{DS}(t)(v_0, u_0, F_0) = L(t; m_0, n_0, E_0)(v_0, u_0, F_0),$$

where $\mathrm{DS}(t)(v_0, u_0, F_0)$ is the differential of $S(t)$ at (m_0, n_0, E_0).

Proposition 3.1 *For arbitrary $R > 0$ and $T < \infty$, there exists $C(R,T)$ for all $(m_0^{(i)}, n_0^{(i)}, E_0^{(i)})$ $(i = 1,2)$ such that $\|(m_0^{(i)}, n_0^{(i)}, E_0^{(i)})\|_{V_1} \le R$, and for all t: $0 < t \le T$, we have*

$$\begin{aligned}&\big\|S(t)\big(m_0^{(2)}, n_0^{(2)}, E_0^{(2)}\big) - S(t)\big(m_0^{(1)}, n_0^{(1)}, E_0^{(1)}\big) - \mathrm{D}S(t)(m_0, n_0, E_0)\big\|_{V_1}\\ &\quad\le C\big\|(m_0, n_0, E_0)\big\|_{V_1}^2,\end{aligned}$$

where $m_0 = m_0^{(2)} - m_0^{(1)}$, $n_0 = n_0^{(2)} - n_0^{(1)}$, $E_0 = E_0^{(2)} - E_0^{(1)}$.

We can obtain this proposition from the continuity of $S(t)$ on V_1. To obtain the solutions to be exponentially damped, we set $p = e^{\sigma t}u$, $q = e^{\sigma t}v$, and $G = e^{\sigma t}F$. Substituting them into (3.1)–(3.3), we obtain

$$iG_t + \Delta G - H^2\Delta^2 G - nG - pE + i(\gamma - \sigma)G = 0, \tag{3.4}$$

$$p_t = (\sigma - \varepsilon)p + q, \tag{3.5}$$

$$\begin{aligned}&q_t + (\alpha - \varepsilon - \sigma)v - \varepsilon(\alpha - \varepsilon)p - \Delta p + H^2\Delta^2 p\\ &\quad - 2\,\mathrm{Re}(G\Delta\bar{E}) - 2\,\mathrm{Re}(\bar{E}\Delta G) - 4\,\mathrm{Re}(\nabla\bar{E}\nabla G) = 0.\end{aligned} \tag{3.6}$$

Now we study the energy estimate of (q, p, G) to investigate the dimension of the attractor Q. At the beginning, we take the scalar product in L^2 of (3.5) by p and obtain

$$\frac{1}{2}\frac{d}{dt}\|p\|^2 = (\sigma - \varepsilon)\|p\|^2 + \int_\Omega pq\,dx. \tag{3.7}$$

Taking the inner product of (3.6) with q yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|q\|^2 + \|\nabla p\|^2 + H^2\|\Delta p\|^2\big) + (\alpha - \varepsilon - \sigma)\|q\|^2\\ &\qquad - \varepsilon(\alpha - \varepsilon)\int_\Omega pq\,dx - (\sigma - \varepsilon)\|\nabla p\|^2 - (\sigma - \varepsilon)H^2\|\Delta p\|^2\\ &\quad = \int_\Omega 2\,\mathrm{Re}(G\Delta\bar{E})q\,dx + \int_\Omega 2\,\mathrm{Re}(\bar{E}\Delta G)q\,dx + 4\int_\Omega \mathrm{Re}(\nabla\bar{E}\nabla G)q\,dx.\end{aligned} \tag{3.8}$$

Taking the inner product of (3.6) with Δq, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla q\|^2 + \|\Delta p\|^2 + H^2\|\nabla\Delta p\|^2\big) + (\alpha - \varepsilon - \sigma)\|\nabla q\|^2\\ &\qquad - \varepsilon(\alpha - \varepsilon)\int_\Omega \nabla p\nabla q\,dx - (\sigma - \varepsilon)\|\Delta p\|^2 - (\sigma - \varepsilon)H^2\|\nabla\Delta p\|^2\\ &\quad = 2\,\mathrm{Re}(\bar{E}\Delta G) + 4\,\mathrm{Re}(\nabla\bar{E}\nabla G) - \int_\Omega \Delta q\big(2\,\mathrm{Re}(G\Delta\bar{E})\big)\,dx.\end{aligned} \tag{3.9}$$

Taking the inner product of (3.4) with G and taking the imaginary parts, we have

$$\frac{1}{2}\frac{d}{dt}\|G\|^2 + (\gamma - \sigma)\|G\|^2 = \mathrm{Im}\int_\Omega pE\bar{G}\,dx. \tag{3.10}$$

Taking the inner product of (3.4) with $(\gamma-\sigma)\times G+G_t$ and taking the real parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla G\|^2+H^2\|\Delta G\|^2+2\operatorname{Re}\int_\Omega nG\bar{G}\,dx+2\operatorname{Re}\int_\Omega pE\bar{G}\,dx\right)\\
&\qquad+(\gamma-\sigma)\left(\|\nabla G\|^2+H^2\|\Delta G\|^2+\operatorname{Re}\int_\Omega nG\bar{G}\,dx+\operatorname{Re}\int_\Omega pE\bar{G}\,dx\right)\\
&\quad=\operatorname{Re}\int_\Omega (nG)_t\bar{G}\,dx+\operatorname{Re}\int_\Omega (pE)_t\bar{G}\,dx.
\end{aligned}\tag{3.11}$$

Taking the inner product of (3.4) with $(\gamma-\sigma)\times\Delta^2G+\Delta^2G_t$ and taking the real parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\Delta G\|^2+H^2\|\Delta^2G\|^2+2\operatorname{Re}\int_\Omega nG\Delta^2\bar{G}\,dx+2\operatorname{Re}\int_\Omega pE\Delta^2\bar{G}\,dx\right)\\
&\qquad+(\gamma-\sigma)\left(\|\nabla\Delta G\|^2+H^2\|\Delta^2G\|^2+\operatorname{Re}\int_\Omega nG\Delta^2\bar{G}\,dx+\operatorname{Re}\int_\Omega pE\Delta^2\bar{G}\,dx\right)\\
&\quad=\operatorname{Re}\int_\Omega (nG)_t\Delta^2\bar{G}\,dx+\operatorname{Re}\int_\Omega (pE)_t\Delta^2\bar{G}\,dx.
\end{aligned}\tag{3.12}$$

Now from $\mu\times(3.7)+(3.8)+(3.9)+\nu\times(3.10)+2\times(3.11)+2\times(3.12)$ we have

$$\frac{d}{dt}I(t)+B(t)=A(t),\tag{3.13}$$

where

$$\begin{aligned}
I(t)={}&\frac{\mu}{2}\|p\|^2+\frac{1}{2}\left(\|q\|^2+\|\nabla p\|^2+H^2\|\Delta p\|^2\right)+\frac{1}{2}\left(\|\nabla q\|^2+\|\Delta p\|^2+H^2\|\nabla\Delta p\|^2\right)\\
&+\frac{\nu}{2}\|G\|^2+\|\nabla G\|^2+H^2\|\Delta G\|^2+\|\nabla\Delta G\|^2+H^2\|\Delta^2G\|^2\\
&+2\operatorname{Re}\int_\Omega nG\bar{G}\,dx+2\operatorname{Re}\int_\Omega pE\bar{G}\,dx+2\operatorname{Re}\int_\Omega nG\Delta^2\bar{G}\,dx+2\operatorname{Re}\int_\Omega pE\Delta^2\bar{G}\,dx,\\
B(t)={}&\mu(\varepsilon-\sigma)\|p\|^2+(\alpha-\varepsilon-\sigma)\|q\|^2+(\alpha-\varepsilon-\sigma)\|\nabla q\|^2-(\sigma-\varepsilon)\|\nabla p\|^2\\
&-(\sigma-\varepsilon)H^2\|\Delta p\|^2-\varepsilon(\alpha-\varepsilon)\int_\Omega pq\,dx-\varepsilon(\alpha-\varepsilon)\int_\Omega \nabla p\nabla q\,dx\\
&-(\sigma-\varepsilon)\|\Delta p\|^2-(\sigma-\varepsilon)H^2\|\nabla\Delta p\|^2+\nu(\gamma-\sigma)\|G\|^2\\
&+2(\gamma-\sigma)\left(\|\nabla G\|^2+H^2\|\Delta G\|^2+\|\nabla\Delta G\|^2+H^2\|\Delta^2G\|^2\right),
\end{aligned}$$

and

$$\begin{aligned}
A(t)={}&\mu\int_\Omega pq\,dx+\int_\Omega 2\operatorname{Re}(G\Delta\bar{E})q\,dx+\int_\Omega 2\operatorname{Re}(\bar{E}\Delta G)q\,dx+4\int_\Omega \operatorname{Re}(\nabla\bar{E}\nabla G)q\,dx\\
&-\int_\Omega \Delta q\left(2\operatorname{Re}(G\Delta\bar{E})+2\operatorname{Re}(\bar{E}\Delta G)+4\operatorname{Re}(\nabla\bar{E}\nabla G)\right)dx+\nu\operatorname{Im}\int_\Omega pE\bar{G}\,dx\\
&+2\operatorname{Re}\int_\Omega (nG)_t\bar{G}\,dx+2\operatorname{Re}\int_\Omega (pE)_t\bar{G}\,dx+2\operatorname{Re}\int_\Omega (nG)_t\Delta^2\bar{G}\,dx\\
&+2\operatorname{Re}\int_\Omega (pE)_t\Delta^2\bar{G}\,dx-2(\gamma-\sigma)\operatorname{Re}\int_\Omega nG\bar{G}\,dx+\operatorname{Re}\int_\Omega pE\bar{G}\,dx\\
&+\operatorname{Re}\int_\Omega nG\Delta^2\bar{G}\,dx+\operatorname{Re}\int_\Omega pE\Delta^2\bar{G}\,dx.
\end{aligned}$$

Since

$$G_t = i\big(\Delta G - H^2\Delta^2 G - nG - pE + i(\gamma-\sigma)G\big),$$

we have

$$\begin{aligned}
&2\operatorname{Re}\int_\Omega (nG)_t\bar{G}\,dx + 2\operatorname{Re}\int_\Omega (pE)_t\bar{G}\,dx\\
&\quad= 2\left(\operatorname{Re}\int_\Omega n_tG\bar{G}\,dx + \operatorname{Re}\int_\Omega nG_t\bar{G}\,dx + \operatorname{Re}\int_\Omega pE_t\bar{G}\,dx + \operatorname{Re}\int_\Omega p_tE\bar{G}\,dx\right)\\
&\quad= 2\left(\operatorname{Re}\int_\Omega n_tG\bar{G}\,dx - \operatorname{Im}\int_\Omega n\bar{G}\Delta G\,dx + H^2\operatorname{Im}\int_\Omega n\bar{G}\Delta^2 G\,dx + \operatorname{Im}\int_\Omega npE\bar{G}\,dx\right.\\
&\qquad\left. - (\gamma-\sigma)\int_\Omega nG\bar{G}\,dx + \operatorname{Re}\int_\Omega \big((\sigma-\varepsilon)p+q\big)E\bar{G}\,dx + \operatorname{Re}\int_\Omega pE_t\bar{G}\,dx\right)
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&2\operatorname{Re}\int_\Omega (nG)_t\Delta^2\bar{G}\,dx + 2\operatorname{Re}\int_\Omega (pE)_t\Delta^2\bar{G}\,dx\\
&\quad= 2\operatorname{Re}\int_\Omega n_tG\Delta^2\bar{G}\,dx - 2\operatorname{Im}\int_\Omega n\Delta G\Delta^2\bar{G}\,dx + 2\operatorname{Im}\int_\Omega n^2G\Delta^2\bar{G}\,dx\\
&\qquad+ 2\operatorname{Im}\int_\Omega npE\Delta^2\bar{G}\,dx - 2(\gamma-\sigma)\operatorname{Re}\int_\Omega nG\Delta^2\bar{G}\,dx\\
&\qquad+ 2\operatorname{Re}\int_\Omega \big(\big((\sigma-\varepsilon)p+q\big)E + pE_t\big)\Delta^2\bar{G}\,dx.
\end{aligned}\tag{3.15}$$

Choosing ε made from the beginning, $B(t)$ infers

$$\begin{aligned}
B(t) &\geq \mu(\varepsilon-\sigma)\|p\|^2 + \left(\frac{\varepsilon}{2}-\sigma\right)\big(\|q\|^2+\|\nabla p\|^2\big) + \left(\frac{\varepsilon}{2}-\sigma\right)\big(\|\nabla q\|^2+\|\Delta p\|^2\big)\\
&\quad+ (\varepsilon-\sigma)H^2\|\Delta p\|^2 + (\varepsilon-\sigma)H^2\|\nabla\Delta p\|^2 + \nu(\gamma-\sigma)\|G\|^2\\
&\quad+ 2(\gamma-\sigma)\big(\|\nabla G\|^2 + H^2\|\Delta G\|^2 + \|\nabla\Delta G\|^2 + H^2\big\|\Delta^2 G\big\|^2\big).
\end{aligned}$$

Moreover, according to (3.14) and (3.15), for the term $A(t)$, we have the estimate

$$A(t) \leq \frac{\varepsilon}{4}\|q\|^2 + \frac{\varepsilon}{4}\|\nabla q\|^2 + \gamma H^2\big\|\Delta^2 G\big\|^2 + \frac{\varepsilon}{4}\|\Delta p\|^2 + c\big(\|p\|^2+\|G\|^2+\|\nabla G\|^2+\|\Delta G\|^2\big),$$

where we used

$$c\|\nabla\Delta G\|^2 \leq \varepsilon\big\|\Delta^2 G\big\|^2 + c\|\Delta G\|^2.$$

Then from (3.13) we obtain

$$\begin{aligned}
&\frac{d}{dt}I(t) + \mu(\varepsilon-\sigma)\|p\|^2 + \left(\frac{\varepsilon}{4}-\sigma\right)\big(\|q\|^2+\|\nabla p\|^2\big) + \left(\frac{\varepsilon}{4}-\sigma\right)\big(\|\nabla q\|^2+\|\Delta p\|^2\big)\\
&\quad+ (\varepsilon-\sigma)H^2\|\Delta p\|^2 + (\varepsilon-\sigma)H^2\|\nabla\Delta p\|^2 + \nu(\gamma-\sigma)\|G\|^2
\end{aligned}$$

$$
\begin{aligned}
&+2(\gamma-\sigma)\left(\|\nabla G\|^2+H^2\|\Delta G\|^2+\|\nabla\Delta G\|^2\right)+2\left(\frac{\gamma}{2}-\sigma\right)H^2\left\|\Delta^2 G\right\|^2\\
&\leq c\left(\|p\|^2+\|G\|^2+\|\nabla G\|^2+\|\Delta G\|^2\right). \qquad (3.16)
\end{aligned}
$$

We can choose $\sigma<\min\{\frac{\varepsilon}{4},\frac{\gamma}{2}\}$. According to (3.16), it is clear that

$$
\frac{d}{dt}I(t)\leq c\left(\|p\|^2+\|G\|^2+\|\nabla G\|^2+\|\Delta G\|^2\right). \qquad (3.17)
$$

Indeed,

$$
\begin{aligned}
I(t)={}&\frac{\mu}{2}\|p\|^2+\frac{1}{2}\left(\|q\|^2+\|\nabla p\|^2+H^2\|\Delta p\|^2\right)+\frac{1}{2}\left(\|\nabla q\|^2+\|\Delta p\|^2+H^2\|\nabla\Delta p\|^2\right)\\
&+\frac{\nu}{2}\|G\|^2+\|\nabla G\|^2+H^2\|\Delta G\|^2+\|\nabla\Delta G\|^2+H^2\left\|\Delta^2 G\right\|^2+2\operatorname{Re}\int_\Omega nG\bar{G}\,dx\\
&+2\operatorname{Re}\int_\Omega pE\bar{G}\,dx+2\operatorname{Re}\int_\Omega nG\Delta^2\bar{G}\,dx+2\operatorname{Re}\int_\Omega pE\Delta^2\bar{G}\,dx
\end{aligned}
$$

with

$$
\begin{aligned}
&2\operatorname{Re}\int_\Omega nG\bar{G}\,dx\leq 2\|n\|_\infty\|G\|\|G\|\leq\frac{\nu}{8}\|G\|^2+\frac{8\|n\|_\infty^2\|G\|^2}{\nu},\\
&2\operatorname{Re}\int_\Omega pE\bar{G}\,dx\leq 2\|E\|_\infty\|p\|\|G\|\leq\frac{\mu}{8}\|p\|^2+\frac{8\|E\|_\infty^2\|G\|^2}{\mu},\\
&2\operatorname{Re}\int_\Omega nG\Delta^2\bar{G}\,dx\leq 2\|n\|_\infty\|G\|\left\|\Delta^2\bar{G}\right\|\leq\frac{\nu}{8}\|G\|^2+\frac{8\|n\|_\infty^2\|\Delta^2\bar{G}\|^2}{\nu},
\end{aligned}
$$

and

$$
2\operatorname{Re}\int_\Omega pE\Delta^2\bar{G}\,dx\leq 2\|p\|\|E\|_\infty\left\|\Delta^2\bar{G}\right\|\leq\frac{\mu}{8}\|p\|^2+\frac{8\|E\|_\infty^2\|\Delta^2\bar{G}\|^2}{\mu}.
$$

From these estimates we have

$$
\begin{aligned}
I(t)\geq{}&\frac{\mu}{4}\|p\|^2+\frac{1}{2}\left(\|q\|^2+\|\nabla p\|^2+H^2\|\Delta p\|^2\right)+\frac{1}{2}\left(\|\nabla q\|^2+\|\Delta p\|^2+H^2\|\nabla\Delta p\|^2\right)\\
&+\left(\frac{\nu}{8}-\frac{8\|n\|_\infty^2}{\nu}-\frac{8\|E\|_\infty^2}{\mu}\right)\|G\|^2+\|\nabla G\|^2+H^2\|\Delta G\|^2+\|\nabla\Delta G\|^2\\
&+\left(H^2-\frac{8\|n\|_\infty^2}{\nu}-\frac{8\|E\|_\infty^2}{\mu}\right)\left\|\Delta^2 G\right\|^2. \qquad (3.18)
\end{aligned}
$$

Then for very large μ and ν, we have

$$
\frac{\nu}{8}-\frac{8\|n\|_\infty^2}{\nu}-\frac{8\|E\|_\infty^2}{\mu}\geq\frac{\nu}{4}>0
$$

and

$$
H^2-\frac{8\|n\|_\infty^2}{\nu}-\frac{8\|E\|_\infty^2}{\mu}\geq\frac{H^2}{2}>0\quad\forall t>0.
$$

Therefore we have $I(t) \geq c\|(q,p,G)\|_{V_1}^2$, with c is the minimum number of the coefficients in (3.18). It is obvious that

$$c\left\|(q,p,G)\right\|_{V_1}^2 \leq I(t) \leq c'\left\|(q,p,G)\right\|_{V_1}^2, \tag{3.19}$$

when μ, ν are chosen. So $I(t)^{\frac{1}{2}}$ is a norm equivalent to the norm in V_1.

We write the bilinear form associated with the quadratic form $I(t)$ as

$$\begin{aligned}
f(\xi_1,\xi_2) = {} & \frac{\mu}{2}\int_\Omega p_1p_2\,dx + \frac{1}{2}\biggl(\int_\Omega q_1q_2\,dx + \int_\Omega \nabla p_1\nabla p_2\,dx + \int_\Omega H^2\Delta p_1\Delta p_2\,dx\biggr) \\
& + \frac{1}{2}\biggl(\int_\Omega \nabla q_1\nabla q_2\,dx + \int_\Omega \Delta p_1\Delta p_2\,dx + H^2\int_\Omega \nabla\Delta p_1\nabla\Delta p_2\,dx\biggr) \\
& + \frac{\nu}{2}\int_\Omega G_1G_2\,dx + \int_\Omega \nabla G_1\nabla G_2\,dx + H^2\int_\Omega \Delta G_1\Delta G_2\,dx \\
& + \int_\Omega \nabla\Delta G_1\nabla\Delta G_2\,dx + H^2\int_\Omega \Delta^2 G_1\Delta^2 G_2\,dx + 2\operatorname{Re}\int_\Omega nG_1\bar{G}_2\,dx \\
& + \operatorname{Re}\int_\Omega p_1E\bar{G}_2\,dx + \operatorname{Re}\int_\Omega p_2E\bar{G}_1\,dx + \operatorname{Re}\int_\Omega nG_1\Delta^2\bar{G}_2\,dx \\
& + \operatorname{Re}\int_\Omega nG_2\Delta^2\bar{G}_1\,dx + \operatorname{Re}\int_\Omega p_1E\Delta^2\bar{G}_2\,dx + \operatorname{Re}\int_\Omega p_2E\Delta^2\bar{G}_1\,dx.
\end{aligned}$$

4 Study of the dimension of Q

Now we study the dimension of the attractor Q via to studying $I(t)$. The attractor Q is considered as an invariant compact subset of V_1. We investigate how the operators $\mathrm{DS}(t)(m_0,n_0,E_0)$ contract the volumes in V_1 for $(m_0,n_0,E_0) \in Q \subset V_2$. We take m elements $U_0^1,\dots,U_0^m$ linearly independent in V_1 and study the evolution of the Gram determinant

$$\left\|U^1 \wedge \cdots \wedge U^m\right\|_{V_1}^2 = \det_{1\leq i,j\leq m}\bigl(U^i(t),U^j(t)\bigr)_{V_1}, \tag{4.1}$$

where $U^i = \mathrm{DS}(t)(m_0,n_0,E_0)U_0^i$ and

$$\begin{aligned}
\bigl(U^i,U^j\bigr)_{V_1} = {} & \int_\Omega v_iv_j\,dx + \int_\Omega \nabla v_i\nabla v_j\,dx + \int_\Omega u_iu_j\,dx + \int_\Omega \nabla u_i\nabla u_j\,dx + \int_\Omega \Delta u_i\Delta u_j\,dx \\
& + \int_\Omega \nabla\Delta u_i\nabla\Delta u_j\,dx + \int_\Omega F_iF_j\,dx + \int_\Omega \nabla F_i\nabla F_j\,dx + \int_\Omega \Delta F_i\Delta F_j\,dx \\
& + \int_\Omega \nabla\Delta F_i\nabla\Delta F_j\,dx + \int_\Omega \Delta^2 F_i\Delta^2 F_j\,dx.
\end{aligned}$$

We will further show that, for sufficiently large m, this determinant exponentially decays as $t \to \infty$. In the proof the bilinear form associated with the functional $I(t)$ plays a key role. We use the functional $I(t)$ instead of the norm in V_1.

In the former, we have

$$(q,p,G) = W(t) = e^{\sigma t}U(t) = e^{\sigma t}(v,u,F).$$

Then we have

$$\bigl(U^i,U^j\bigr) = e^{-2\sigma t}\bigl(W^i,W^j\bigr)$$

and

$$\det_{1\le i,j\le m}\big(U^i,U^j\big) = e^{-2\sigma mt}\det_{1\le i,j\le m}\big(W^i,W^j\big).$$

We set $K_m(t) = \det_{1\le i,j\le m}(W^i, W^j)$ and $H_m(t) = \det_{1\le i,j\le m} f(t; W^i, W^j)$, where f is the bilinear form associated with $I(t)$. From (3.19) we obtain the following relation between K_m and H_m:

$$c^m K_m(t) \le H_m(t) \le c'^m K_m(t). \tag{4.2}$$

Then we can estimate $H_m(t)$. We have

$$\frac{d}{dt}H_m(t) = \sum_{l=1}^{m} \det f\big(t; W^i, W^j\big)_l, \tag{4.3}$$

where

$$\det f\big(t; W^i, W^j\big)_l = (1-\delta_{jl})\det f\big(t; W^i, W^j\big) + \delta_{jl}\frac{d}{dt}f\big(t; W^i, W^j\big),$$

δ_{jl} being the Kronecker symbol. In addition, we have

$$\frac{d}{dt}f\big(t; W^i, W^j\big) = \frac{1}{4}\left[\frac{d}{dt}I\big(t; W^i + W^j\big) - \frac{d}{dt}I\big(t; W^i - W^j\big)\right].$$

Due to (3.13), we have

$$\frac{d}{dt}I(t) = A - B = R(t).$$

Hence we have

$$\frac{d}{dt}f\big(t; W^i, W^j\big) = \frac{R(t; W^i + W^j) - R(t; W^i - W^j)}{4} = \rho\big(t; W^i, W^j\big),$$

where ρ is the bilinear form associated with R. According to the result used by Ghidaglia [13], which is in fact a lemma of linear algebra, we obtain

$$\frac{d}{dt}H_m(t) = H_m(t)\sum_{l=1}^{m}\max_{\substack{F\subset R^m\\ \dim F = l}}\min_{\substack{x\in F\\ x\ne 0}}\frac{R(t;\sum_{j=1}^{m}x_j W^j(t))}{I(t;\sum_{j=1}^{m}x_j W^j(t))}, \tag{4.4}$$

where

$$I\left(t;\sum_{j=1}^{m}x_j W^j(t)\right) = I\left(t;\sum_{j=1}^{m}x_j DS(t)W_0^j\right) = I\big(t; DS(t)W_0\big), \quad W_0 = \sum_{j=1}^{m}x_j W_0^j.$$

According to (3.17), we have

$$\begin{aligned}R\big(t;(q,p,G)\big) &\le c\big(\|p\|^2 + \|G\|^2 + \|\nabla G\|^2 + \|\Delta G\|^2\big)\\ &\le c\big(\|\nabla p\|^2 + \|G\|^2 + \|\nabla G\|^2 + \|\Delta G\|^2\big) = \big(M(q,p,G),(q,p,G)\big)_{V_1},\end{aligned}$$

where M is defined by

$$M(q,p,G) = \big(0,(-\Delta)^{-2}p,(-\Delta)^{-2}G\big), \quad (q,p,G) \in V_1,$$

which is a symmetric compact operator. Hence, using (3.13), we deduce from (4.4) that

$$\frac{d}{dt}H_m(t) \le H_m(t)\sum_{l=1}^{m} \max_{\substack{F\subset R^m\\ \dim F=l}} \min_{\substack{x\in F\\ x\neq 0}} \frac{c_2(M(\sum x_j W^j(t)), \sum x_j W^j(t))_{V_1}}{c\|\sum x_j W^j(t)\|_{V_1}^2}. \tag{4.5}$$

Moreover,

$$\min_{\substack{x\in F\\ x\neq 0}} \frac{(M(\sum x_j W^j(t)), \sum x_j W^j(t))_{V_1}}{\|\sum x_j W^j(t)\|_{V_1}^2} \le \max_{\substack{\mathbf{F}\subset R^m\\ \dim \mathbf{F}=l}} \min_{\substack{\xi\in \mathbf{F}\\ \xi\neq 0}} \frac{(M(\xi),\xi)_{V_1}}{\|\xi\|_{V_1}^2},$$

which is the lth eigenvalue of the operator M, denoted by $\omega_l = (\lambda_l)^{-2}$, where λ_l is the lth eigenvalue of the Laplace operator in dimension d, and $\lambda_l \sim l^{\frac{2}{d}}$.

Consequently, (4.5) can be rewritten as

$$\frac{d}{dt}H_m(t) \le H_m(t)\frac{c_2}{c}\sum_{l=1}^{m}\big((\lambda_l)^{-2}\big) \le H_m(t)\frac{c_2}{c}\sum_{l=1}^{m}\frac{1}{l^{\frac{4}{d}}}.$$

Set $s_m = \sum_{l=1}^{m}(\lambda_l)^{-2}$. Using the Gronwall lemma, we have

$$H_m(t) \le H_m(0)\exp\left(\frac{c_2}{c}s_m t\right).$$

From (4.3) we obtain

$$K_m(t) \le \left(\frac{c'}{c}\right)^m K_m(0)\exp\left(\frac{c_2}{c}s_m t\right).$$

Due to the definition of K_m, this yields

$$\begin{aligned}\det_{1\le i,j\le m}\big(U^i,U^j\big)_{V_1} &\le \left(\frac{c'}{c}\right)^m K_m(0)\exp\left(\frac{c_2}{c}s_m - 2\sigma m\right)t \\ &\le \left(\frac{c'}{c}\right)^m K_m(0)\exp\left(\frac{c_2}{c}s - 2\sigma m\right)t.\end{aligned}$$

It is well known that when $d < 4$, $s_m \le$ constant $= s$. According to the preceding analysis, we obtain the following theorem.

Theorem 4.1 *For arbitrary $\xi_0 = (m_0, n_0, E_0) \in Q$, $m \ge 1$, $t > 0$, we have that, for all $U_0^i \in V_1$,*

$$\begin{aligned}&\big\|\mathrm{DS}(t)(\xi_0)U_0^1 \wedge \cdots \wedge \mathrm{DS}(t)(\xi_0)U_0^m\big\|_{V_1}^2 \\ &\quad \le \big\|U_0^1 \wedge \cdots \wedge U_0^m\big\|_{V_1}^2 \left(\frac{c'}{c}\right)^m \exp\left(\frac{c_2}{c}s - 2\sigma m\right)t.\end{aligned} \tag{4.6}$$

Therefore we have obtained the principle assumption of Theorem 3.1 of Constantin et al. [14]. Then we can show that Q has finite fractal dimension. We set

$$L = \mathrm{DS}(nt_0)(m_0, n_0, E_0), \qquad \xi_0 = (m_0, n_0, E_0), \qquad U^i(t) = \mathrm{DS}(nt_0)\xi_0 \cdot U_0^i$$

and define

$$\omega_m\bigl(L(\xi_0)\bigr) = \sup_{\substack{(\eta^i)_{i=1}^m \\ \det(\eta^i,\eta^j)\le 1}} \bigl(\det\bigl(L\eta^i, L\eta^j\bigr)\bigr)^{\frac{1}{2}}.$$

From Theorem 4.1 we derive

$$\omega_m\bigl(L(\xi_0)\bigr) \le \left(\frac{c'}{c}\right)^{\frac{m}{2}} \exp\left(\frac{c_2}{c}s - 2\sigma m\right)\frac{nt_0}{2}.$$

So if we choose m such that $m\sigma > \frac{c_2 s}{2c}$ for sufficiently large t, then we have $\omega_m(L(\xi_0)) \le 1$ uniformly with respect to ξ_0. Then we obtain that $d_H(Q) \le m$ and $d_F(Q) \le 2m$, that is, they are finite. Similarly to the proof method in [13], we claim the following theorem.

Theorem 4.2 *The maximal attractor Q of the dissipative modified Zakharov system has finite Hausdorff dimension and fractal dimension in V_1.*

We investigate the bound of Hausdorff dimension and fractal dimension of the attractor for the modified Zakharov system, which represent some dynamics behaviors of finite dimension of the attractors of the system. The idea of the proof of our results is different from the proof of the existence of the attractors of the dissipative quantum Zakharov equations in [11]. Here we firstly obtained the linearized system of the modified Zakharov system, which is needed to consider the Hausdorff dimension and fractal dimension of the attractors. Secondly, we consider energy estimates of Lyapunov exponents for the linearized system, which is given by exponential transformation on the basis of the above linearized system. Finally, our results are obtained by studying the bilinear form associated with the quadratic form $f(\xi_1, \xi_2)$ and evolution of the Gram determinant in finite solution spaces.

Our results are also different from the existence of global strong solutions of higher-dimensional and vector forms of dissipative quantum Zakharov equations, which have not been given until now. Therefore we are interested in investigation of global strong solutions of higher-dimensional dissipative quantum Zakharov equations in vector form, which is our further goal of studying the nonlinear phenomenon of quantum Zakharov equations in vector form in the two- and three-dimensional cases.

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