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Approximating trigonometric functions by using exponential inequalities

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Abstract

In this paper, some exponential inequalities are derived from the inequalities containing trigonometric functions. Numerical examples show that one can achieve much tighter bounds than those of prevailing methods, which are presented by Cusa, Huygens, Chen and Sándor.

Keywords: Trigonometric function; Cusa–Huygens's inequality; Chen–Sándor's inequality; Monotonically increasing function; Monotonically decreasing function

1 Introduction

Inequalities involving trigonometric and inverse trigonometric functions play an important role and have many applications in science and engineering [2, 8, 12, 17–19, 27]. The sinc function, defined as $\frac{\sin(x)}{x}$, is often used in signal processing, optics, radio transmission, sound recording [12], has been studied in many references [1, 3–5, 7, 10, 13–17, 19, 21, 26, 28–30]. The study starts from the Jordan's inequality [19], namely

$$\frac{2}{\pi} \le \frac{\sin(x)}{x} \le 1, \quad 0 \le x \le \frac{\pi}{2}. \tag{1}$$

Later, the sinc function is bounded by using polynomials [7, 10, 17, 24], or by using exponential bounds [3, 4, 25].

Cusa–Huygens's inequality is studied in [3, 4, 11, 20, 22, 23, 25], and gives

$$\cos^{\frac{1}{3}}(x) < \frac{\sin(x)}{x} < \frac{2+\cos(x)}{3}, \quad 0 < x < \frac{\pi}{2}, \tag{2}$$

$$\left(\frac{2+\cos(x)}{3}\right)^{\theta} < \frac{\sin(x)}{x} < \left(\frac{2+\cos(x)}{3}\right)^{\vartheta}, \tag{3}$$

where the constants $\theta = \frac{\ln(\pi/2)}{\ln(3/2)} \approx 1.113$ and $\vartheta = 1$ are the best possible.

Becker–Stark's inequality, namely

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan(x)}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \quad 0 < x < \frac{\pi}{2}, \tag{4}$$

is studied in [3, 4, 10, 19, 29, 31]. In [32], Zhu provided improved bounds:

$$t_1(x) < \frac{\tan(x)}{x} < t_2(x), \quad 0 < x < \frac{\pi}{2}, \tag{5}$$

where $t_1(x) = \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{\pi^2-9}{6\pi^4}(\pi^2 - 4x^2)$ and $t_2(x) = \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{(10-\pi^2)}{\pi^4}(\pi^2 - 4x^2)$. Later, there in [6] further improved bounds were given, namely

$$t_3(x) = \frac{\bar{t}_3(x)}{45\pi^6(\pi^2 - 4x^2)} < \frac{\tan(x)}{x} < \frac{\bar{t}_4(x)}{3\pi^6(\pi^2 - 4x^2)} = t_4(x), \tag{6}$$

where $\bar{t}_3(x) = 45\pi^8 + (-2\pi^8 - 3660\pi^6 + 36000\pi^4)x^2 + (16\pi^7 + 21000\pi^5 - 208800\pi^3)x^3 + (-48\pi^6 - 49440\pi^4 + 492480\pi^2)x^4 + (64\pi^5 + 54240\pi^3 - 541440\pi)x^5 + (-32\pi^4 - 23040\pi^2 + 230400)x^6$ and $\bar{t}_4(x) = 3\pi^8 + (-12\pi^6 + \pi^8)x^2 + (5280\pi^3 - 456\pi^5 - 8\pi^7)x^3 + (-24768\pi^2 + 2272\pi^4 + 24\pi^6)x^4 + (40704\pi - 3808\pi^3 - 32\pi^5)x^5 + (-23040 + 2176\pi^2 + 16\pi^4)x^6$.

Chen and Cheng established the following exponential bounds [3]:

$$\left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^{\frac{\pi^2}{12}} \le \frac{\tan(x)}{x} \le \frac{\pi^2}{\pi^2 - 4x^2}, \quad 0 < x < \frac{\pi}{2}. \tag{7}$$

Recently, Nishizawa established [25]

$$\left(\cos(x)\right)^{\frac{\bar{\theta}_1(x)}{3}} \le \frac{\sin(x)}{x} \le \left(\cos(x)\right)^{\frac{\bar{\theta}_2(x)}{3}}, \quad 0 < x < \frac{\pi}{2}, \tag{8}$$

$$\left(\frac{4\pi}{\pi^2 - 4x^2}\right)^{(2x/\pi)^2} \le \frac{1}{\cos(x)} \le \left(\frac{4\pi}{\pi^2 - 4x^2}\right)^{\bar{\theta}_3(x)}, \quad 0 < x < \frac{\pi}{2}, \tag{9}$$

where

$$\bar{\theta}_1(x) = \begin{cases} 1, & 0 < x \le 1, \\ 2 - x, & 1 < x \le 3/2, \\ 1/2, & 3/2 < x < \pi/2, \end{cases} \qquad \bar{\theta}_2(x) = 1 - 2x/\pi, \quad \text{and}$$

$$\bar{\theta}_3(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 < x < \pi/2. \end{cases}$$

Motivated by Eqs. (5), (8) and (9), we provide some inequalities with much tighter bounds by using power exponential functions, which are described in Theorems 1.1–1.2.

Theorem 1.1 *For every* $0 < x < \pi/2$, *we have*

$$a(x)^{-\theta_1(x)} \le \frac{1}{\cos(x)} \le a(x)^{-\theta_2(x)}, \tag{10}$$

where $a(x) = 1 + \frac{\pi-4}{\pi}x - \frac{2(\pi-2)}{\pi^2}x^2$, $\theta_1(x) = -\frac{\pi}{2(\pi-4)}x - \frac{\pi^2-4\pi+8}{4(\pi-4)^2}x^2 + \frac{\pi^4+8\pi^2-128\pi+256}{2\pi^3(\pi-4)^2}x^3$ *and* $\theta_2(x) = -\frac{\pi}{2(\pi-4)}x + \frac{2(\pi^2+6\pi-24)}{\pi^2(\pi-4)}x^2 - \frac{2(\pi^2+8\pi-32)}{(\pi-4)\pi^3}x^3$.

Theorem 1.2 *For every* $0 < x < \pi/2$, *we have*

$$\cos(x)^{\frac{1}{3} - \frac{2}{45}x^2 + \frac{5}{124}x^3 - \frac{41}{1000}x^4} \le \frac{\sin(x)}{x} \le \cos(x)^{\frac{1}{3} - \frac{4}{3\pi^2}x^2}. \tag{11}$$

2 Proof of Theorem 1.1

Proof Equation (10) is equivalent to

$$-\theta_1(x)\cdot\ln\bigl(a(x)\bigr)\le-\ln\bigl(\cos(x)\bigr)\le-\theta_2(x)\cdot\ln\bigl(a(x)\bigr). \tag{12}$$

Step 1. Firstly, we prove that for every $0<x<\pi/2$,

$$-\theta_1(x)\cdot\ln\bigl(a(x)\bigr)\le-\ln\bigl(\cos(x)\bigr),$$

which is equivalent to

$$D_1(x)=\ln\bigl(\cos(x)\bigr)-\theta_1(x)\cdot\ln\bigl(a(x)\bigr)\le 0. \tag{13}$$

Let $x_1=0.9$ and $x_2=\frac{111\cdot\pi}{256}\approx 1.362$.

Case 1.1. Proof of $D_1(x)\le 0$, $\forall x\in(0,x_1]$.

Combining with Eq. (6), for every $0<x\le x_1$, we have that

$$\begin{aligned} D_1'(x) &= -\tan(x)-\frac{\theta_1(x)\cdot a'(x)}{a(x)}-\theta_1'(x)\cdot\ln\bigl(a(x)\bigr)\\ &\le -t_3(x)\cdot x-\frac{\theta_1(x)\cdot a'(x)}{a(x)}-\theta_1'(x)\cdot\ln\bigl(a(x)\bigr)=D_2(x). \end{aligned} \tag{14}$$

It can be verified that $\theta_1'(x)=\alpha_2x^2+\alpha_1x+\alpha_0=\alpha_2(x+\frac{\alpha_1}{2\alpha_2})^2+\alpha_3$, where $\alpha_0\approx 1.82$, $\alpha_1\approx -3.59$, $\alpha_2\approx 1.98>0$ and $\alpha_3\approx 0.19>0$, so we have $\theta_1'(x)>0$, $\forall x\in(0,\pi/2)$. Let $D_3(x)=\frac{D_2(x)}{\theta_1'(x)}$. It can be verified that

$$D_3'(x)=\frac{(\sum_{i=0}^{9}\gamma_{1,i}B_{1,i}(x))\cdot x^2}{W_1(x)}, \tag{15}$$

where $W_1(x)=((\pi x-2x+\pi)\cdot(2x+\pi)\cdot(-768x^2-3\pi^4x^2-24\pi^2x^2+384\pi x^2+\pi^5x-4\pi^4x+8\pi^3x-4\pi^4+\pi^5)\cdot\pi)^2(\pi-2x)$, $B_{1,i}(x)=\frac{C_9^i\cdot x^i\cdot(x_1-x)^{9-i}}{(x_1-0)^9}$, and $\gamma_{1,0}\approx-3.5\cdot 10^7<0$, $\gamma_{1,1}\approx-3.2\cdot 10^7<0$, $\gamma_{1,2}\approx-2.9\cdot 10^7<0$, $\gamma_{1,3}\approx-2.4\cdot 10^7<0$, $\gamma_{1,4}\approx-2.0\cdot 10^7<0$, $\gamma_{1,5}\approx-1.5\cdot 10^7<0$, $\gamma_{1,6}\approx-1.1\cdot 10^7<0$, $\gamma_{1,7}\approx-7.7\cdot 10^6<0$, $\gamma_{1,8}\approx-4.5\cdot 10^6<0$, $\gamma_{1,9}\approx-1.7\cdot 10^6<0$. Note that $W_1(x)>0$, $\forall x\in[0,\pi/2)$ and $B_{1,i}(x)\ge 0$, $\forall x\in[0,x_1]$, and from Eq. (15) we have that $D_3'(x)\le 0$, $\forall x\in[0,x_1]$. So

$$D_3(x)\le D_3(0)=0. \tag{16}$$

Combining Eq. (16) with $\theta_1'(x)>0$, we have that $D_2(x)\le 0$, $\forall x\in(0,x_1]$. Combining with Eq. (14) yields

$$D_1'(x)\le D_2(x)\le 0,\quad \forall x\in(0,x_1]. \tag{17}$$

From Eq. (17), we have that

$$D_1(x)\le D_1(0)=0,\quad \forall x\in(0,x_1]. \tag{18}$$

Case 1.2. Proof of $D_1(x) \le 0$, $\forall x \in (x_1, x_2]$.
For every $x_1 < x \le x_2$, we have $\theta_1'(x) \ge 0$, $\ln(a(x)) \le 0$, $\theta_1(x) \le \theta_1(x_2)$ and

$$D_1(x) \le \ln\bigl(\cos(x)\bigr) - \theta_1(x_2) \cdot \ln\bigl(a(x)\bigr) = D_4(x). \tag{19}$$

Combining with Eq. (5) gives

$$\begin{aligned} D_4'(x) &= -\tan(x) - \theta_1(x_2)\frac{a'(x)}{a(x)} \\ &\le -x \cdot t_1(x) - \theta_1(x_2)\frac{a'(x)}{a(x)} \\ &= \frac{(\sum_{i=0}^{6} \gamma_{2,i} \cdot x^i)}{(\pi x - 2x + \pi)(-2x + \pi)(\pi - 4)^2 \pi^4 (2x + \pi)} = D_5(x), \end{aligned} \tag{20}$$

where $\gamma_{2,6} \approx 1.95 > 0$, $\gamma_{2,5} \approx 5.36 > 0$, $\gamma_{2,4} \approx 56.7 > 0$, $\gamma_{2,3} \approx 156.2 > 0$, $\gamma_{2,2} \approx -265.4 < 0$, $\gamma_{2,1} \approx -1050 < 0$, $\gamma_{2,0} \approx 502.8 > 0$. So for every $x_1 < x \le x_2$,

$$\begin{aligned} \sum_{i=0}^{6} \gamma_{2,i} \cdot x^i &< \gamma_{2,6} \cdot x_2^6 + \gamma_{2,5} \cdot x_2^5 + \gamma_{2,4} \cdot x_2^4 + \gamma_{2,3} \cdot x_2^3 + \gamma_{2,0} + \gamma_{2,2} \cdot x_1^2 + \gamma_{2,1} \cdot x_1 \\ &\approx -29.68 < 0. \end{aligned}$$

Combining with Eq. (20) yields

$$D_4'(x) \le D_5(x) \le 0. \tag{21}$$

Combining with Eqs. (19)–(21), we have that

$$D_1(x) \le D_4(x) \le D_4(x_1) \approx -0.0058 < 0, \quad \forall x \in (x_1, x_2]. \tag{22}$$

Case 1.3. Proof of $D_1(x) \le 0$, $\forall x \in (x_2, \pi/2)$.
Combining with Eq. (5), for every $x_2 < x < \pi/2$, we have

$$\begin{aligned} D_1'(x) &= -\tan(x) - \frac{\theta_1(x) \cdot a'(x)}{a(x)} - \theta_1'(x) \cdot \ln\bigl(a(x)\bigr) \\ &\ge -t_2(x) \cdot x - \frac{\theta_1(x) \cdot a'(x)}{a(x)} - \theta_1'(x) \cdot \ln\bigl(a(x)\bigr) = D_6(x). \end{aligned} \tag{23}$$

Let $D_7(x) = \frac{D_6(x)}{\theta_1'(x)}$. It can be verified that

$$D_7'(x) = \frac{(\sum_{i=0}^{7} \gamma_{3,i} B_{2,i}(x)) \cdot x^2}{W_1(x)}, \tag{24}$$

where $B_{2,i}(x) = \frac{C_7^i \cdot (x - x_2)^i \cdot (\pi/2 - x)^{7-i}}{(\pi/2 - x_2)^7} \ge 0$, $\forall x \in [x_2, \pi/2)$, $\gamma_{3,0} \approx 8.8 \cdot 10^6$, $\gamma_{3,1} \approx 9.7 \cdot 10^6$, $\gamma_{3,2} \approx 1.0 \cdot 10^7$, $\gamma_{3,3} \approx 1.1 \cdot 10^7$, $\gamma_{3,4} \approx 1.3 \cdot 10^7$, $\gamma_{3,5} \approx 1.4 \cdot 10^7$, $\gamma_{3,6} \approx 1.6 \cdot 10^7$ and $\gamma_{3,7} \approx 1.8 \cdot 10^7$, such that $\gamma_{3,i} > 0$, $i = 0, 1, \ldots, 7$, and $D_7'(x) \ge 0$, $\forall x \in (x_2, \pi/2)$. So we have

$$D_7(x) \ge D_7(x_2) \approx 6.7 > 0, \quad \forall x \in (x_2, \pi/2). \tag{25}$$

Note that $\theta_1'(x) > 0$, $\forall x \in (0, \pi/2)$, and, combining with Eqs. (23)–(25), we have that

$$D_1'(x) \geq D_6(x) \geq 0, \quad \forall x \in (x_2, \pi/2). \tag{26}$$

From Eq. (26), we obtain

$$D_1(x) \leq D_1(\pi/2) = 0, \quad \forall x \in (x_2, \pi/2). \tag{27}$$

Using Eqs. (18), (22) and (27), we have completed the proof of Eq. (13).

Step 2. Now we prove that for every $0 < x < \pi/2$,

$$-\ln\bigl(\cos(x)\bigr) \leq -\theta_2(x) \cdot \ln\bigl(a(x)\bigr),$$

which is equivalent to

$$E_1(x) = \ln\bigl(\cos(x)\bigr) - \theta_2(x) \cdot \ln\bigl(a(x)\bigr) \geq 0. \tag{28}$$

Let $x_3 = \frac{98\cdot\pi}{256} \approx 1.202$.

Case 2.1. Proof of $E_1(x) \geq 0$, $\forall x \in (0, 1]$.

It can be verified that for every $x \in (0, \pi/2)$,

$$\begin{cases} 0 < a(x) \leq 1, \\ \ln(a(x)) \leq (a(x) - 1) - \frac{(a(x)-1)^2}{2} + \frac{(a(x)-1)^3}{3}. \end{cases} \tag{29}$$

Combining Eq. (29) with Eq. (5), for every $0 < x \leq 1$, we have that

$$\begin{aligned} E_1'(x) &= -\tan(x) - \frac{\theta_2(x) \cdot a'(x)}{a(x)} - \theta_2'(x) \cdot \ln\bigl(a(x)\bigr) \\ &\geq -t_2(x) \cdot x - \frac{\theta_2(x) \cdot a'(x)}{a(x)} - \theta_2'(x) \cdot \ln\bigl(a(x)\bigr) \\ &\geq -t_2(x) \cdot x - \frac{\theta_2(x) \cdot a'(x)}{a(x)} - \theta_2'(x) \cdot \left(\bigl(a(x) - 1\bigr) - \frac{(a(x) - 1)^2}{2} + \frac{(a(x) - 1)^3}{3}\right) \\ &= \frac{(\sum_{i=0}^{8} \gamma_{4,i} B_{3,i}(x)) \cdot x^2}{((\pi - 2)x + \pi)(2x + \pi)}, \end{aligned} \tag{30}$$

where $B_{3,i}(x) = C_8^i x^i (1 - x)^{8-i}$, $\gamma_{4,0} \approx 5.54$, $\gamma_{4,1} \approx 5.28$, $\gamma_{4,2} \approx 4.91$, $\gamma_{4,3} \approx 4.43$, $\gamma_{4,4} \approx 3.84$, $\gamma_{4,5} \approx 3.16$, $\gamma_{4,6} \approx 2.38$, $\gamma_{4,7} \approx 1.52$ and $\gamma_{4,8} \approx 0.632$. In Eq. (30), for every $x \in (0, 1)$, we have $B_{3,i}(x) > 0$, $\gamma_{4,i} > 0$ and $((\pi - 2)x + \pi)(2x + \pi) > 0$, which means that

$$E_1'(x) \geq 0, \quad \forall x \in [0, 1]. \tag{31}$$

From Eq. (31), we obtain

$$E_1(x) \geq E_1(0) = 0, \quad \forall x \in [0, 1]. \tag{32}$$

Case 2.2. Proof of $E_1(x) \geq 0$, $\forall x \in (1, x_3]$.

For every $1 < x \leq x_3$, note that $\theta_2'(x) > 0$ and $\ln(a(x)) < 0$, hence

$$E_1(x) \geq \ln\bigl(\cos(x)\bigr) - \theta_2(1) \cdot \ln\bigl(a(x)\bigr) = E_2(x). \tag{33}$$

Combining with Eq. (5), we have that

$$\begin{aligned} E_2'(x) &= -\tan(x) - \theta_2(1) \cdot \frac{a'(x)}{a(x)} \\ &\leq -x \cdot t_1(x) - \theta_2(1) \cdot \frac{a'(x)}{a(x)} \\ &= \frac{(\sum_{i=0}^{6} \gamma_{5,i} \cdot B_{4,i}(x))}{(\pi x - 2x + \pi)(-2x + \pi)(2x + \pi)} = E_3(x), \end{aligned} \tag{34}$$

where $B_{4,i}(x) = \frac{C_6^i (x-1)^i (x_3-x)^{6-i}}{(x_3-1)^6}$, $\gamma_{5,0} \approx -3.98$, $\gamma_{5,1} \approx -4.23$, $\gamma_{5,2} \approx -4.45$, $\gamma_{5,3} \approx -4.65$, $\gamma_{5,4} \approx -4.81$, $\gamma_{5,5} \approx -4.94$ and $\gamma_{5,6} \approx -5.03$. In Eq. (34), for every $1 < x < x_3$, we have $\gamma_{5,i} < 0$, $B_{4,i}(x) > 0$ and $(\pi x - 2x + \pi)(-2x + \pi)(2x + \pi) > 0$, which means that $E_3(x) \leq 0$, $\forall x \in (1, x_3)$. Combining with Eq. (33), we get

$$E_2'(x) \leq E_3(x) < 0, \quad \forall x \in (1, x_3). \tag{35}$$

Combining Eq. (35) with Eq. (33) yields

$$E_1(x) \geq E_2(x) \geq E_2(x_3) \approx 0.0026 > 0, \quad \forall x \in (1, x_3). \tag{36}$$

Case 2.3. Proof of $E_1(x) \geq 0$, $\forall x \in (x_3, \pi/2)$.
Combining with Eq. (5), for every $x_3 < x < \pi/2$, we have

$$\begin{aligned} E_1'(x) &= -\tan(x) - \frac{\theta_2(x) \cdot a'(x)}{a(x)} - \theta_2'(x) \cdot \ln\bigl(a(x)\bigr) \\ &< -t_1(x) \cdot x - \frac{\theta_2(x) \cdot a'(x)}{a(x)} - \theta_2'(x) \cdot \ln\bigl(a(x)\bigr) = E_4(x). \end{aligned} \tag{37}$$

Let $E_5(x) = \frac{E_4(x)}{\theta_2'(x)}$. It can be verified that

$$E_5'(x) = \frac{\sum_{i=0}^{8} \gamma_{6,i} B_{5,i}(x)}{W_2(x)}, \tag{38}$$

where $W_2(x) = ((\pi x - 2x + \pi)(-2x + \pi)(2x + \pi)(-48\pi x + 192x - 6\pi^2 x + \pi^3))^2 \geq 0$, $B_{5,i}(x) = \frac{C_8^i \cdot (x-x_3)^i \cdot (\pi/2-x)^{8-i}}{(\pi/2-x_3)^8} \geq 0$, $\forall x \in [x_3, \pi/2)$, $\gamma_{6,0} \approx -1.34 \cdot 10^5$, $\gamma_{6,1} \approx -1.46 \cdot 10^5$, $\gamma_{6,2} \approx -1.54 \cdot 10^5$, $\gamma_{6,3} \approx -1.5 \cdot 10^5$, $\gamma_{6,4} \approx -1.5 \cdot 10^5$, $\gamma_{6,5} \approx -1.4 \cdot 10^5$, $\gamma_{6,6} \approx -1.25 \cdot 10^5$, $\gamma_{6,7} \approx -98875$ and $\gamma_{6,8} \approx -64163$. In Eq. (38), for every $x_3 < x < \pi/2$, noting that $W_2(x) > 0$, $\gamma_{6,i} < 0$ and $B_{5,i}(x) > 0$, we have

$$E_5'(x) \leq 0, \quad \forall x \in (x_3, \pi/2). \tag{39}$$

From Eq. (39), we obtain

$$E_5(x) \leq E_5(x_3) \approx -0.25 < 0, \quad \forall x \in (x_3, \pi/2). \tag{40}$$

Combining Eq. (40) with Eq. (37), for every $x \in (x_3, \pi/2)$, and noting that $\theta_2'(x) > 0$, we get

$$\begin{cases} E_4(x) < 0, & \forall x \in (x_3, \pi/2), \\ E_1(x) > E_1(\pi/2) = 0, & \forall x \in (x_3, \pi/2). \end{cases} \tag{41}$$

Using Eqs. (32), (36) and (41), we have completed the proof of Eq. (28).

Now combining Eq. (13) with Eq. (28), we have completed the proof of Eq. (10), and hence of Theorem 1.1. □

3 Proof of Theorem 1.2

3.1 Lemmas

We recall Theorem 3.5.1 in [9, Chap. 3.5, p. 67] as follows.

Theorem 3.1 *Let $w_0, w_1, \dots, w_r$ be $r+1$ distinct points in $[a,b]$, and $n_0, \dots, n_r$ be $r+1$ integers ≥ 0. Let $N = n_0 + \cdots + n_r + r$. Suppose that $g(t)$ is a polynomial of degree N such that $g^{(i)}(w_j) = f^{(i)}(w_j)$, $i = 0, \dots, n_j$, $j = 0, \dots, r$. Then there exists $\xi_1(t) \in [a,b]$ such that $f(t) - g(t) = \frac{f^{(N+1)}(\xi_1(t))}{(N+1)!} \prod_{i=0}^{r} (t - w_i)^{n_i+1}$.*

Lemma 3.2 *For every $0 < x < \pi/2$, we have that*

$$\sin(x) \leq \frac{1}{120}x^5 - \frac{1}{6}x^3 + x = c(x).$$

Proof Let $H_1(x) = \sin(x) - (\frac{1}{120}x^5 - \frac{1}{6}x^3 + x)$. It can be verified that $H_1(0) = H_1'(0) = \cdots = H^{(6)}(0) = 0$ and $H^{(7)}(0) = -1 \neq 0$. For every $0 < x < \pi/2$, using Theorem 3.1, there exists $\psi(x) \in (0, \pi/2)$ such that

$$H_1(x) = \frac{H^{(7)}(\psi(x))}{7!}(x-0)^7 = \frac{-\cos(\psi(x))}{7!}x^7 \leq 0,$$

completing the proof. □

Lemma 3.3 *For every $0 < x < \pi/2$, we have*

$$\ln^{(7)}\bigl(\cos(x)\bigr) < 0,$$

where $\ln^{(7)}(\cos(x))$ denotes the seventh derivative.

Proof For every $0 < x < \pi/2$, it can be verified that

$$\begin{aligned} \ln^{(7)}\bigl(\cos(x)\bigr) &= \frac{-16(45 - 30 \cdot \cos(x)^2 + 2 \cdot \cos(x)^4)) \cdot \sin(x)}{(\cos(x))^7} \\ &< \frac{-16(45 - 30) \cdot \sin(x)}{(\cos(x))^7} < 0. \end{aligned}$$

This completes the proof. □

Lemma 3.4 *For every* $0 < x < \pi/2$, *we have*

$$\varphi_1(x) = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \frac{x^8}{315} - \frac{2x^{10}}{14175} + \frac{2x^{12}}{467775} - \frac{4x^{14}}{42567525} \le \cos(x)^2 = \kappa_1(x).$$

Proof For every $0 < x < \pi/2$, it can be verified that $\bar{\kappa}_1^{(i)}(0) = 0$, $i = 0, 1, \ldots, 15$ and $\bar{\kappa}_1^{(16)}(0) = 32768 > 0$, where $\bar{\kappa}_1(x) = \kappa_1(x) - \varphi_1(x)$. Employing Theorem 3.1, for every $0 < x < \pi/2$, there exists $\xi_2(x) \in (0, \pi/2)$ such that

$$\begin{aligned}\kappa_1(x) - \varphi_1(x) &= \frac{\kappa_1^{(16)}(\xi_2(x))}{16!}(x-0)^{16} \\ &= \frac{32768\cos(2\xi_2(x))}{16!}(x-0)^{16}, \quad \forall x \in (0, \pi/2). \end{aligned} \tag{42}$$

Note that $\bar{\kappa}_1(\pi/2) = \frac{32768\cos(2\xi_2(\pi/2))}{16!}(\pi/2)^{16} \approx 0.0000020 > 0$, and, on the other hand, $\cos(2x) > 0$, $\forall x \in (0, \pi/4)$ and $\cos(2x) < 0$, $\forall x \in (\pi/4, \pi/2)$, hence we have that $\xi_2(\pi/2) \in (0, \pi/4)$ and then $\xi_2(x) \in (\xi_2(0), \xi_2(\pi/2)) \in (0, \pi/4)$. Combining with Eq. (42), we get $\kappa_1(x) - \varphi_1(x) > 0$, $\forall x \in (0, \pi/2)$, completing the proof. □

Lemma 3.5 *For every* $0 < x < \pi/2$, *we have*

$$\ln^{(6)}\left(\frac{x}{\sin(x)}\right) > 0.$$

Proof It can be verified that

$$\ln^{(6)}\left(\frac{x}{\sin(x)}\right) = \frac{8 \cdot \kappa_2(x)}{\sin(x)^6 x^6}, \tag{43}$$

where $\kappa_2(x) = (2\cos(x)^4 + 11\cos(x)^2 + 2) + \cos(x)^2(45 - 45\cos(x)^2 + 15\cos(x)^4) - 15$. Combining with Lemma 3.4, we have

$$\kappa_2(x) \ge \left(2\varphi_1(x)^2 + 11\varphi_1(x) + 2\right) + \varphi_1(x)\left(45 - 45\varphi_1(x) + 15\varphi_1(x)^2\right) - 15 = \kappa_3(x),$$

where $\kappa_3(x) = \frac{x^{10}}{514214051692706052187 5}\bar{\kappa}_3(x)$, and

$$\begin{aligned}\bar{\kappa}_3(x) &= -64x^{30} + 8736x^{28} - 685776x^{26} + 38749256x^{24} - 1619962344x^{22} \\ &\quad + 53283946716x^{20} - 1386097749036x^{18} + 28235771493930x^{16} \\ &\quad - 435529376632350x^{14} + 4560089491932975x^{12} - 17910555851588400x^{10} \\ &\quad - 349231679183512125x^8 + 7231910390065486125x^6 \\ &\quad - 58866255075932679375x^4 + 179566811702214811875x^2 \\ &\quad + 163242556092922556250 \\ &\ge \left(-64\left(\frac{\pi}{2}\right)^2 + 8736\right)x^{28} + \left(-685776\left(\frac{\pi}{2}\right)^2 + 38749256\right)x^{24} \\ &\quad + \left(-1619962344\left(\frac{\pi}{2}\right)^2 + 53283946716\right)x^{20}\end{aligned}$$

$$
\begin{aligned}
&+\left(-1386097749036\left(\frac{\pi}{2}\right)^2+28235771493930\right)x^{16}\\
&+\left(-435529376632350\left(\frac{\pi}{2}\right)^2+4560089491932975\right)x^{12}\\
&+\left(-17910555851588400\left(\frac{\pi}{2}\right)^4+7231910390065486125\right)x^6\\
&+\left(-349231679183512125\left(\frac{\pi}{2}\right)^6-58866255075932679375\left(\frac{\pi}{2}\right)^2\right.\\
&\left.+179566811702214811875\right)x^2+163242556092922556250\\
\approx{}& 8578x^{28}+3.7\times10^7x^{24}+4.9\times10^{10}x^{20}+2.4\times10^{13}x^{16}+3.4\times10^{15}x^{12}\\
&+7.1\times10^{18}x^6+2.9\times10^{19}x^2+1.6\times10^{20}>0,\quad \forall x\in(0,\pi/2).
\end{aligned}
$$

Combining with Eq. (43), for every $0 < x < \pi/2$, and noting that $\sin(x)^6x^6 > 0$, we have $\ln^{(6)}(\frac{x}{\sin(x)}) > 0$, which completes the proof. □

Lemma 3.6 *For every* $0 < x < \pi/3$, *we have*

$$\varphi_2(x) \geq \ln(\cos(x)),$$

where $\varphi_2(x) = \frac{-x^2}{2} - \frac{x^4}{12} + \frac{(162\sqrt{3}\pi+108\pi^2+\pi^4-2916\ln(2))\cdot x^5}{2\pi^5} - \frac{3(324\sqrt{3}\pi+162\pi^2+\pi^4-4860\ln(2))\cdot x^6}{4\pi^6}$.

Proof Let $\kappa_4(x) = \ln(\cos(x)) - \varphi_2(x)$. It can be verified that

$$\kappa_4^{(i)}(0) = 0, \quad i = 0,1,\dots,4, \quad \text{and} \quad \kappa_4^{(j)}(\pi/3) = 0, \quad j = 0,1.$$

Using Theorem 3.1 and Lemma 3.3, for $0 < x < \pi/3$, there exists $\xi_3(x) \in (0,\pi/3)$ such that

$$\kappa_4(x) = \frac{\kappa_4^{(7)}(\xi_3(x))}{7!}(x-\pi/3)^2\cdot x^5 = \frac{\ln^{(7)}(\cos(\xi_3(x)))}{7!}(x-\pi/3)^2\cdot x^5 < 0,$$

which means that $\ln(\cos(x)) - \varphi_2(x) \leq 0$, and we complete the proof. □

Lemma 3.7 *For every* $0 < x < \pi/3$, *we have*

$$\varphi_3(x) = \frac{x^2}{6} + \frac{x^4}{180} - \frac{(-14580\ln(\frac{2\sqrt{3}\cdot\pi}{9}) + 270\pi^2 + \pi^4)\cdot x^5}{60\pi^5} \geq \ln\left(\frac{x}{\sin(x)}\right).$$

Proof Let $\kappa_5(x) = \ln(\frac{x}{\sin(x)}) - \varphi_3(x)$. It can be verified that

$$\kappa_5^{(i)}(0) = 0, \quad i = 0,1,\dots,4, \quad \text{and} \quad \kappa_5(\pi/3) = 0.$$

Now by Theorem 3.1 and Lemma 3.5, for $0 < x < \pi/3$, there exists $\xi_4(x) \in (0,\pi/3)$ such that

$$\kappa_5(x) = \frac{\kappa_5^{(6)}(\xi_4(x))}{6!}(x-\pi/3)\cdot x^5 = \frac{\ln^{(6)}(\frac{\xi_4(x)}{\sin(\xi_4(x))})}{6!}(x-\pi/3))\cdot x^5 < 0,$$

which means that $\ln(\frac{x}{\sin(x)}) - \varphi_3(x) < 0$, $\forall x \in (0,\pi/3)$, and we complete the proof. □

3.2 Proof of Theorem 1.2

Proof of Theorem 1.2 Step 1. Firstly, we prove that $\frac{\sin(x)}{x} \le \cos(x)^{\theta_3(x)}$, $\forall x \in (0, \pi/2)$, where $\theta_3(x) = \frac{1}{3} - \frac{4}{3\pi^2}x^2$. This is equivalent to

$$F_1(x) = \ln\bigl(\sin(x)\bigr) - \ln(x) - \theta_3(x) \cdot \ln\bigl(\cos(x)\bigr) \le 0, \quad \forall x \in (0, \pi/2). \tag{44}$$

Combining with Lemma 3.2, we have that

$$F_1(x) \le \ln\bigl(c(x)\bigr) - \ln(x) - \theta_3(x) \cdot \ln\bigl(\cos(x)\bigr) = F_2(x), \quad \forall x \in (0, \pi/2). \tag{45}$$

For every $0 < x < \pi/2$, noting that $\theta_3'(x) = -\frac{8x}{3\pi^2} < 0$ and $\theta_3(x) > 0$, and combining with Eq. (4), we have

$$\begin{aligned} F_2'(x) &= \frac{c'(x)}{c(x)} - \frac{1}{x} + \theta_3(x) \cdot \tan(x) - \theta_3'(x) \cdot \ln\bigl(\cos(x)\bigr) \\ &\le \frac{c'(x)}{c(x)} - \frac{1}{x} + \theta_3(x) \cdot \tan(x) \\ &< \frac{c'(x)}{c(x)} - \frac{1}{x} + \theta_3(x) \cdot \frac{\pi^2 \cdot x}{\pi^2 - 4x^2} \\ &= \frac{(x^2 - 8) \cdot x^3}{3(x^4 - 20x^2 + 120)} < 0, \quad \forall x \in (0, \pi/2). \end{aligned} \tag{46}$$

Combining Eq. (45) with Eq. (46), we obtain

$$F_1(x) \le F_2(x) < F_2(0) = 0, \quad \forall x \in (0, \pi/2). \tag{47}$$

This completes the proof of Eq. (44), and hence proves $\frac{\sin(x)}{x} \le \cos(x)^{\theta_3(x)}$.

Step 2. Secondly, we prove that $\cos(x)^{\theta_4(x)} \le \frac{\sin(x)}{x}$, $\forall x \in (0, \pi/2)$, where $\theta_4(x) = \frac{1}{3} - \frac{2}{45}x^2 + \frac{5}{124}x^3 - \frac{41}{1000}x^4$. This is equivalent to

$$F_3(x) = \theta_4(x) \cdot \ln\bigl(\cos(x)\bigr) + \ln\left(\frac{x}{\sin(x)}\right) \le 0, \quad \forall x \in (0, \pi/2). \tag{48}$$

Case 2.1. $0 < x < \pi/3$.

Noting that $\theta_4(x) \ge 0$, and combining with Lemmas 3.6 and 3.7, we have that

$$F_3(x) \le \theta_4(x) \cdot \varphi_2(x) + \varphi_3(x) = \left(\sum_{i=0}^{5} \gamma_{7,i} B_{6,i}(x)\right) \cdot x^5 = G_1(x), \tag{49}$$

where $B_{6,i} = \frac{C_5^i \cdot x^i \cdot (\pi/3 - x)^{5-i}}{(\pi/3)^5}$, and $\gamma_{7,0} \approx -0.0068$, $\gamma_{7,1} \approx -0.0067$, $\gamma_{7,2} \approx -0.0071$, $\gamma_{7,3} \approx -0.0072$, $\gamma_{7,4} \approx -0.0070$, $\gamma_{7,5} \approx -0.0041$. Noting that $B_{6,i} \ge 0$ and $\gamma_{7,i} < 0$, $i = 0, 1, \ldots, 5$, and combining with Eq. (49), we obtain

$$F_3(x) \le G_1(x) < 0, \quad \forall x \in (0, \pi/3]. \tag{50}$$

Case 2.2. $\pi/3 < x < 1.36$.

Let $\varphi_4(x)$ be the sextic interpolation polynomial such that

$$\ln^{(i)}\bigl(\cos(\pi/3)\bigr) = \varphi_4^{(i)}(\pi/3), \quad i = 0,1,\ldots,4, \quad \text{and}$$
$$\ln^{(j)}\bigl(\cos(1.36)\bigr) = \varphi_4^{(j)}(1.36), \quad j = 0,1,$$

and $\kappa_6(x) = \ln(\cos(x)) - \varphi_4(x)$. We have that

$$\kappa_6^{(i)}(\pi/3) = 0, \quad i = 0,1,\ldots,4, \quad \text{and} \quad \kappa_6^{(j)}(1.36) = 0, \quad j = 0,1.$$

Similar as in the proof of Lemma 3.6, by Theorem 3.1 and Lemma 3.3, for $\pi/3 < x < 1.36$, there exists $\xi_5(x) \in (\pi/3, 1.36)$ such that

$$\kappa_6(x) = \frac{\ln^{(7)}(\cos(\xi_5(x)))}{7!}(x-1.36)^2 \cdot (x-\pi/3)^5 < 0,$$

which means that $\ln(\cos(x)) - \varphi_4(x) \le 0$.

On the other hand, let $\varphi_5(x)$ be the quintic interpolation polynomial such that

$$\ln^{(i)}\left(\frac{\pi/3}{\sin(\pi/3)}\right) = \varphi_5^{(i)}(\pi/3), \quad i = 0,1,\ldots,4, \quad \text{and} \quad \ln\left(\frac{1.36}{\sin(1.36)}\right) = \varphi_5(1.36).$$

Similarly, let $\kappa_7(x) = \ln(\frac{x}{\sin(x)}) - \varphi_5(x)$, and then, for every $\pi/3 < x < 1.36$, there exists $\xi_6(x) \in (\pi/3, 1.36)$ such that

$$\kappa_7^{(i)}(\pi/3) = 0, \quad i = 0,1,\ldots,4, \quad \text{and} \quad \kappa_7(1.36) = 0,$$
$$\kappa_7(x) = \frac{\ln^{(6)}(\frac{\xi_6(x)}{\sin(\xi_6(x))})}{6!}(x-\pi/3)^5 \cdot (x-1.36) < 0,$$

which means that $\ln(\frac{x}{\sin(x)}) \le \varphi_5(x)$. Finally, for every $\pi/3 < x < 1.36$, we have

$$F_3(x) \le \theta_4(x) \cdot \varphi_4(x) + \varphi_5(x) = \sum_{i=0}^{10} \gamma_{8,i} B_{7,i}(x) = G_2(x), \tag{51}$$

where $B_{7,i}(x) \approx \frac{C_{10}^i \cdot (x-\pi/3)^i \cdot (1.36-x)^{10-i}}{(1.36-\pi/3)^{10}}$, $\gamma_{8,0} \approx -0.0052$, $\gamma_{8,1} \approx -0.0054$, $\gamma_{8,2} \approx -0.0055$, $\gamma_{8,3} \approx -0.0056$, $\gamma_{8,4} \approx -0.0055$, $\gamma_{8,5} \approx -0.0052$, $\gamma_{8,6} \approx -0.0048$, $\gamma_{8,7} \approx -0.0041$, $\gamma_{8,8} \approx -0.0033$, $\gamma_{8,9} \approx -0.0026$ and $\gamma_{8,10} \approx -0.0022$. Noting that $B_{7,i}(x) \ge 0$ and $\gamma_{8,i} < 0$, and combining with Eq. (51), we have

$$F_3(x) \le G_2(x) < 0, \quad \forall x \in (\pi/3, 1.36]. \tag{52}$$

Case 2.3. $1.36 < x < 1.54$.

Let $\varphi_6(x)$ be the sextic interpolation polynomial such that

$$\ln^{(i)}\bigl(\cos(1.36)\bigr) = \varphi_6^{(i)}(\pi/3), \quad i = 0,1,\ldots,4, \quad \text{and}$$
$$\ln^{(j)}\bigl(\cos(1.54)\bigr) = \varphi_6^{(j)}(1.54), \quad j = 0,1,$$

and $\kappa_8(x) = \ln(\cos(x)) - \varphi_6(x)$. We have that

$$\kappa_8^{(i)}(1.36) = 0, \quad i = 0, 1, \dots, 4, \quad \text{and} \quad \kappa_8^{(j)}(1.54) = 0, \quad j = 0, 1.$$

Similar as in the proof of Lemma 3.6, by Theorem 3.1 and Lemma 3.3, for $1.36 < x < 1.54$, there exists $\xi_7(x) \in (1.36, 1.54)$ such that

$$\kappa_8(x) = \frac{\ln^{(7)}(\cos(\xi_7(x)))}{7!}(x - 1.54)^2 \cdot (x - 1.36)^5 < 0,$$

which means that $\ln(\cos(x)) \le \varphi_6(x)$.

On the other hand, let $\varphi_7(x)$ be the quintic interpolation polynomial such that

$$\ln^{(i)}\left(\frac{1.36}{\sin(1.36)}\right) = \varphi_7^{(i)}(1.36), \quad i = 0, 1, \dots, 4, \quad \text{and} \quad \ln\left(\frac{1.54}{\sin(1.54)}\right) = \varphi_7(1.54).$$

Similarly, letting $\kappa_9(x) = \ln(\frac{x}{\sin(x)}) - \varphi_7(x)$, for every $1.36 < x < 1.54$, there exists $\xi_8(x) \in (1.36, 1.54)$ such that

$$\kappa_9^{(i)}(1.36) = 0, \quad i = 0, 1, \dots, 4, \quad \text{and} \quad \kappa_9(1.54) = 0,$$
$$\kappa_9(x) = \frac{\ln^{(6)}(\frac{\xi_8(x)}{\sin(\xi_8(x))})}{6!}(x - 1.36)^5 \cdot (x - 1.54) < 0,$$

which means that $\ln(\frac{x}{\sin(x)}) \le \varphi_7(x)$. Finally, for every $1.36 < x < 1.54$, we have

$$F_3(x) \le \theta_4(x) \cdot \varphi_6(x) + \varphi_7(x) = \sum_{i=0}^{10} \gamma_{9,i} \cdot B_{8,i}(x) = G_3(x), \tag{53}$$

where $B_{8,i}(x) = \frac{C_{10}^i \cdot (x-1.36)^i \cdot (1.54-x)^{10-i}}{(1.54-1.36)^{10}}$, $\gamma_{9,0} \approx -0.0022$, $\gamma_{9,1} \approx -0.0019$, $\gamma_{9,2} \approx -0.0018$, $\gamma_{9,3} \approx -0.0020$, $\gamma_{9,4} \approx -0.0025$, $\gamma_{9,5} \approx -0.0029$, $\gamma_{9,6} \approx -0.0024$, $\gamma_{9,7} \approx -0.0021$, $\gamma_{9,8} \approx -0.0068$, $\gamma_{9,9} \approx -0.025$ and $\gamma_{9,10} \approx -0.071$. Noting that $B_{8,i}(x) > 0$, $\forall x \in (1.36, 1.54)$, and $\gamma_{9,i} < 0$, and combining with Eq. (53), we have that

$$F_3(x) \le G_3(x) < 0, \quad \forall x \in [1.36, 1.54]. \tag{54}$$

Case 2.3. $1.54 < x < \pi/2$.

Let $\varphi_8(x)$ be the quintic interpolation polynomial such that

$$\ln^{(i)}\left(\frac{1.54}{\sin(1.54)}\right) = \varphi_8^{(i)}(1.54), \quad i = 0, 1, \dots, 4, \quad \text{and} \quad \ln\left(\frac{\pi/2}{\sin(\pi/2)}\right) = \varphi_8(\pi/2).$$

Similarly, letting $\kappa_{10}(x) = \ln(\frac{x}{\sin(x)}) - \varphi_8(x)$, for every $1.54 < x < \pi/2$, there exists $\xi_9(x) \in (1.54, \pi/2)$ such that

$$\kappa_{10}^{(i)}(1.54) = 0, \quad i = 0, 1, \dots, 4, \quad \text{and} \quad \kappa_{10}(\pi/2) = 0,$$
$$\kappa_{10}(x) = \frac{\ln^{(6)}(\frac{\xi_9(x)}{\sin(\xi_9(x))})}{6!}(x - 1.54)^5 \cdot (x - \pi/2) < 0,$$

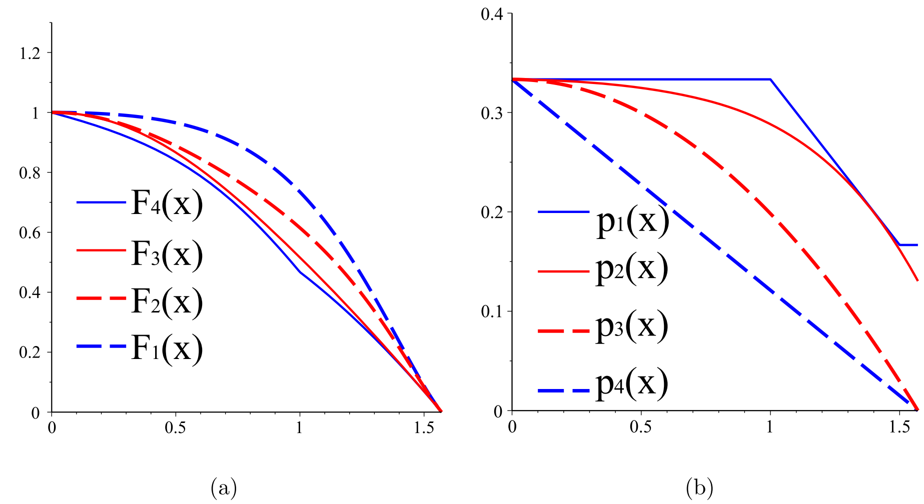

Figure 1 Plots of (**a**) $F_i(x)$ and (**b**) $p_i(x)$, $i = 1, 2, 3, 4$

which means that $\ln(\frac{x}{\sin(x)}) \le \varphi_8(x)$. Finally, for every $1.54 < x < \pi/2$, we have

$$F_3(x) \le \theta_4(x) \cdot \ln\bigl(\cos(1.54)\bigr) + \varphi_8(x) = \sum_{i=0}^{5} \gamma_{10,i} \cdot B_{9,i}(x) = G_4(x), \tag{55}$$

where $B_{9,i}(x) = \frac{C_5^i \cdot (x-1.54)^i \cdot (\pi/2-x)^{5-i}}{(\pi/2-1.54)^5}$, $\gamma_{10,0} = -0.071$, $\gamma_{10,1} = -0.057$, $\gamma_{10,2} = -0.043$, $\gamma_{10,3} = -0.030$, $\gamma_{10,4} = -0.01$ and $\gamma_{10,5} = -0.0020$. Noting that $B_{9,i}(x) > 0$, $\forall x \in (1.54, \pi/2)$, and $\gamma_{10,i} < 0$, and combining with Eq. (55), we have

$$F_3(x) \le G_4(x) < 0, \quad \forall x \in [1.54, \pi/2]. \tag{56}$$

Using Eqs. (50), (52), (54) and (56), we have completed the proof of Eq. (48).

Combining Eqs. (44) and (48), we have completed the proof of Theorem 1.2. □

4 Comparisons and conclusion

Let $f_1(x) = (\frac{4\pi}{\pi^2-4x^2})^{(2x/\pi)^2}$, $f_2(x) = a(x)^{-\theta_1(x)}$, $f_3(x) = a(x)^{-\theta_2(x)}$ and $f_4(x) = (\frac{4\pi}{\pi^2-4x^2})^{\bar{\theta}_3(x)}$, and $F_i(x) = \frac{1}{f_i(x)}$, $i = 1, 2, 3, 4$. As shown in Fig. 1(a), $F_1(x) \ge F_2(x) \ge F_3(x) \ge F_4(x)$, which means that Theorem 1.1 achieves much tighter bounds than those of Eq. (9).

Let $p_1(x) = \bar{\theta}_1(x)/3$, $p_2(x) = \frac{1}{3} - \frac{2}{45}x^2 + \frac{5}{124}x^3 - \frac{41}{1000}x^4$, $p_3(x) = \frac{1}{3} - \frac{4}{3\pi^2}x^2$ and $p_4(x) = \bar{\theta}_2(x)/3$. As shown in Fig. 1(b), we have that $p_1(x) \ge p_2(x) \ge p_3(x) \ge p_4(x)$, $\forall x \in (0, \pi/2)$, combining with $\cos(x) < 1$, $\forall x \in (0, \pi/2)$, we have that

$$\cos(x)^{p_1(x)} \le \cos(x)^{p_2(x)} \le \frac{\sin(x)}{x} \le \cos(x)^{p_3(x)} \le \cos(x)^{p_4(x)}, \quad \forall x \in (0, \pi/2).$$

which means that the bounds in Theorem 1.2 are tighter than in Eq. (8).

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The authors declare that they have no competing interests.

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