Zhao et al. *Journal of Inequalities and Applications* (2019) 2019:42
https://doi.org/10.1186/s13660-019-1991-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On approximating the quasi-arithmetic mean

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Abstract

In this article, we prove that the double inequalities

$$\begin{aligned}&\alpha_1\left[\frac{7C(a,b)}{16}+\frac{9H(a,b)}{16}\right]+(1-\alpha_1)\left[\frac{3A(a,b)}{4}+\frac{G(a,b)}{4}\right]\\&\quad< E(a,b)\\&\quad<\beta_1\left[\frac{7C(a,b)}{16}+\frac{9H(a,b)}{16}\right]+(1-\beta_1)\left[\frac{3A(a,b)}{4}+\frac{G(a,b)}{4}\right],\\&\left[\frac{7C(a,b)}{16}+\frac{9H(a,b)}{16}\right]^{\alpha_2}\left[\frac{3A(a,b)}{4}+\frac{G(a,b)}{4}\right]^{1-\alpha_2}\\&\quad< E(a,b)\\&\quad<\left[\frac{7C(a,b)}{16}+\frac{9H(a,b)}{16}\right]^{\beta_2}\left[\frac{3A(a,b)}{4}+\frac{G(a,b)}{4}\right]^{1-\beta_2}\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 3/16=0.1875$, $\beta_1\geq 64/\pi^2-6=0.484555\ldots$, $\alpha_2\leq 3/16=0.1875$ and $\beta_2\geq(5\log 2-\log 3-2\log\pi)/(\log 7-\log 6)=0.503817\ldots$, where $E(a,b)=(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a\cos^2\theta+b\sin^2\theta}\,d\theta)^2$, $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $A(a,b)=(a+b)/2$ and $C(a,b)=(a^2+b^2)/(a+b)$ are the quasi-arithmetic, harmonic, geometric, arithmetic and contra-harmonic means of a and b, respectively.

MSC: 26E60; 33C05

Keywords: Quasi-arithmetic mean; Harmonic mean; Geometric mean; Arithmetic mean; Contra-harmonic mean

1 Introduction

Let $a,b>0$, $p:(0,\infty)\mapsto(0,\infty)$ be a strictly monotone real-valued function, $\theta\in(0,2\pi)$ and

$$r_n(\theta)=\begin{cases}(a^n\cos^2\theta+b^n\sin^2\theta)^{1/n}, & n\neq 0,\\ a^{\cos^2\theta}b^{\sin^2\theta}, & n=0.\end{cases}\tag{1.1}$$

Then the class of quasi-arithmetic mean [1] is defined by

$$M_{p,n}(a,b)=p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi}p\big(r_n(\theta)\big)\,d\theta\right)$$

$$= p^{-1}\left(\frac{2}{\pi}\int_0^{\pi/2} p\big(r_n(\theta)\big)\,d\theta\right), \tag{1.2}$$

where p^{-1} is the inverse function of p.

Many important means are the special cases of the quasi-arithmetic mean $M_{p,n}(a,b)$. For example, from (1.1) and (1.2) we clearly see that

$$M_{1/x,2}(a,b) = \frac{\pi}{2\int_0^{\pi/2}(a^2\cos^2\theta + b^2\sin^2\theta)^{-1/2}\,d\theta} = AGM(a,b)$$

is the Gaussian arithmetic–geometric mean [2–9], which is related to the complete elliptic integral of the first kind $\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2}(1 - r^2\sin^2\theta)^{-1/2}\,d\theta$ $(0 < r < 1)$,

$$T(a,b) = M_{x,2}(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$$

is the Toader mean [10–12], which can be expressed in terms of the complete elliptic integral of the second kind $\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1 - r^2\sin^2\theta}\,d\theta$ $(0 < r < 1)$, and

$$TQ(a,b) = M_{x,0}(a,b) = \frac{\pi}{2}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta}\,d\theta$$

is the Toader–Qi mean [13–15], which is related to the modified Bessel function of the first kind $I_0(x) = \sum_{n=0}^{\infty}(x/2)^{2n}/(n!)^2$ $(x > 0)$.

It is well-known that $\mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$ and $\mathcal{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1,\pi/2)$. Moreover, $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the following Landen identities and derivative formulas (see [16, Appendix E, pp. 474–475])

$$\begin{aligned}
&\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}, \qquad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E} - r'^2\mathcal{K}}{1+r},\\
&\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},\\
&\frac{d(\mathcal{E} - r'^2\mathcal{K})}{dr} = r\mathcal{K}, \qquad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2}.
\end{aligned}$$

In particular, $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the special cases of the Gaussian hypergeometric function [17–26] as follows:

$$\mathcal{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right), \qquad \mathcal{E}(r) = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right), \tag{1.3}$$

and the Gaussian hypergeometric function $F(a,b;c;x)$ with real parameters a, b, and c $(c \neq 0,-1,-2,\dots)$ is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!} \tag{1.4}$$

for $x \in (-1,1)$, where $(a)_0 = 1$ for $a \neq 0$, $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt$ $(x > 0)$ is the classical gamma function [27–35].

Recently, the bounds for the complete elliptic integrals have attracted the attention of many researchers. In particular, many remarkable inequalities and properties for $\mathcal{K}(r)$, $\mathcal{E}(r)$ and $F(a,b;c;x)$ can be found in the literature [36–66].

In this article, we focus on the special quasi-arithmetic mean $E(a,b)$ obtained by substituting $p=\sqrt{x}$ and $n=1$ into (1.2), more explicitly,

$$E(a,b)=M_{\sqrt{x},1}(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a\cos^2\theta+b\sin^2\theta}\,d\theta\right)^2, \tag{1.5}$$

which can be rewritten in terms of complete elliptic integral of the second kind as

$$E(a,b)=\begin{cases}\frac{4a\mathcal{E}(\sqrt{1-b/a})^2}{\pi^2}, & a\ge b,\\ \frac{4b\mathcal{E}(\sqrt{1-a/b})^2}{\pi^2}, & a<b.\end{cases} \tag{1.6}$$

Very recently, Meng [67], and Yuan, Yu and Wang [68] proved that the double inequalities

$$\lambda_1A(a,b)+(1-\lambda_1)G(a,b)<E(a,b)<\mu_1A(a,b)+(1-\mu_1)G(a,b), \tag{1.7}$$

$$\lambda_2C(a,b)+(1-\lambda_2)H(a,b)<E(a,b)<\mu_2C(a,b)+(1-\mu_2)H(a,b) \tag{1.8}$$

hold for $a,b>0$ with $a\neq b$ if and only if $\lambda_1\le 3/4$, $\mu_1\ge 8/\pi^2$, $\lambda_2\le 4/\pi^2$ and $\mu_2\ge 7/16$, where $A(a,b)=(a+b)/2$, $G(a,b)=\sqrt{ab}$, $H(a,b)=2ab/(a+b)$ and $C(a,b)=(a^2+b^2)/(a+b)$ are the arithmetic, geometric, harmonic and contra-harmonic means of a and b, respectively.

Qian and Chu [69] showed that the double inequality

$$\begin{aligned}&G^p\bigl[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a\bigr]A^{1-p}(a,b)\\&\quad<E(a,b)\\&\quad<G^p\bigl[\mu a+(1-\mu)b,\mu b+(1-\mu)a\bigr]A^{1-p}(a,b)\end{aligned}$$

holds for any $p\in[1,\infty)$ and all $a,b>0$ with $a\neq b$ if and only if $\lambda\le 1/2-\sqrt{1-(2\sqrt{2}/\pi)^{4/p}}/2$ and $\mu\ge 1/2-\sqrt{p}/(4p)$.

From (1.7) and (1.8) we clearly see that

$$\frac{3A(a,b)}{4}+\frac{G(a,b)}{4}<E(a,b)<\frac{7C(a,b)}{16}+\frac{9H(a,b)}{16} \tag{1.9}$$

for $a,b>0$ with $a\neq b$.

We define

$$M_1(a,b)=\frac{3A(a,b)+G(a,b)}{4},\qquad M_2(a,b)=\frac{7C(a,b)+9H(a,b)}{16}. \tag{1.10}$$

Motivated by inequality (1.9), it is natural to ask what are the best possible parameters $\alpha_i,\beta_i\in(0,1)$ $(i=1,2)$ such that the double inequalities

$$\alpha_1M_2(a,b)+(1-\alpha_1)M_1(a,b)<E(a,b)<\beta_1M_2(a,b)+(1-\beta_1)M_1(a,b),$$

$$M_2(a,b)^{\alpha_2}M_1(a,b)^{1-\alpha_2} < E(a,b) < M_2(a,b)^{\beta_2}M_1(a,b)^{1-\beta_2}$$

hold for all $a,b>0$ with $a\neq b$? The main purpose of this article is to answer this question.

2 Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 (See [16, Theorem 1.25]) *Let $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (See [70]) *Suppose that the power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n$ have the radius of convergence $r>0$ with $b_n>0$ for all $n\in\{0,1,2,\ldots\}$. If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for all $n>0$, then $f(x)/g(x)$ is strictly increasing (decreasing) on $(0,r)$.*

Lemma 2.3 *The following assertions hold true:*

(1) *The function $r\to(\mathcal{E}-r'^2\mathcal{K})/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$;*
(2) *The function $r\to 2\mathcal{E}-r'^2\mathcal{K}$ is strictly increasing from $(0,1)$ onto $(\pi/2,2)$;*
(3) *The function $r\to[\mathcal{K}-\mathcal{E}-(\mathcal{E}-r'^2\mathcal{K})]/r^4$ is strictly increasing from $(0,1)$ onto $(\pi/16,+\infty)$.*

Proof Parts (1) and (2) can be found in the literature [16, Theorem 3.21(1) and Exercise 3.43(13)].

For part (3), we clearly see that

$$\frac{\mathcal{K}-\mathcal{E}-(\mathcal{E}-r'^2\mathcal{K})}{r^4}=\frac{\mathcal{K}-\mathcal{E}-(\mathcal{E}-r'^2\mathcal{K})}{(\mathcal{E}-r'^2\mathcal{K})^2}\cdot\left(\frac{\mathcal{E}-r'^2\mathcal{K}}{r^2}\right)^2.$$

Therefore, part (3) follows easily from part (1) and [16, Exercise 3.43(25)]. □

Lemma 2.4 *The function*

$$f(r)=\frac{8/\pi^2(1+r^2)(2\mathcal{E}-r'^2\mathcal{K})^2-(r^2+1)(r^2+2)}{r^4}$$

is strictly increasing from $(0,1)$ onto $(3/16, 64/\pi^2-6)$.

Proof Let $f_1(r)=8/\pi^2(1+r^2)(2\mathcal{E}-r'^2\mathcal{K})^2-(r^2+1)(r^2+2)$ and $f_2(r)=r^4$, then $f_1(0^+)=f_2(0^+)=0$ and $f(r)=f_1(r)/f_2(r)$.

A simple calculation yields

$$\frac{f_1'(r)}{f_2'(r)}=\frac{f_{11}(r)}{f_{22}(r)}, \tag{2.1}$$

where

$$f_{11}(r) = 16\big(2\mathcal{E} - r'^2\mathcal{K}\big)^2 + 16\big(1+r^2\big)\big(2\mathcal{E} - r'^2\mathcal{K}\big)\big(\mathcal{E} - r'^2\mathcal{K}\big)/r^2 - \big(4r^2+6\big),$$
$$f_{22}(r) = 4r^2.$$

Moreover,

$$f_{11}\big(0^+\big) = f_{22}\big(0^+\big) = 0, \tag{2.2}$$

$$\begin{aligned}\frac{f_{11}'(r)}{f_{22}'(r)} &= 8\big(2\mathcal{E} - r'^2\mathcal{K}\big)\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2} + 2\big(1+r^2\big)\left(\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2}\right)^2 \\ &\quad + 2\big(1+r^2\big)\big(2\mathcal{E} - r'^2\mathcal{K}\big)\frac{\mathcal{K} - \mathcal{E} - (\mathcal{E} - r'^2\mathcal{K})}{r^4} - 1.\end{aligned} \tag{2.3}$$

From Lemma 2.3 and (2.3), we clearly see that $f_{11}'(r)/f_{22}'(r)$ is strictly increasing on $(0,1)$. Equations (2.1)–(2.2) and Lemma 2.1 lead to the conclusion that $f(r)$ is strictly increasing on $(0,1)$.

Therefore, Lemma 2.4 follows from the monotonicity of $f(r)$, together with the facts that $f(0^+) = 3/16$ and $f(1^-) = 64/\pi^2 - 6$. □

Lemma 2.5 *The function*

$$g(r) = \frac{(2r^6 + 5r^4 + 5r^2 + 2)[2(\mathcal{E} - r'^2\mathcal{K}) - r^2\mathcal{E}]}{r^4(3r^2+4)(2\mathcal{E} - r'^2\mathcal{K})}$$

is strictly increasing from $(0,1)$ *onto* $(3/16,1)$.

Proof Let $g_1(r) = (2r^6 + 5r^4 + 5r^2 + 2)[2(\mathcal{E} - r'^2\mathcal{K}) - r^2\mathcal{E}]$ and $g_2(r) = r^4(3r^2+4)(2\mathcal{E} - r'^2\mathcal{K})$, then $g(r) = g_1(r)/g_2(r)$.

Making use of (1.3) and (1.4), we get

$$\frac{2}{\pi}\big[2\big(\mathcal{E} - r'^2\mathcal{K}\big) - r^2\mathcal{E}\big] = \sum_{n=0}^{\infty}\frac{3(\frac{1}{2},n)(\frac{1}{2},n+1)}{2n!(n+2)!}r^{2n+4}, \tag{2.4}$$

$$\frac{2}{\pi}\big(2\mathcal{E} - r'^2\mathcal{K}\big) = 1 + \sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)^2}{4[(n+1)!]^2}r^{2n+2}. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}\frac{2}{\pi}g_1(r) &= \big(2r^6 + 5r^4 + 5r^2 + 2\big)\sum_{n=0}^{\infty}\frac{3(\frac{1}{2},n)(\frac{1}{2},n+1)}{2n!(n+2)!}r^{2n+4} \\ &= \sum_{n=0}^{\infty}\frac{3(\frac{1}{2},n)(\frac{1}{2},n+1)}{n!(n+2)!}r^{2n+4} + \sum_{n=0}^{\infty}\frac{15(\frac{1}{2},n)(\frac{1}{2},n+1)}{2n!(n+2)!}r^{2n+6} \\ &\quad + \sum_{n=0}^{\infty}\frac{15(\frac{1}{2},n)(\frac{1}{2},n+1)}{2n!(n+2)!}r^{2n+8} + \sum_{n=0}^{\infty}\frac{3(\frac{1}{2},n)(\frac{1}{2},n+1)}{n!(n+2)!}r^{2n+10} \\ &= r^4\left(\frac{3}{4} + \frac{33}{16}r^2 + \frac{1245}{512}r^4 + \sum_{n=0}^{\infty}\widetilde{A}_n r^{2n+6}\right)\end{aligned}$$

$$= r^4\sum_{n=0}^{\infty}A_n r^{2n} \tag{2.6}$$

and

$$\begin{aligned}\frac{2}{\pi}g_2(r) &= r^4\left(3r^2+4\right)\left(1+\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)^2}{4[(n+1)!]^2}r^{2n+2}\right)\\ &= r^4\left(4+3r^2+\sum_{n=0}^{\infty}\frac{(\frac{1}{2},n)^2}{[(n+1)!]^2}r^{2n+2}+\sum_{n=0}^{\infty}\frac{3(\frac{1}{2},n)^2}{4[(n+1)!]^2}r^{2n+4}\right)\\ &= r^4\left(4+4r^2+\frac{13}{16}r^4+\sum_{n=0}^{\infty}\widetilde{B}_n r^{2n+6}\right)\\ &= r^4\sum_{n=0}^{\infty}B_n r^{2n},\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}&A_0=\frac{3}{4},\qquad A_1=\frac{33}{16},\qquad A_2=\frac{1245}{512},\qquad A_n=\widetilde{A}_{n-3}\quad (n\geq 3),\\ &B_0=4,\qquad B_1=4,\qquad B_2=\frac{13}{16},\qquad B_n=\widetilde{B}_{n-3}\quad (n\geq 3),\\ &\widetilde{A}_n=\frac{3(\frac{1}{2},n)(\frac{1}{2},n+1)}{64(n+3)!(n+5)!}\big(45{,}765+152{,}928n+192{,}838n^2\\ &\qquad +120{,}672n^3+40{,}024n^4+6720n^5+448n^6\big),\\ &\widetilde{B}_n=\frac{(\frac{1}{2},n+1)^2(7n^2+30n+36)}{[4(n+3)!]^2}\end{aligned}$$

for $n \geq 0$.

It follows from (2.6) and (2.7) that

$$g(r)=\frac{\sum_{n=0}^{\infty}A_n r^{2n}}{\sum_{n=0}^{\infty}B_n r^{2n}} \tag{2.8}$$

for $r \in (0,1)$.

In order to prove the monotonicity of $g(r)$, Lemma 2.2 and (2.8) enable us to conclude that it suffices to show the monotonicity of $\{A_n/B_n\}_{n=0}^{\infty}$.

A simple calculation leads to

$$\frac{A_0}{B_0}=\frac{3}{16},\qquad \frac{A_1}{B_1}=\frac{33}{64},\qquad \frac{A_2}{B_2}=\frac{1245}{416},\qquad \frac{A_3}{B_3}=\frac{3051}{128} \tag{2.9}$$

and

$$\begin{aligned}\frac{A_{n+3}}{B_{n+3}}=\frac{\widetilde{A}_n}{\widetilde{B}_n}&=\frac{3}{8(n+4)(n+5)(2n+1)(36+30n+7n^2)}\big(45{,}765+152{,}928n\\ &\quad +192{,}838n^2+120{,}672n^3+40{,}024n^4+6720n^5+448n^6\big),\end{aligned}$$

$$\frac{\widetilde{A}_{n+1}}{\widetilde{B}_{n+1}} - \frac{\widetilde{A}_n}{\widetilde{B}_n} = \frac{3\Delta_1(n)}{8\Delta_2(n)} > 0, \tag{2.10}$$

for $n \geq 0$, where

$$\begin{aligned}\Delta_1(n) &= 20{,}417{,}670 + 119{,}034{,}009n + 234{,}552{,}870n^2 \\ &\quad + 238{,}084{,}434n^3 + 144{,}127{,}820n^4 \\ &\quad + 55{,}145{,}420n^5 + 13{,}474{,}832n^6 + 2{,}036{,}720n^7 + 172{,}928n^8 + 6272n^9, \\ \Delta_2(n) &= (n+4)(n+5)(n+6)(2n+1)(2n+3)\bigl(36+30n+7n^2\bigr)\bigl(73+44n+7n^2\bigr).\end{aligned}$$

It follows from Lemma 2.2 and (2.8)–(2.10) that $g(r)$ is strictly increasing on $(0,1)$. Therefore, Lemma 2.5 follows easily from the monotonicity of $g(r)$, together with the facts that $g(0^+) = A_0/B_0 = 3/16$ and $g(1^-) = 1$. □

3 Main results

Theorem 3.1 *The double inequality*

$$\alpha_1 M_2(a,b) + (1-\alpha_1)M_1(a,b) < E(a,b) < \beta_1 M_2(a,b) + (1-\beta_1)M_1(a,b)$$

holds for $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \leq 3/16$ and $\beta_1 \geq 64/\pi^2 - 6$.

Proof Since $M_1(a,b), M_2(a,b)$ and $E(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b > 0$. Let $r = (1-\sqrt{b/a})/(1+\sqrt{b/a}) \in (0,1)$, then (1.6) and (1.10), together with Landen identities, lead to

$$E(a,b) = A(a,b)\frac{4(1+r)^2}{\pi^2(1+r^2)}\mathcal{E}^2\left(\frac{2\sqrt{r}}{1+r}\right) = A(a,b)\frac{4}{\pi^2}\frac{(2\mathcal{E}-r'^2\mathcal{K})^2}{1+r^2}, \tag{3.1}$$

$$M_1(a,b) = A(a,b)\frac{r^2+2}{2(1+r^2)}, \qquad M_2(a,b) = A(a,b)\frac{2+3r^2+2r^4}{2(1+r^2)^2} \tag{3.2}$$

and

$$\begin{aligned}&E(a,b) - pM_2(a,b) - (1-p)M_1(a,b) \\ &\quad = A(a,b)\left[\frac{4}{\pi^2}\frac{(2\mathcal{E}-r'^2\mathcal{K})^2}{1+r^2} - p\frac{2+3r^2+2r^4}{2(1+r^2)^2} - (1-p)\frac{r^2+2}{2(1+r^2)}\right] \\ &\quad = \frac{A(a,b)r^4}{2(1+r^2)^2}\bigl[f(r)-p\bigr],\end{aligned} \tag{3.3}$$

where $f(r)$ is defined as in Lemma 2.4.

Therefore, Theorem 3.1 follows from Lemma 2.4 and (3.3) immediately. □

Theorem 3.2 *The double inequality*

$$M_2(a,b)^{\alpha_2}M_1(a,b)^{1-\alpha_2} < E(a,b) < M_2(a,b)^{\beta_2}M_1(a,b)^{1-\beta_2}$$

holds for $a,b>0$ with $a \neq b$ if and only if $\alpha_2 \leq 3/16$ and $\beta_2 \geq \log[32/(3\pi^2)]/\log(7/6)$.

Proof Without loss of generality, we may assume that $a > b > 0$. Let $r = (1 - \sqrt{b/a})/(1 + \sqrt{b/a}) \in (0,1)$, then (3.1) and (3.2) lead to

$$\begin{aligned} &\log E(a,b) - \lambda \log M_2(a,b) - (1-\lambda)\log M_1(a,b) \\ &\quad = \log\frac{8}{\pi^2} + \log\frac{(2\mathcal{E} - r'^2\mathcal{K})^2}{r^2+2} - \lambda\log\frac{2r^4+3r^2+2}{(r^2+1)(r^2+2)} \\ &\quad \triangleq \varphi(r). \end{aligned} \tag{3.4}$$

Elaborated computations lead to

$$\varphi(0) = 0, \qquad \varphi(1) = \log\frac{32}{3\pi^2} - \lambda\log\frac{7}{6}, \tag{3.5}$$

$$\varphi'(r) = \frac{2r(3r^2+4)}{(r^2+1)(r^2+2)(2r^4+3r^2+2)}\bigl[g(r) - \lambda\bigr], \tag{3.6}$$

where $g(r)$ is defined as in Lemma 2.5.

We divide the proof into three cases.

Case 1. $\lambda_1 = 3/16$. We clearly see from Lemma 2.5 that

$$g(r) > \lambda_1 \tag{3.7}$$

for $r \in (0,1)$. It follows from (3.5)–(3.7) that $\varphi(r) > 0$ for $r \in (0,1)$. This, in conjunction with (3.4), yields

$$E(a,b) > M_2(a,b)^{\lambda_1} M_1(a,b)^{1-\lambda_1}$$

for all $a, b > 0$ with $a \neq b$.

Case 2. $\lambda_2 = \log[32/(3\pi^2)]/\log(7/6)$. It follows from Lemma 2.5 that there exists $\delta \in (0,1)$ such that $g(r) < \lambda_2$ for $r \in (0,\delta)$ and $g(r) > \lambda_2$ for $r \in (\delta,1)$. This, in conjunction with (3.6), implies that $\varphi(r)$ is strictly decreasing on $(0,\delta)$ and is strictly increasing on $(\delta,1)$. Moreover, we clearly see from (3.5) that

$$\varphi(0) = \varphi(1) = 0. \tag{3.8}$$

The piecewise monotonicity property of $g(r)$ and (3.8) lead to the conclusion that $\varphi(r) < 0$ for $r \in (0,1)$. Therefore,

$$E(a,b) < M_2(a,b)^{\lambda_2} M_1(a,b)^{1-\lambda_2}$$

for all $a, b > 0$ with $a \neq b$ follows from (3.4).

Case 3. $3/16 < \lambda_3 < \log[32/(3\pi^2)]/\log(7/6)$. By the locally sign-preserving property of limit, Lemma 2.5 and (3.6) enable us to know that there exists $\tau_1 \in (0,1)$ such that $\varphi(r)$ is strictly decreasing on $(0,\tau_1)$. This, in conjunction with (3.5), implies that $\varphi(r) < 0$ for

$0 < r < \tau_1$. Therefore,

$$E(a,b) < M_2(a,b)^{\lambda_3} M_1(a,b)^{1-\lambda_3}$$

for $b < a < [(1+\tau_1)/(1-\tau_1)]^2 b$ follows from (3.4).

On the other hand, we clearly see from (3.5) that $\varphi(1) > 0$. This, in conjunction with the continuity of $\varphi(r)$, implies that there exists $\tau_2 \in (0,1)$ such that $\varphi(r) > 0$ for $\tau_2 < r < 1$. Therefore, it follows from (3.4) that

$$E(a,b) > M_2(a,b)^{\lambda_3} M_1(a,b)^{1-\lambda_3}$$

for $a > [(1+\tau_2)/(1-\tau_2)]^2 b$. □

Let $a = 1$ and $b = 1 - r^2 = r'^2$, then (1.6), and Theorems 3.1 and 3.2 give rise to Corollary 3.3 immediately.

Corollary 3.3 *The double inequalities*

$$\begin{aligned} &\frac{3(7+18r'^2+7r'^4)}{256(1+r'^2)} + \frac{13(1+6r'+r'^2)}{128} \\ &\quad < \mathcal{E}(r) \\ &\quad < \frac{(64-6\pi^2)(7+18r'^2+7r'^4)}{16\pi^2(1+r'^2)} + \frac{(7\pi^2-64)(1+6r'+r'^2)}{8\pi^2}, \\ &\left[\frac{7+18r'^2+7r'^4}{16(1+r'^2)}\right]^{3/16}\left(\frac{1+6r'+r'^2}{8}\right)^{13/16} \\ &\quad < \mathcal{E}(r) \\ &\quad < \left[\frac{7+18r'^2+7r'^4}{16(1+r'^2)}\right]^{\frac{\log 32/(3\pi^2)}{\log(7/6)}}\left(\frac{1+6r'+r'^2}{8}\right)^{\frac{\log(7\pi^2/64)}{\log(7/6)}} \end{aligned}$$

hold for all $r \in (0,1)$.

4 Results and discussion

In this article, we find the best possible parameters α_1, β_1, α_2 and β_2 on the interval $(0,1)$ such that the double inequalities

$$\begin{aligned} &\alpha_1\left[\frac{7C(a,b)}{16} + \frac{9H(a,b)}{16}\right] + (1-\alpha_1)\left[\frac{3A(a,b)}{4} + \frac{G(a,b)}{4}\right] \\ &\quad < E(a,b) \\ &\quad < \beta_1\left[\frac{7C(a,b)}{16} + \frac{9H(a,b)}{16}\right] + (1-\beta_1)\left[\frac{3A(a,b)}{4} + \frac{G(a,b)}{4}\right], \\ &\left[\frac{7C(a,b)}{16} + \frac{9H(a,b)}{16}\right]^{\alpha_2}\left[\frac{3A(a,b)}{4} + \frac{G(a,b)}{4}\right]^{1-\alpha_2} \\ &\quad < E(a,b) \end{aligned}$$

$$< \left[\frac{7C(a,b)}{16} + \frac{9H(a,b)}{16}\right]^{\beta_2} \left[\frac{3A(a,b)}{4} + \frac{G(a,b)}{4}\right]^{1-\beta_2}$$

hold for all $a, b > 0$ with $a \neq b$. Our results improve and refine the results given in [67, 68].

5 Conclusion

We present several sharp bounds for the quasi-arithmetic mean in terms of the combination of harmonic, geometric, arithmetic and contra-harmonic means. Our approach may have further applications in the theory of bivariate means.

Funding

This work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant no. Y201635325)

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 18 October 2018 Accepted: 8 February 2019 Published online: 20 February 2019

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