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M-positive semi-definiteness and M-positive definiteness of fourth-order partially symmetric Cauchy tensors

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Abstract

Inspired by symmetric Cauchy tensors, we define fourth-order partially symmetric Cauchy tensors with their generating vectors. In this article, we focus on the necessary and sufficient conditions for the M-positive semi-definiteness and M-positive definiteness of fourth-order Cauchy tensors. Moreover, the necessary and sufficient conditions of the strong ellipticity conditions for fourth-order Cauchy tensors are obtained. Furthermore, fourth-order Cauchy tensors are M-positive semi-definite if and only if the homogeneous polynomial for fourth-order Cauchy tensors is monotonically increasing. Several M-eigenvalue inclusion theorems and spectral properties of fourth-order Cauchy tensors are discussed. A power method is proposed to compute the smallest and the largest M-eigenvalues of fourth-order Cauchy tensors. The given numerical experiments show the effectiveness of the proposed method.

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Keywords: Cauchy tensor; M-positive semi-definite; M-positive definite; Spectral property; Homogeneous polynomial; M-eigenvalue; Power method

1 Introduction

Let $\mathbb{R}^n$ be an n-dimensional real Euclidean space and denote the set consisting of all natural numbers by N. Suppose that m and n are positive natural numbers and write $[n] = \{1, 2, \ldots, n\}$. The nonlinear elastic materials analysis and entanglement studies in quantum physics can be formulated as the following optimization problem:

$$\begin{cases} \max f(\mathbf{x}, \mathbf{y}) = \sum_{i,k\in[m]} \sum_{j,l\in[n]} c_{ijkl} x_i y_j x_k y_l, \\ \text{s.t. } \mathbf{x}^T\mathbf{x} = 1, \mathbf{y}^T\mathbf{y} = 1, \\ \quad \mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the coefficients c_{ijkl} satisfy the following symmetric property:

$$c_{ijkl} = c_{kjil} = c_{ilkj} = c_{klij}, \quad i, k \in [m], j, l \in [n].$$

Then, the tensor $\mathcal{C} = (c_{ijkl})$ is called to be a partially symmetric fourth-order tensor. In the nonlinear elastic materials analysis, one approach is to consider an elastic material in terms of a fourth-order three-dimensional elastic module tensor that satisfies the partially symmetric property [1]. Thus, the partially symmetric property of tensor is becoming to be an interesting subject. Inspired by Cauchy matrix and Cauchy tensor, we will construct a new kind of tensor which satisfies the partially symmetric property.

An $m \times n$ Cauchy matrix assigned to $m + n$ parameters $x_1, x_2, \ldots, x_m, y_1, \ldots, y_n$ was introduced by [2] as follows:

$$C = \left[\frac{1}{x_i + y_j} \right], \quad i \in [m], j \in [n]. \tag{1.2}$$

The Cauchy matrix has played an important role in algorithm designing [3–5]. If $x_i = y_i$ and $m = n$ in (1.2), then it reduces to a real symmetric Cauchy matrix. Motivated by symmetric Cauchy matrices, Chen and Qi [6] proposed the definition of Cauchy tensors.

Definition 1 Let a vector $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$. Suppose that a real tensor $\mathcal{C} = (c_{i_1 i_2 \cdots i_m})$ is defined by

$$c_{i_1 i_2 \cdots i_m} = \frac{1}{c_{i_1} + c_{i_2} + \cdots + c_{i_m}}, \quad i_j \in [n], j \in [m].$$

Then $\mathcal{C}$ is called an order m dimension n symmetric Cauchy tensor and the vector $\mathbf{c} \in \mathbb{R}^n$ is called the generating vector of $\mathcal{C}$.

Following the ideas of Cauchy matrix [2] and Cauchy tensor [6], we present the definition of fourth-order Cauchy tensors.

Definition 2 Let a vector $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ and a vector $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$. Suppose that a real tensor $\mathcal{C} = (c_{ijkl})$ is defined by

$$c_{ijkl} = \frac{1}{a_i + b_j + a_k + b_l}, \quad i, k \in [m], j, l \in [n].$$

Then we claim that $\mathcal{C}$ is a fourth-order Cauchy tensor and the vectors $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$ are called generating vectors of $\mathcal{C}$.

Obviously, the fourth-order Cauchy tensor has the following partially symmetric property:

$$c_{ijkl} = c_{kjil} = c_{ilkj} = c_{klij} = \frac{1}{a_i + b_j + a_k + b_l}, \quad i, k \in [m], j, l \in [n].$$

Furthermore, if $\mathbf{a} = \mathbf{b}$ and $m = n$, then the fourth-order partially symmetric Cauchy tensor reduces to the fourth-order symmetric Cauchy tensor. In Definition 2, we point out that, for the generating vectors a and b, it should satisfy

$$a_i + b_j + a_k + b_l \neq 0, \quad i, k \in [m], j, l \in [n].$$

In this paper, we always consider the fourth-order real partially symmetric Cauchy tensors. Hence, they can be called fourth-order Cauchy tensors for simplicity.

Recently, a lot of researchers have focused on structured tensors [6–28] such as M-tensors, Hankel tensors, Hilbert tensors, Cauchy tensors, completely positive tensors, B-tensors, and P-tensors. These papers not only gave some results on positive semi-definiteness property and spectral theory of structured tensors, but also revealed some important applications in data fitting and stochastic process [10, 29].

In this article, we focus on the M-positive semi-definiteness and M-positive definiteness conditions for fourth-order Cauchy tensors. Several spectral properties of M-positive semi-definite fourth-order Cauchy tensors are discussed. A power method is proposed to compute the smallest and the largest M-eigenvalues of fourth-order Cauchy tensors. In Sect. 2, the necessary and sufficient conditions for M-positive semi-definiteness and M-positive definiteness of fourth-order Cauchy tensors are obtained. Moreover, the necessary and sufficient conditions of the strong ellipticity condition for fourth-order Cauchy tensors are obtained. Furthermore, fourth-order Cauchy tensors are M-positive semi-definite if and only if the homogeneous polynomial of fourth-order Cauchy tensors is monotonically increasing in the nonnegative orthant of $\mathbb{R}^m \times \mathbb{R}^n$, and the homogeneous polynomial is strictly monotone increasing when fourth-order Cauchy tensors are M-positive definite. In Sect. 3, several spectral inequalities are presented on the M-eigenvalue of fourth-order Cauchy tensors. We introduce a power method to compute the smallest and the largest M-eigenvalues of fourth-order Cauchy tensors, and numerical experiments show the effectiveness of the proposed method in Sect. 4.

At the end of the introduction, we make some notations that will be applied to the sequel. Denote vectors by lowercase boldface letters, i.e., $\mathbf{x}$, $\mathbf{y}$, $\ldots$, and tensors are written to calligraphic capitals such as $\mathcal{A}$, $\mathcal{T}$, $\ldots$. For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $\mathbf{x} \geq \mathbf{y}$ ($\mathbf{x} \leq \mathbf{y}$) means $x_i \geq y_i$ ($x_i \leq y_i$) for all $i \in [n]$.

2 M-positive semi-definiteness and M-positive definiteness of fourth-order Cauchy tensors

Let

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{C}\mathbf{x}\mathbf{y}\mathbf{x}\mathbf{y} = \sum_{i,k \in [m], j,l \in [n]} c_{ijkl} x_i y_j x_k y_l.$$

The tensor $\mathcal{C}$ is called M-positive semi-definite if $f(\mathbf{x}, \mathbf{y}) \geq 0$ for any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$. The tensor $\mathcal{C}$ is called M-positive definite if $f(\mathbf{x}, \mathbf{y}) > 0$ for any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq \mathbf{0}$. Similarly, the tensor $\mathcal{C}$ is M-negative semi-definite (M-negative definite) if $f(\mathbf{x}, \mathbf{y}) \leq 0$ for any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ ($f(\mathbf{x}, \mathbf{y}) < 0$ for any vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq \mathbf{0}$).

Now, we will show some necessary and sufficient conditions for fourth-order Cauchy tensors to be M-positive semi-definite.

Theorem 2.1 *Let the vectors* $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$ *be generating vectors of the fourth-order Cauchy tensor* $\mathcal{C}$. *Then the tensor* $\mathcal{C}$ *is M-positive semi-definite if and only if* $a_i + b_j > 0$ *for all* $i \in [m]$, $j \in [n]$.

Proof From the fact that the Cauchy tensor $\mathcal{C}$ is M-positive semi-definite, taking $\mathbf{x} = \mathbf{e}_i \in \mathbb{R}^m$ and $\mathbf{y} = \mathbf{e}_j \in \mathbb{R}^n$, we have

$$\begin{aligned} f(\mathbf{e}_i, \mathbf{e}_j) &= \mathcal{C}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_j \\ &= \frac{1}{2(a_i + b_j)} \geq 0, \quad i \in [m], j \in [n], \end{aligned}$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the ith and jth coordinate vectors, respectively. Thus, we have $a_i + b_j > 0$ for all $i \in [m], j \in [n]$.

On the other hand, suppose that $a_i + b_j > 0$ for all $i \in [m], j \in [n]$. For any $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$, one has

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) = \mathcal{C}\mathbf{xyxy} &= \sum_{i,k \in [m], j,l \in [n]} c_{ijkl} x_i y_j x_k y_l \\ &= \sum_{i,k \in [m], j,l \in [n]} \frac{x_i y_j x_k y_l}{a_i + b_j + a_k + b_l} \\ &= \sum_{i,k \in [m], j,l \in [n]} \int_0^1 t^{a_i + b_j + a_k + b_l - 1} x_i y_j x_k y_l \, dt \\ &= \int_0^1 \left(\sum_{i \in [m]} t^{a_i - \frac{1}{4}} x_i \right)^2 \left(\sum_{j \in [n]} t^{b_j - \frac{1}{4}} y_j \right)^2 dt \\ &\geq 0, \end{aligned}$$

which means that the tensor $\mathcal{C}$ is M-positive semi-definite and the conclusion follows. □

From Theorem 2.1, we deduce the following corollary directly.

Corollary 2.1 *Assume that the fourth-order Cauchy tensor $\mathcal{C}$ and its generating vectors $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$ are defined in Theorem 2.1. Then the tensor $\mathcal{C}$ is M-negative semi-definite if and only if $a_i + b_j < 0$ for all $i \in [m], j \in [n]$.*

Corollary 2.2 *Assume that the fourth-order Cauchy tensor $\mathcal{C}$ and its generating vectors $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$ are defined in Theorem 2.1. Then the tensor $\mathcal{C}$ is not M-positive semi-definite if and only if there exist at least $i \in [m], j \in [n]$, $a_i + b_j < 0$ holds.*

Next, we will reveal some necessary and sufficient conditions for fourth-order Cauchy tensors to be M-positive definite.

Theorem 2.2 *Assume that the vectors $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$ are generating vectors of the fourth-order Cauchy tensor $\mathcal{C}$. For all $i \in [m], j \in [n]$, if $a_i + b_j > 0$ and the elements of generating vectors $\mathbf{a}$, $\mathbf{b}$ are mutually distinct, respectively, then the tensor $\mathcal{C}$ is M-positive definite.*

Proof It follows from Theorem 2.1 that the tensor $\mathcal{C}$ is M-positive semi-definite. We prove by contradiction that the tensor $\mathcal{C}$ is M-positive definite. Assume that there exists nonzero vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ such that

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{C}\mathbf{xyxy} = 0.$$

Following the proof of Theorem 2.1, we deduce

$$\int_0^1 \left(\sum_{i\in[m]} t^{a_i-\frac{1}{4}} x_i\right)^2 \left(\sum_{j\in[n]} t^{b_j-\frac{1}{4}} y_j\right)^2 dt = 0,$$

which implies

$$\sum_{i\in[m]} t^{a_i-\frac{1}{4}} x_i \equiv 0, \quad t \in (0,1],$$

or

$$\sum_{j\in[m]} t^{b_j-\frac{1}{4}} y_j \equiv 0, \quad t \in (0,1].$$

Without loss of generality, we can assume

$$\sum_{i\in[m]} t^{a_i-\frac{1}{4}} x_i \equiv 0, \quad t \in (0,1].$$

Then

$$x_1 + t^{a_2-a_1} x_2 + \cdots + t^{a_m-a_1} x_m \equiv 0, \quad t \in (0,1].$$

By the continuity and the fact that $a_1, a_2, \ldots, a_m$ are mutually distinct, it yields that

$$x_1 = 0,$$

and

$$x_2 + t^{a_3-a_2} x_3 + \cdots + t^{a_m-a_2} x_m \equiv 0, \quad t \in (0,1].$$

Applying the same argument, one has

$$x_1 = x_2 = \cdots = x_m = 0,$$

which is a contradiction with $\mathbf{x} \neq \mathbf{0}$. So, we conclude that the tensor $\mathcal{C}$ is M-positive definite. □

Moreover, the following conclusion shows that the conditions in Theorem 2.2 are necessary and sufficient conditions.

Theorem 2.3 *Assume that the fourth-order Cauchy tensor $\mathcal{C}$ and its generating vectors* $\mathbf{a}$, $\mathbf{b}$ *are defined in Theorem 2.2. The tensor $\mathcal{C}$ is M-positive definite if and only if $a_i + b_j > 0$ for all $i \in [m]$, $j \in [n]$, and the elements of generating vectors* $\mathbf{a}$, $\mathbf{b}$ *are mutually distinct, respectively.*

Proof From Theorem 2.2, if $a_i + b_j > 0$ for all $i \in [m], j \in [n]$ and the elements of generating vectors **a**, **b** are mutually distinct, respectively, then we can obtain that the tensor $\mathcal{C}$ is M-positive definite. Next, suppose that the tensor $\mathcal{C}$ is M-positive definite, we will reveal that $a_i + b_j > 0$ for all $i \in [m], j \in [n]$ and the elements of generating vectors **a**, **b** are mutually distinct, respectively. Indeed, from Theorem 2.1, we know that the tensor $\mathcal{C}$ is M-positive semi-definite; therefore, $a_i + b_j > 0$ for all $i \in [m], j \in [n]$. By contradiction, we can assume that two elements of the vector **a** are equal. Without loss of generality, suppose that $a_1 = a_2 = \overline{a}$. Let $\mathbf{x} = (1, -1, 0, \dots, 0) \in \mathbb{R}^m$ and $\mathbf{y} = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$. Thus, we have

$$\begin{aligned}
\mathcal{C}\mathbf{xyxy} &= \sum_{i,k\in[m],j,l\in[n]} c_{ijkl} x_i y_j x_k y_l \\
&= \sum_{i,k\in[m],j,l\in[n]} \frac{x_i y_j x_k y_l}{a_i + b_j + a_k + b_l} \\
&= \frac{1}{2(\overline{a} + b_1)} \sum_{i,k\in[2]} x_i x_k \\
&= \frac{1}{2(\overline{a} + b_1)} \bigl[1 \cdot 1 + (-1) \cdot 1 + 1 \cdot (-1) + (-1) \cdot (-1)\bigr] \\
&= 0,
\end{aligned}$$

which contradicts the assumption that the tensor $\mathcal{C}$ is M-positive definite and the proof is completed. □

In what follows, we will give the definition of the monotonicity of a homogeneous polynomial with respect to fourth-order Cauchy tensors.

For any $\mathbf{x}, \overline{\mathbf{x}} \in \mathbb{R}^m$ and $\mathbf{y}, \overline{\mathbf{y}} \in \mathbb{R}^n$, if $f(\mathbf{x}, \mathbf{y}) \geq f(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ when $\mathbf{x} \geq \overline{\mathbf{x}}$ and $\mathbf{y} \geq \overline{\mathbf{y}}$, ($\mathbf{x} \leq \overline{\mathbf{x}}$ and $\mathbf{y} \leq \overline{\mathbf{y}}$), then $f(\mathbf{x}, \mathbf{y})$ is called monotonically increasing (monotonically decreasing respectively). If $f(\mathbf{x}, \mathbf{y}) > f(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ when $\mathbf{x} \geq \overline{\mathbf{x}}$, $\mathbf{x} \neq \overline{\mathbf{x}}$ and $\mathbf{y} \geq \overline{\mathbf{y}}$, $\mathbf{y} \neq \overline{\mathbf{y}}$ ($\mathbf{x} \leq \overline{\mathbf{x}}$, $\mathbf{x} \neq \overline{\mathbf{x}}$ and $\mathbf{y} \leq \overline{\mathbf{y}}$, $\mathbf{y} \neq \overline{\mathbf{y}}$), then $f(\mathbf{x}, \mathbf{y})$ is called strictly monotone increasing (strictly monotone decreasing respectively).

The following conclusions reveal the relationships between M-positive semi-definiteness of fourth-order Cauchy tensor and the monotonicity of a homogeneous polynomial with respect to the proposed Cauchy tensor.

Theorem 2.4 *Let $\mathcal{C}$ be a fourth-order Cauchy tensor with generating vectors $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$. Then the tensor $\mathcal{C}$ is M-positive semi-definite if and only if the homogeneous polynomial $f(\mathbf{x}, \mathbf{y})$ is monotonically increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+$.*

Proof For sufficiency, let $\mathbf{x} = \mathbf{e}_i \in \mathbb{R}^m_+$, $\overline{\mathbf{x}} = \mathbf{0} \in \mathbb{R}^m_+$ and $\mathbf{y} = \mathbf{e}_j \in \mathbb{R}^n_+$, $\overline{\mathbf{y}} = \mathbf{0} \in \mathbb{R}^n_+$, one has

$$\frac{1}{2(a_i + b_j)} = \mathcal{C}\mathbf{xyxy} = f(\mathbf{x}, \mathbf{y}) \geq f(\overline{\mathbf{x}}, \overline{\mathbf{y}}) = \mathcal{C}\overline{\mathbf{x}}\overline{\mathbf{y}}\overline{\mathbf{x}}\overline{\mathbf{y}} = 0,$$

which implies that $a_i + b_j > 0$ for all $i \in [m], j \in [n]$. By Theorem 2.1, it yields that the tensor $\mathcal{C}$ is M-positive semi-definite.

On the other hand, suppose $\mathbf{x}, \overline{\mathbf{x}} \in \mathbb{R}^m_+$, $\mathbf{y}, \overline{\mathbf{y}} \in \mathbb{R}^n_+$, and $\mathbf{x} \geq \overline{\mathbf{x}}$ and $\mathbf{y} \geq \overline{\mathbf{y}}$. From Theorem 2.1, we obtain that $a_i + b_j > 0$ for all $i \in [m]$, $j \in [n]$. Furthermore,

$$\begin{aligned} f(\mathbf{x},\mathbf{y}) - f(\overline{\mathbf{x}},\overline{\mathbf{y}}) &= \mathcal{C}\mathbf{xyxy} - \mathcal{C}\overline{\mathbf{xyxy}} \\ &= \sum_{i,k\in[m],j,l\in[n]} c_{ijkl}(x_i y_j x_k y_l - \overline{x}_i \overline{y}_j \overline{x}_k \overline{y}_l) \\ &= \sum_{i,k\in[m],j,l\in[n]} \frac{1}{a_i + b_j + a_k + b_l}(x_i y_j x_k y_l - \overline{x}_i \overline{y}_j \overline{x}_k \overline{y}_l) \\ &\geq 0, \end{aligned}$$

which implies that $f(\mathbf{x},\mathbf{y})$ is monotonically increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+$ and the desired result holds. □

Theorem 2.5 *Let $\mathcal{C}$ be a fourth-order Cauchy tensor with generating vectors $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$. If the tensor $\mathcal{C}$ is M-positive definite, then the homogeneous polynomial $f(\mathbf{x},\mathbf{y})$ is strictly monotone increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+$.*

Proof From Theorem 2.3, one has, for all $i \in [m]$, $j \in [n]$,

$$a_i + b_j > 0.$$

For any $\mathbf{x} \geq \overline{\mathbf{x}}$, $\mathbf{x} \neq \overline{\mathbf{x}}$ and $\mathbf{y} \geq \overline{\mathbf{y}}$, $\mathbf{y} \neq \overline{\mathbf{y}}$, then there exist indexes $i_0 \in [m]$ and $j_0 \in [n]$ such that

$$x_{i_0} > \overline{x}_{i_0} \geq 0$$

and

$$y_{j_0} > \overline{y}_{j_0} \geq 0.$$

Thus,

$$\begin{aligned} f(\mathbf{x},\mathbf{y}) - f(\overline{\mathbf{x}},\overline{\mathbf{y}}) &= \mathcal{C}\mathbf{xyxy} - \mathcal{C}\overline{\mathbf{xyxy}} \\ &= \sum_{i,k\in[m],j,l\in[n]} c_{ijkl}(x_i y_j x_k y_l - \overline{x}_i \overline{y}_j \overline{x}_k \overline{y}_l) \\ &= \sum_{i,k\in[m],j,l\in[n],(i,j,k,l)\neq(i_0,j_0,i_0,j_0)} c_{ijkl}(x_i y_j x_k y_l - \overline{x}_i \overline{y}_j \overline{x}_k \overline{y}_l) \\ &\quad + c_{i_0 j_0 i_0 j_0}\left(x_{i_0}^2 y_{j_0}^2 - \overline{x}_{i_0}^2 \overline{y}_{j_0}^2\right) \\ &= \sum_{i,k\in[m],j,l\in[n],(i,j,k,l)\neq(i_0,j_0,i_0,j_0)} \frac{1}{a_i + b_j + a_k + b_l}(x_i y_j x_k y_l - \overline{x}_i \overline{y}_j \overline{x}_k \overline{y}_l) \\ &\quad + \frac{1}{2(a_{i_0} + b_{j_0})}\left(x_{i_0}^2 y_{j_0}^2 - \overline{x}_{i_0}^2 \overline{y}_{j_0}^2\right) \\ &> 0, \end{aligned}$$

which means that the homogeneous polynomial $f(\mathbf{x},\mathbf{y})$ is strictly monotone increasing in $\mathbb{R}^m_+ \times \mathbb{R}^n_+$. □

Now, we are in a position to propose an example to reveal that the strictly monotone increasing property for the polynomial $f(\mathbf{x},\mathbf{y})$ is only a necessary condition for the M-positive definiteness property of the tensor $\mathcal{C}$ but not a sufficient condition.

Example 2.1 Let the tensor $\mathcal{C} = (c_{ijkl})$ be a fourth-order Cauchy tensor with generating vectors $\mathbf{a} = (2,2,2)$ and $\mathbf{b} = (4,4,4,4)$. Then one has

$$c_{ijkl} = \frac{1}{12}, \quad i,k \in [3], j,l \in [4]$$

and the homogeneous polynomial

$$f(\mathbf{x},\mathbf{y}) = \mathcal{C}\mathbf{xyxy} = \frac{1}{12} \sum_{i,k\in[3],j,l\in[4]} x_i y_j x_k y_l.$$

It is easy to check that $f(\mathbf{x},\mathbf{y})$ is strictly monotone increasing in $\mathbb{R}^3_+ \times \mathbb{R}^4_+$. However, it follows from Theorem 2.3 that the tensor $\mathcal{C}$ is not M-positive definite.

Copositive tensors have some important applications in polynomial optimization [18], vacuum stability of a general scalar potential [30], tensor generalized eigenvalue complementarity problem [31], and tensor complementarity problem [32, 33]. The tensor $\mathcal{C}$ is called copositive if $\mathcal{C}\mathbf{x}^m \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$. Consider the tensor complement problem (TCP($\mathbf{q},\mathcal{C}$)) of finding a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{x} \geq \mathbf{0}, \qquad \mathbf{q} + \mathcal{C}\mathbf{x}^{m-1} \geq \mathbf{0}, \qquad \mathbf{x}^T\left(\mathbf{q} + \mathcal{C}\mathbf{x}^{m-1}\right) = 0.$$

Applying the above definitions, we have the following technical conclusion.

Theorem 2.6 *Let $\mathcal{C}$ be a fourth-order Cauchy tensor with generating vectors $\mathbf{a},\mathbf{b} \in \mathbb{R}^n_+$. Then the following statements are equivalent*:

(i) *the tensor $\mathcal{C}$ is copositive*;

(ii) *for every $\mathbf{q} > \mathbf{0}$, TCP($\mathbf{q},\mathcal{C}$) has a unique solution*;

(iii) *for every index set $N \subset [n]$, the system*

$$\mathcal{C}^{|N|}\left(\mathbf{x}^N\right)^3 < \mathbf{0}, \quad \mathbf{x}^N \geq \mathbf{0}$$

has no solution, where $\mathbf{x}^N \in \mathbb{R}^{|N|}$;

(iv) *for all $i \in [n]$, $a_i + b_i > 0$.*

Proof (i) ⇒ (ii). It follows from $\mathbf{q} > \mathbf{0}$ that $\mathbf{0}$ is a solution of TCP($\mathbf{q},\mathcal{C}$). Suppose, to the contrary, that there exists a vector $\mathbf{q}' > \mathbf{0}$ such that TCP($\mathbf{q}',\mathcal{C}$) has a solution $\mathbf{x} \neq \mathbf{0}$. Since $\mathcal{C}$ is copositive, for $\mathbf{x} \in \mathbb{R}^n_+$, one has

$$\mathcal{C}\mathbf{x}^4 \geq 0.$$

On the other hand,

$$\begin{aligned}(\mathcal{C}\mathbf{x}^3)_i &= \sum_{k,j,k\in[n]} c_{ijkl}x_jx_kx_l \\ &= \sum_{k,j,k\in[n]} \frac{x_jx_kx_l}{a_i+b_j+a_k+b_l} \\ &\geq \mathbf{0},\end{aligned}$$

which means $\mathbf{q}' + \mathcal{C}\mathbf{x}^3 > \mathbf{0}$. Then

$$\mathbf{x}^T(\mathbf{q}' + \mathcal{C}\mathbf{x}^3) = \mathbf{x}^T\mathbf{q}' + \mathcal{C}\mathbf{x}^4 > 0,$$

which contradicts the assumption that x solves TCP($\mathbf{q}', \mathcal{C}$). Thus, for every $\mathbf{q} > \mathbf{0}$, we obtain TCP($\mathbf{q}, \mathcal{C}$) has a unique solution.

(ii) ⇒ (iii). Following the proof of Theorem 3.1 [34], we have the desired result.

(iii) ⇒ (iv). Let $N = \{i\}$, $i \in [n]$, $\mathbf{x}^N = 1$. Then

$$(\mathcal{C}^{|N|}(\mathbf{x}^N)^3)_i = c_{iiii}x_i^3 = c_{iiii} = \frac{1}{2(a_i+b_i)} > 0.$$

Then, for all $i \in [n]$, one has $a_i + b_i > 0$.

(iv) ⇒ (i). By Theorem 2.1, we obtain that the tensor $\mathcal{C}$ is M-positive semi-definite, which means that the tensor $\mathcal{C}$ is copositive. □

3 Spectral properties for fourth-order Cauchy tensors

In this section, we discuss M-eigenvalue inclusion theorems and spectral properties of fourth-order Cauchy tensors. M-eigenvalue problem has a close relationship with the strong ellipticity condition, which plays an important role in nonlinear elasticity and in materials, since it can ensure an elastic material to satisfy some mechanical properties. Thus, to identify whether the strong ellipticity condition of a given material holds or not becomes an important problem in mechanics [29, 35–38].

The strong ellipticity condition for a partially symmetric fourth-order tensor $\mathcal{C}$ is stated by

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{C}\mathbf{xyxy} = \sum_{i,k,j,l\in[n]} c_{ijkl}x_iy_jx_ky_l > 0$$

for any vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} \neq 0$. In [39], Qi, Dai, and Han revealed the necessary and sufficient condition of the strong ellipticity condition by introducing the following definition of an M-eigenvalue of the tensor $\mathcal{C}$. For $\lambda \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$, it holds that

$$\begin{cases} \mathcal{C}\cdot\mathbf{yxy} = \lambda\mathbf{x}, \\ \mathcal{C}\mathbf{xyx}\cdot = \lambda\mathbf{y}, \\ \mathbf{x}^T\mathbf{x} = 1, \\ \mathbf{y}^T\mathbf{y} = 1, \end{cases} \tag{3.1}$$

where $(\mathcal{C}\cdot\mathbf{yxy})_i = \sum_{k\in[m],j,l\in[n]} c_{ijkl} y_j x_k y_l$, and $(\mathcal{C}\mathbf{xyx}\cdot)_l = \sum_{i,k\in[m],j\in[n]} c_{ijkl} x_i y_j x_k$. The scalar λ is called an M-eigenvalue of the tensor $\mathcal{C}$, and $\mathbf{x}$ and $\mathbf{y}$ are called left and right M-eigenvectors of $\mathcal{C}$, respectively, associated with the M-eigenvalue λ. When $m = n$, Qi, Dai, and Han [39] gave the following technical theorem.

Theorem 3.1 ([39]) *The strong ellipticity condition holds if and only if the smallest M-eigenvalue of the elasticity tensor is positive.*

From Theorems 2.3 and 3.1, we can obtain the necessary and sufficient conditions of the strong ellipticity condition for fourth-order Cauchy tensors.

Theorem 3.2 *Let the vectors* $\mathbf{a}\in\mathbb{R}^n$, $\mathbf{b}\in\mathbb{R}^n$ *be generating vectors of the fourth-order Cauchy tensor* $\mathcal{C}$. *The strong ellipticity condition holds if and only if the smallest M-eigenvalue of the tensor* $\mathcal{C}$ *is positive.*

Theorem 3.3 *Let the vectors* $\mathbf{a}\in\mathbb{R}^n$, $\mathbf{b}\in\mathbb{R}^n$ *be generating vectors of the fourth-order Cauchy tensor* $\mathcal{C}$. *The strong ellipticity condition holds if and only if* $a_i + b_j > 0$ *for all* $i, j \in [n]$, *and the elements of generating vectors a, b are mutually distinct, respectively.*

The spectral radius $\rho(\mathcal{C})$ of $\mathcal{C}$ is defined as

$$\rho(\mathcal{C}) = \max\bigl\{|\lambda| : \lambda\in\sigma(\mathcal{C})\bigr\},$$

where $\sigma(\mathcal{C})$ is the spectrum of $\mathcal{C}$, which contains all M-eigenvalues of $\mathcal{C}$.

Now, inspired by the idea of H-eigenvalue inclusion theorem [40], we establish the following M-eigenvalue inclusion theorems for fourth-order Cauchy tensors.

Theorem 3.4 *Suppose that the tensor* $\mathcal{C}$ *is a fourth-order Cauchy tensor with generating vectors* $\mathbf{a}\in\mathbb{R}^m$, $\mathbf{b}\in\mathbb{R}^n$. *Then*

$$\sigma(\mathcal{C}) \subseteq \Gamma(\mathcal{C}) = \bigcup_{i\in[m]} \Gamma_i(\mathcal{C}),$$

where $\Gamma_i(\mathcal{C}) = \{z\in\mathbb{C} : |z| \le \sum_{k\in[m],j,l\in[n]} \frac{1}{a_i+b_j+a_k+b_l}\}$.

Proof Let λ be an M-eigenvalue of the tensor $\mathcal{C}$ with left M-eigenvector $\mathbf{x}\in\mathbb{R}^m$ and right M-eigenvector $\mathbf{y}\in\mathbb{R}^n$. Since $\mathbf{x}$ is a left M-eigenvector of the tensor $\mathcal{C}$ with $\mathbf{x}^T\mathbf{x} = 1$, we know that it has at least one nonzero component. Assume that

$$|x_t| = \max_{i\in[m]} |x_i| > 0.$$

It follows from (3.1) that

$$\begin{aligned}\lambda x_t &= (\mathcal{C}\cdot\mathbf{yxy})_t \\ &= \sum_{k\in[m],j,l\in[n]} c_{tjkl} y_j x_k y_l \\ &= \sum_{k\in[m],j,l\in[n]} \frac{y_j x_k y_l}{a_t + b_j + a_k + b_l}.\end{aligned}$$

Since $\mathbf{y}^T\mathbf{y} = 1$, we obtain $|y_j| \leq 1$ for any $j \in [n]$. Furthermore,

$$|\lambda| \leq \sum_{k\in[m],j,l\in[n]} \frac{1}{|a_t + b_j + a_k + b_l|} \frac{|x_k|}{|x_t|} |y_j y_l| \\ \leq \sum_{k\in[m],j,l\in[n]} \frac{1}{|a_t + b_j + a_k + b_l|}.$$

This implies that $\lambda \in \Gamma(\mathcal{C})$. □

Theorem 3.5 *Suppose that the tensor* $\mathcal{C}$ *is a fourth-order Cauchy tensor with generating vectors* $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$. *If there exists an index* $i \in [m]$ *such that* $c_{i1i1} = c_{i2i2} = \cdots = c_{inin} = d$, *then*

$$\sigma(\mathcal{C}) \subseteq \mathcal{K}(\mathcal{C}) = \bigcup_{i\in[m]} \mathcal{K}_i(\mathcal{C}),$$

where $\mathcal{K}_i(\mathcal{C}) = \{z \in \mathbb{C} : |z - d| \leq \sum_{j,l\in[n],j\neq l} \frac{1}{|a_i+b_j+a_i+b_l|} + \sum_{k\neq i,k\in[m],j,l\in[n]} \frac{1}{a_i+b_j+a_k+b_l}\}$.

Proof Let λ be an M-eigenvalue of the tensor $\mathcal{C}$ with corresponding left M-eigenvector $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x}^T\mathbf{x} = 1$ and right M-eigenvector $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y}^T\mathbf{y} = 1$. Then left M-eigenvector $\mathbf{x}$ has at least one nonzero component. Assume that

$$|x_t| = \max_{i\in[m]} |x_i| > 0.$$

It follows from (3.1) that

$$\lambda x_t = (\mathcal{C} \cdot \mathbf{yxy})_t \\ = \sum_{k\in[m],j,l\in[n]} c_{tjkl} y_j x_k y_l \\ = \sum_{j\in[n]} c_{tjtj} y_j^2 x_t + \sum_{j,l\in[n],j\neq l} c_{tjtl} y_j x_t y_l + \sum_{k\neq t,k\in[m],j,l\in[n]} c_{tjkl} y_j x_k y_l \\ = d x_t + \sum_{j,l\in[n],j\neq l} c_{tjtl} y_j x_t y_l + \sum_{k\neq t,k\in[m],j,l\in[n]} c_{tjkl} y_j x_k y_l.$$

Furthermore,

$$|\lambda - d| \leq \sum_{j,l\in[n],j\neq l} \frac{1}{|a_t + b_j + a_t + b_l|} |y_j y_l| + \sum_{k\neq t,k\in[m],j,l\in[n]} \frac{1}{|a_t + b_j + a_k + b_l|} \frac{|x_k|}{|x_t|} |y_j y_l| \\ \leq \sum_{j,l\in[n],j\neq l} \frac{1}{|a_t + b_j + a_t + b_l|} + \sum_{k\neq t,k\in[m],j,l\in[n]} \frac{1}{|a_t + b_j + a_k + b_l|}.$$

This implies that $\lambda \in \mathcal{K}(\mathcal{C})$. □

Next, we will reveal several spectral properties for fourth-order Cauchy tensors.

Theorem 3.6 *Suppose that the tensor $\mathcal{C}$ is a fourth-order Cauchy tensor with generating vectors $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$, and for all $i \in [m]$, $j \in [n]$ such that $a_i + b_j > 0$. Then the tensor $\mathcal{C}$ is M-positive definite if and only if its M-eigenvalues are positive.*

Proof Suppose that λ, $\mathbf{x}$, and $\mathbf{y}$ satisfy (3.1). It is easy to obtain that $\lambda = \mathcal{C}\mathbf{xyxy}$. Furthermore, (3.1) is the optimality condition of

$$\min\{\mathcal{C}\mathbf{xyxy} : \mathbf{x}^T\mathbf{x} = 1, \mathbf{y}^T\mathbf{y} = 1, \mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n\}. \tag{3.2}$$

From the feasible set is compact and the objective function is continuous, we obtain that the global maximizer and minimizer always exist. This shows that $\mathcal{C}$ always has M-eigenvalues. Since $\mathcal{C}$ is M-positive definite (M-positive semidefinite) if and only if the optimal value of (3.2) is positive (nonnegative), we obtain the desired result. □

Theorem 3.7 *Suppose that the fourth-order Cauchy tensor $\mathcal{C}$ and its generating vectors* $\mathbf{a}$, $\mathbf{b}$ *are defined as in Theorem 3.6 with the elements of generating vectors* $\mathbf{a}$, $\mathbf{b}$ *mutually distinct, respectively. If $\lambda \in \sigma(\mathcal{C})$ is an M-eigenvalue of the tensor $\mathcal{C}$ with non-negative left M-eigenvector* $\mathbf{x}$ *or non-negative right M-eigenvector* $\mathbf{y}$, *then $\lambda \neq 0$.*

Proof Since $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, we know that it has at least one nonzero component $x_t > 0$. From the definitions of M-eigenvalue and M-eigenvector of the tensor $\mathcal{C}$, one has

$$\begin{aligned}
\lambda x_i &= (\mathcal{C} \cdot \mathbf{yxy})_i \\
&= \sum_{k\in[m],j,l\in[n]} \frac{y_j x_k y_l}{a_t + b_j + a_k + b_l} \\
&= \sum_{k\in[m],j,l\in[n]} \int_0^1 t^{a_t+b_j+a_k+b_l-1} y_j x_k y_l \, dt \\
&= \int_0^1 t^{a_t-\frac{1}{4}} \Bigl(\sum_{k\in[m]} t^{a_k-\frac{1}{4}} x_k\Bigr)\Bigl(\sum_{j\in[n]} t^{b_j-\frac{1}{4}} y_j\Bigr)^2 \, dt.
\end{aligned} \tag{3.3}$$

Suppose, to the contrary, that the tensor $\mathcal{C}$ has M-eigenvalue $\lambda = 0$ with left M-eigenvector $\mathbf{x}$. Then, from (3.3), we have

$$\int_0^1 t^{a_t-\frac{1}{4}} \Bigl(\sum_{k\in[m]} t^{a_k-\frac{1}{4}} x_k\Bigr)\Bigl(\sum_{j\in[n]} t^{b_j-\frac{1}{4}} y_j\Bigr)^2 \, dt \equiv 0.$$

Using the properties of integration, one has

$$\Bigl(\sum_{k\in[m]} t^{a_k-\frac{1}{4}} x_k\Bigr)\Bigl(\sum_{j\in[n]} t^{b_j-\frac{1}{4}} y_j\Bigr)^2 \equiv 0, \quad t \in (0,1],$$

which implies

$$\sum_{k\in[m]} t^{a_k-\frac{1}{4}} x_k \equiv 0, \quad t \in (0,1],$$

or

$$\sum_{j\in[n]} t^{b_j-\frac{1}{4}} y_j \equiv 0, \quad t \in (0,1].$$

Without loss of generality, we assume

$$\sum_{k\in[m]} t^{a_k-\frac{1}{4}} x_k \equiv 0, \quad t \in (0,1].$$

Furthermore,

$$x_1 + t^{a_2-a_1} x_2 + \cdots + t^{a_m-a_1} x_m \equiv 0, \quad t \in (0,1].$$

By the continuity and the fact that $a_1, a_2, \ldots, a_m$ are mutually distinct, it yields that

$$x_1 = 0$$

and

$$x_2 + t^{a_3-a_2} x_3 + \cdots + t^{a_m-a_2} x_m \equiv 0, \quad t \in (0,1].$$

Applying the same argument, one has

$$x_1 = x_2 = \cdots = x_m = 0,$$

which contradicts with $\mathbf{x}^T\mathbf{x} = 1$. Then the tensor $\mathcal{C}$ has no zero M-eigenvalue. Similarly, if right M-eigenvector $\mathbf{y} \geq \mathbf{0}$, using the second equation of (3.1), we can also obtain that the tensor $\mathcal{C}$ has no zero M-eigenvalue and the desired conclusion follows. □

4 Power method of fourth-order Cauchy tensors

In this section, a power method is proposed to compute the smallest and the largest M-eigenvalues of fourth-order Cauchy tensors. It is well known that the power method is an efficient method to solve the largest eigenvalue of a matrix [41]. The method has successfully extended to compute the largest Z-eigenvalue in magnitude of higher-order tensors [40] and the largest M-eigenvalue of a fourth-order partially symmetric tensor [42]. Motivated by these, we first propose a power method to compute the smallest M-eigenvalue of fourth-order Cauchy tensors.

We introduce the following identity tensor $\mathcal{I} \in \mathbb{R}^{m\times n\times m\times n}$:

$$\mathcal{I}_{ijkl} = \begin{cases} 1, & \text{if } i = k \text{ and } j = l, \\ 0, & \text{otherwise.} \end{cases}$$

Choose a suitable $\alpha \in \mathbb{R}$ such that $\alpha > |\lambda|$, where $\lambda \in \sigma(\mathcal{C})$, and take

$$\bar{f}(\mathbf{x},\mathbf{y}) = \alpha\mathcal{I}\mathbf{xyxy} - \mathcal{C}\mathbf{xyxy} \triangleq \overline{\mathcal{C}}\mathbf{xyxy}.$$

It is easy to check that the tensor $\overline{\mathcal{C}}$ is M-positive definite on $\mathbb{R}^m \times \mathbb{R}^n$ with the same symmetry property of the tensor $\mathcal{C}$. Moreover, Theorem 3.4 suggests that we can take

$$\alpha = (1+\varepsilon)\max_{i\in[m]} \sum_{k\in[m],j,l\in[n]} |c_{ijkl}|,$$

where $\varepsilon > 0$ is a sufficiently small number. Furthermore, if $\mathbf{x}$ and $\mathbf{y}$ constitute a pair of M-eigenvectors of the tensor $\overline{\mathcal{C}}$ associated with M-eigenvalue λ, then they are also a pair of M-eigenvectors of tensor $\mathcal{C}$ associated with M-eigenvalue $\alpha - \lambda$.

Now, we are in a position to propose a power method to compute the smallest M-eigenvalue of a fourth-order Cauchy tensor $\mathcal{C}$.

Algorithm 4.1

Initialization step: Choose initial points $\mathbf{x}_0 \in \mathbb{R}^m$, $\mathbf{y}_0 \in \mathbb{R}^n$, and let $k = 0$.

Iterative step: Carry out the following formula alternatively until certain convergence criterion is satisfied and output $\mathbf{x}^*$, $\mathbf{y}^*$:

$$\begin{aligned}
&\overline{x}_{k+1} = \overline{\mathcal{C}} \cdot y_k x_k y_k, \qquad x_{k+1} = \frac{\overline{x}_{k+1}}{\|\overline{x}_{k+1}\|},\\
&\overline{y}_{k+1} = \overline{\mathcal{C}} x_{k+1} y_k x_{k+1} \cdot, \qquad y_{k+1} = \frac{\overline{y}_{k+1}}{\|\overline{y}_{k+1}\|},\\
&k = k+1.
\end{aligned}$$

Final step: Output the smallest M-eigenvalue $\alpha - \overline{f}(\mathbf{x}^*, \mathbf{y}^*)$ of the tensor $\mathcal{C}$ and the associated M-eigenvectors $\mathbf{x}^*$, $\mathbf{y}^*$.

Similarly, we can compute the largest M-eigenvalue of fourth-order Cauchy tensors. Take

$$\alpha = (1+\varepsilon)\max_{i\in[m]} \sum_{k\in[m],j,l\in[n]} |c_{ijkl}|$$

and define

$$\widehat{f}(x,y) = \alpha\mathcal{I}\mathbf{xyxy} + \mathcal{C}\mathbf{xyxy} \triangleq \widehat{\mathcal{C}}\mathbf{xyxy}.$$

Obviously, if $\mathbf{x}$ and $\mathbf{y}$ constitute a pair of M-eigenvectors of the tensor $\widehat{\mathcal{C}}$ associated with M-eigenvalue $\widehat{\lambda}$, then they are also a pair of M-eigenvectors of the tensor $\mathcal{C}$ associated with M-eigenvalue $\widehat{\lambda} - \alpha$. Thus, we can apply Algorithm 4.1 to compute the largest M-eigenvalue of a fourth-order Cauchy tensor $\mathcal{C}$.

The following numerical experiments show the effectiveness of the proposed method. The whole program was written in Matlab 7.0. All the numerical results were carried out on a personal Lenovo Thinkpad computer with Intel(R) Core(TM) i7-6500U CPU 2.50 GHz and RAM 8.00 GB. In the implementation, we choose $\|x_{k+1} - x_k\| + \|y_{k+1} - y_k\| \leq 10^{-10}$ as the stopping criterion, and take the parameter $\varepsilon = 0.0001$.

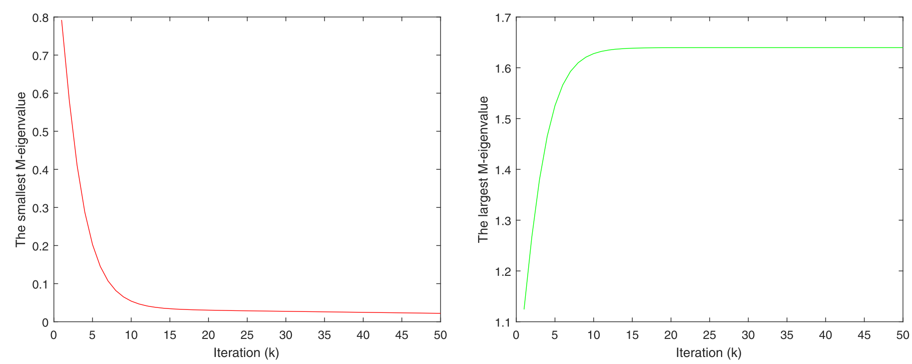

Figure 1 Numerical result of Example 4.1

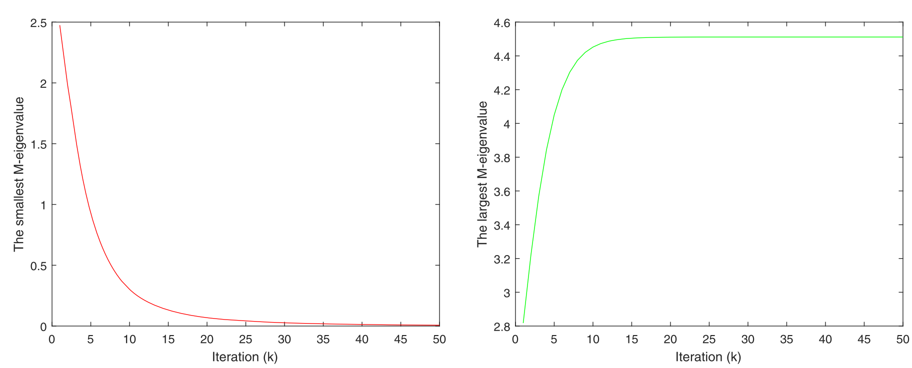

Figure 2 Numerical result of Example 4.2

Example 4.1 Consider a fourth-order Cauchy tensor $\mathcal{C}$ with generating vectors $\mathbf{a} = [4, 5, 3]$ and $\mathbf{b} = [-1, -2, 1]$. The variation of the objective function value corresponding to the tensor $\mathcal{C}$ during the iteration process can be shown in Fig. 1. For the tensor $\mathcal{C}$, its smallest M-eigenvalue is 0.0221 and the largest M-eigenvalue is 1.6397.

Example 4.2 Consider a fourth-order Cauchy tensor $\mathcal{C}$ with generating vectors $\mathbf{a} = [1.4424, 1.1837, 1.2492]$ and $\mathbf{b} = [0.0285, 0.0530, 0.0560, 0.0415]$. The variation of the objective function value corresponding to the tensor $\mathcal{C}$ during the iteration process can be shown in Fig. 2. For the tensor $\mathcal{C}$, its smallest M-eigenvalue is 0.0069 and the largest M-eigenvalue is 4.5117.

Example 4.3 Consider a fourth-order Cauchy tensor $\mathcal{C}$ with generating vectors $\mathbf{a} = \mathrm{rand}(20, 1) + 10$ and $\mathbf{b} = 30 * \mathrm{rand}(30, 1)$. The variation of the objective function value corresponding to the tensor $\mathcal{C}$ during the iteration process can be shown in Fig. 4. For the tensor $\mathcal{C}$, its smallest M-eigenvalue is 0.0594 and the largest M-eigenvalue is 12.4184.

Example 4.4 Consider a fourth-order Cauchy tensor $\mathcal{C}$ with generating vectors $\mathbf{a} = 5 * \mathrm{rand}(30, 1) + 10$ and $\mathbf{b} = \mathrm{rand}(40, 1) + 8$. The variation of the objective function value corresponding to the tensor $\mathcal{C}$ during the iteration process can be shown in Fig. 4. For the tensor $\mathcal{C}$, its smallest M-eigenvalue is 0.0229 and the largest M-eigenvalue is 28.9232.

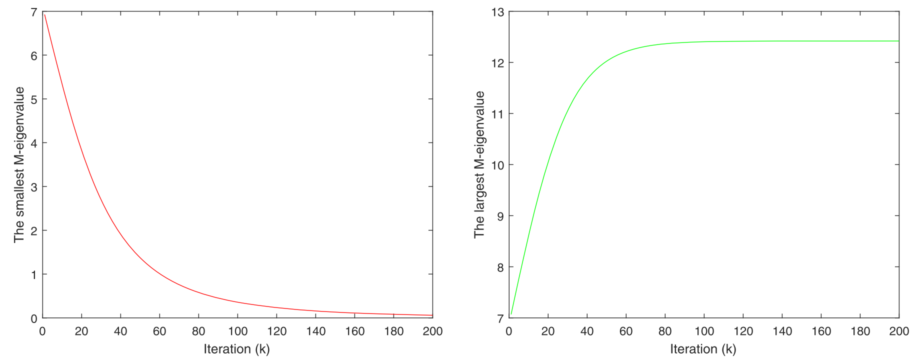

Figure 3 Numerical result of Example 4.3

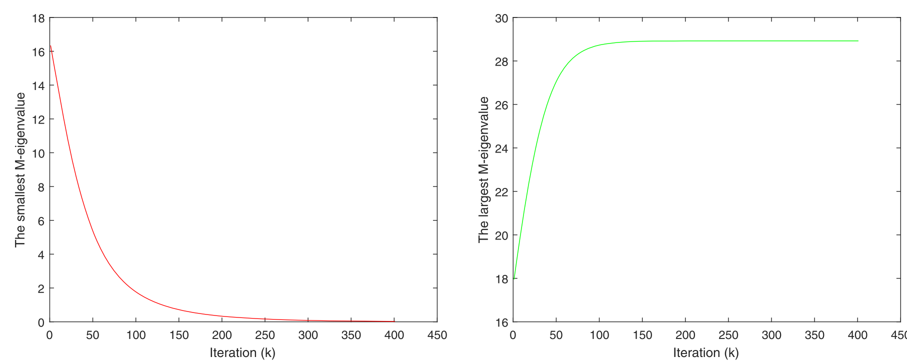

Figure 4 Numerical result of Example 4.4

Figures 1, 2, 3, and 4 show that the smallest M-eigenvalue sequence generated by Algorithm 4.1 is decreasing, and the largest M-eigenvalue sequence generated by Algorithm 4.1 is nondecreasing. From Theorems 3.2 and 3.3, it is easy to see that M-eigenvalues always exist and the strong ellipticity condition holds if and only if the smallest M-eigenvalue of $\mathcal{C}$ is positive; thus Example 4.1 verifies the strong ellipticity condition. From Theorem 2.3, we know that the tensor $\mathcal{C}$ is M-positive definite if it satisfies that $a_i + b_j > 0$ for all $i, j \in [n]$ and the elements of generating vectors $\mathbf{a}$, $\mathbf{b}$ are mutually distinct, respectively; then Example 4.2 shows that the tensor $\mathcal{C}$ is M-positive definite. Furthermore, Examples 4.3 and 4.4 reclaim that our algorithm is also suitable for the tensors with high dimensions.

5 Final remarks

In this article, the necessary and sufficient conditions for the M-positive semi-definiteness and M-positive definiteness of fourth-order Cauchy tensors are discussed. Moreover, the necessary and sufficient conditions of the strong ellipticity condition for fourth-order Cauchy tensors are obtained. Furthermore, we reveal that fourth-order Cauchy tensors are M-positive semi-definite if and only if there is a monotone increasing homogeneous polynomial defined in the nonnegative orthant of $\mathbb{R}^m \times \mathbb{R}^n$. Several M-eigenvalue inclusion theorems and spectral properties of fourth-order Cauchy tensors are discussed. A power method is proposed to compute the smallest and the largest M-eigenvalues of fourth-order

Cauchy tensors. The given numerical experiments show the effectiveness of the proposed method.

However, there are still some questions that we are not sure about now. Can we have the type of Cauchy–Toeplitz tensors with the partially symmetric property? If so, how about their spectral properties? What are the necessary and sufficient conditions for their M-positive semi-definiteness?

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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