https://doi.org/10.1186/s13660-019-1980-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

An improved version of a result of Chandra, Li, and Rosalsky

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Abstract

For an array of rowwise pairwise negative quadrant dependent, mean 0 random variables, Chandra, Li, and Rosalsky provided conditions under which weighted averages converge in $\mathcal{L}_1$ to 0. The Chandra, Li, and Rosalsky result is extended to $\mathcal{L}_r$ convergence ($1 \le r < 2$) and is shown to hold under weaker conditions by applying a mean convergence result of Sung and an inequality of Adler, Rosalsky, and Taylor.

MSC: 60F25; 60F05

Keywords: Array of rowwise pairwise negative quadrant dependent random variables; Weighted averages; Degenerate mean convergence; Stochastic domination

1 Introduction

For an array of mean 0 random variables $\{X_{n,j}, 1 \le j \le k_n, n \ge 1\}$ and an array of constants $\{a_{n,j}, 1 \le j \le k_n, n \ge 1\}$, Chandra, Li, and Rosalsky [2, Theorem 3.1] recently provided conditions under which the weighted averages $\sum_{j=1}^{k_n} a_{n,j}X_{n,j}$ obey the degenerate mean convergence law

$$\sum_{j=1}^{k_n} a_{n,j}X_{n,j} \xrightarrow{\mathcal{L}_1} 0.$$

The random variables comprising the array are assumed to be (i) rowwise pairwise negative quadrant dependent and (ii) stochastically dominated by a random variable. (Technical definitions such as these will be reviewed in Sect. 2.) In this note, Theorem 3.1 of Chandra, Li, and Rosalsky [2] is extended to $\mathcal{L}_r$ convergence where $1 \le r < 2$ and is shown to hold under weaker conditions. This is accomplished by applying a result of Sung [3] and an inequality of Adler, Rosalsky, and Taylor [1]. This note owes much to the work of Sung [3].

2 Preliminaries

In this section, some definitions will be reviewed and the needed results of Sung [3] and Adler, Rosalsky, and Taylor [1] will be stated.

Definition 2.1 The random variables comprising an array $\{X_{n,j}, 1 \le j \le k_n, n \ge 1\}$ are said to be rowwise *pairwise negative quadrant dependent* (PNQD) if for all $n \ge 1$ and all $i, j \in$

$\{1,\ldots,k_n\}$ $(i \neq j)$,

$$\mathbb{P}(X_{n,i} \leq x, X_{n,j} \leq y) \leq \mathbb{P}(X_{n,i} \leq x)\mathbb{P}(X_{n,j} \leq y) \quad \text{for all } x, y \in \mathbb{R}.$$

Definition 2.2 The random variables comprising an array $\{Y_{n,j}, 1 \leq j \leq k_n, n \geq 1\}$ are said to be *stochastically dominated* by a random variable Y if there exists a constant D such that

$$\mathbb{P}\bigl(|Y_{n,j}| > y\bigr) \leq D\mathbb{P}\bigl(|DY| > y\bigr), \quad y \geq 0, 1 \leq j \leq k_n, n \geq 1. \tag{2.1}$$

Lemma 2.1 (Adler, Rosalsky, and Taylor [1, Lemma 2.3]) *If the random variables in the array* $\{Y_{n,j}, 1 \leq j \leq k_n, n \geq 1\}$ *are stochastically dominated by a random variable* Y*, then for all* $n \geq 1$ *and* $j \in \{1,\ldots,k_n\}$,

$$\mathbb{E}\bigl(|Y_{n,j}|I\bigl(|Y_{n,j}| > y\bigr)\bigr) \leq D^2\mathbb{E}\bigl(|Y|I\bigl(|DY| > y\bigr)\bigr) \quad \textit{for all } y \geq 0,$$

where D *is as in* (2.1).

Proposition 2.1 (Sung [3, Theorem 2.1]) *Let* $\{X_{n,j}, 1 \leq j \leq k_n, n \geq 1\}$ *be an array of rowwise PNQD random variables and let* $r \in [1,2)$. *Let* $\{a_{n,j}, 1 \leq j \leq k_n, n \geq 1\}$ *be an array of constants. Suppose that*

$$\sup_{n \geq 1} \sum_{j=1}^{k_n} |a_{n,j}|^r \mathbb{E}|X_{n,j}|^r < \infty \tag{2.2}$$

and

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} |a_{n,j}|^r \mathbb{E}\bigl(|X_{n,j}|^r I\bigl(|a_{n,j}|^r |X_{n,j}|^r > \varepsilon\bigr)\bigr) = 0 \quad \textit{for all } \varepsilon > 0. \tag{2.3}$$

Then

$$\sum_{j=1}^{k_n} a_{n,j}(X_{n,j} - \mathbb{E}X_{n,j}) \xrightarrow{\mathcal{L}_r} 0$$

and, a fortiori,

$$\sum_{j=1}^{k_n} a_{n,j}(X_{n,j} - \mathbb{E}X_{n,j}) \xrightarrow{\mathbb{P}} 0.$$

3 Improved version of the Chandra, Li, and Rosalsky [2] result

We will now use Lemma 2.1 and Proposition 2.1 to present the following improved version of Theorem 3.1 of Chandra, Li, and Rosalsky [2].

Theorem 3.1 *Let* $\{X_{n,j}, 1 \leq j \leq k_n, n \geq 1\}$ *be an array of rowwise PNQD mean* 0 *random variables which are stochastically dominated by a random variable* X *with* $\mathbb{E}|X|^r < \infty$ *for* *some* $r \in [1,2)$. *Let* $\{a_{n,j}, 1 \le j \le k_n, n \ge 1\}$ *be an array of constants such that*

$$\sup_{n\ge 1} \sum_{j=1}^{k_n} |a_{n,j}|^r < \infty \tag{3.1}$$

and

$$\lim_{n\to\infty} \sup_{1\le j\le k_n} |a_{n,j}| = 0. \tag{3.2}$$

Then

$$\sum_{j=1}^{k_n} a_{n,j} X_{n,j} \xrightarrow{\mathcal{L}_r} 0 \tag{3.3}$$

and, a fortiori,

$$\sum_{j=1}^{k_n} a_{n,j} X_{n,j} \xrightarrow{\mathbb{P}} 0.$$

Remark 3.1 Before proving Theorem 3.1, we point out that Theorem 3.1 of Chandra, Li, and Rosalsky [2]

(i) only treated the case $r = 1$,

(ii) had the additional condition

$$\text{for each } n \ge 1, \text{either } \min_{1\le j\le k_n} a_{n,j} \ge 0 \text{ or } \max_{1\le j\le k_n} a_{n,j} \le 0,$$

(iii) had the condition

$$\sup_{n\ge 1} \sum_{j=1}^{k_n} |a_{n,j}| < \infty \quad \text{and} \quad \lim_{n\to\infty} \sum_{j=1}^{k_n} a_{n,j}^2 = 0,$$

the second half of which is clearly stronger than (3.2).

Proof of Theorem 3.1 Letting D be as in (2.1) with $Y_{n,j}$ replaced by $X_{n,j}$, $1 \le j \le k_n$, $n \ge 1$ and Y replaced by X, it follows that

$$\mathbb{E}|X_{n,j}|^r \le D^{r+1}\mathbb{E}|X|^r, \quad 1 \le j \le k_n, n \ge 1.$$

Thus

$$\sup_{n\ge 1} \sum_{j=1}^{k_n} |a_{n,j}|^r \mathbb{E}|X_{n,j}|^r \le D^{r+1}\left(\sup_{n\ge 1} \sum_{j=1}^{k_n} |a_{n,j}|^r\right)\mathbb{E}|X|^r < \infty$$

by (3.1) and $\mathbb{E}|X|^r < \infty$, thereby verifying (2.2).

Next, we show that (2.3) holds. Let

$$\lambda_n = D \sup_{1\le j\le k_n} |a_{n,j}|, \quad n \ge 1.$$

Then $\lim_{n\to\infty} \lambda_n = 0$ by (3.2). Now the stochastic domination hypothesis ensures that

$$\mathbb{P}\big(|X_{n,j}|^r > x\big) \le D\mathbb{P}\big(|DX|^r > x\big) = D\mathbb{P}\big(D\big(D^{r-1}|X|^r\big) > x\big), \quad x \ge 0, 1 \le j \le k_n, n \ge 1$$

and so by Lemma 2.1 with $Y_{n,j}$ replaced by $|X_{n,j}|^r$, $1 \le j \le k_n$, $n \ge 1$ and Y replaced by $D^{r-1}|X|^r$,

$$\begin{aligned}
&\mathbb{E}\big(|X_{n,j}|^r I\big(|X_{n,j}|^r > x\big)\big) \\
&\quad \le D^2\mathbb{E}\big(D^{r-1}|X|^r I\big(D^r|X|^r > x\big)\big) \\
&\quad = D^{r+1}\mathbb{E}\big(|X|^r I\big(D^r|X|^r > x\big)\big), \quad x \ge 0, 1 \le j \le k_n, n \ge 1.
\end{aligned} \tag{3.4}$$

Then for arbitrary $\varepsilon > 0$,

$$\begin{aligned}
\sum_{j=1}^{k_n} |a_{n,j}|^r \mathbb{E}\big(|X_{n,j}|^r I\big(|a_{n,j}|^r|X_{n,j}|^r > \varepsilon\big)\big) &\le D^{r+1}\sum_{j=1}^{k_n} |a_{n,j}|^r \mathbb{E}\left(|X|^r I\left(D^r|X|^r > \frac{\varepsilon}{|a_{n,j}|^r}\right)\right) \\
&\le D^{r+1}\left(\sum_{j=1}^{k_n} |a_{n,j}|^r\right)\mathbb{E}\left(|X|^r I\left(|X|^r > \frac{\varepsilon}{\lambda_n^r}\right)\right) \\
&\le D^{r+1}\left(\sup_{m\ge1}\sum_{j=1}^{k_m} |a_{m,j}|^r\right)\mathbb{E}\left(|X|^r I\left(|X|^r > \frac{\varepsilon}{\lambda_n^r}\right)\right) \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}$$

by (3.1), $\lambda_n \to 0$, and $\mathbb{E}|X|^r < \infty$. Thus (2.3) holds, and conclusion (3.3) follows from Proposition 2.1. □

Remark 3.2 See Chandra, Li, and Rosalsky [2] for examples

(i) showing that Theorem 3.1 can fail if the PNQD hypothesis is dispensed with,

(ii) showing that $\sum_{j=1}^{k_n} a_{n,j}X_{n,j} \to 0$ almost surely does not necessarily hold under the hypotheses of Theorem 3.1.

4 Conclusions

For an array of rowwise PNQD random variables $\{X_{n,j}, 1 \le j \le k_n, n \ge 1\}$, conditions are provided under which the following degenerate mean convergence law holds:

$$\sum_{j=1}^{k_n} a_{n,j}X_{n,j} \xrightarrow{\mathcal{L}_r} 0,$$

where $1 \le r < 2$, $\mathbb{E}X_{n,j} = 0$, $1 \le j \le k_n$, $n \ge 1$, and $\{a_{n,j}, 1 \le j \le k_n, n \ge 1\}$ is an array of constants. This result is an improved version of Theorem 3.1 of Chandra, Li, and Rosalsky [2] in that $\mathcal{L}_1$ convergence is extended to $\mathcal{L}_r$ convergence and the hypotheses are weakened.

The result is obtained by applying a result of Sung [3] and an inequality of Adler, Rosalsky, and Taylor [1].

Funding
The research of Deli Li was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada (Grant #: RGPIN-2014-05428).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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Received: 1 October 2018 Accepted: 17 January 2019 Published online: 04 February 2019

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