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An exact estimate result for p-biharmonic equations with Hardy potential and negative exponents

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Abstract

In this paper, p-biharmonic equations involving Hardy potential and negative exponents with a parameter λ are considered. By means of the structure and properties of Nehari manifold, we give uniform lower bounds for $\Lambda > 0$, which is the supremum of the set of λ. When $\lambda \in (0, \Lambda)$, the above problems admit at least two positive solutions.

Keywords: p-biharmonic equation; Nehari manifold; Positive solution; Negative exponents

1 Introduction and preliminaries

In this paper, we consider a p-biharmonic equation with Hardy potential and negative exponents:

$$\begin{cases} \Delta_p^2 u - \mu \frac{|u|^{p-2}u}{|x|^{2p}} = f(x)u^{-q} + \lambda g(x)u^{\gamma} & \text{in } \Omega \setminus \{0\}, \\ u(x) > 0 & \text{in } \Omega \setminus \{0\}, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $1 < p < \frac{N}{2}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator. $\lambda > 0$ is a parameter, $0 < \mu < \mu_{N,p} = (\frac{(p-1)N(N-2p)}{p^2})^p$, $0 < q < 1$ and $p - 1 < \gamma < p^* - 1$, where $p^* = \frac{Np}{N-2p}$ is called the critical Sobolev exponent. $f(x) \geq 0$, $f(x) \not\equiv 0$, $g(x)$ satisfies the requirement that the set $\{x \in \Omega : g(x) > 0\}$ has positive measures, $\operatorname{supp} f \cap \{x \in \Omega : g(x) > 0\} \neq \emptyset$ and $f, g \in C(\overline{\Omega})$. Biharmonic equations describe the sport of a rigid body and the deformations of an elastic beam. For example, this type of equation provides a model for considering traveling wave in suspension bridges [5, 16, 27, 30, 36]. Various methods and tools have been adopted to deal with singular problems, such that fixed point theorems [14], topological methods [37], Fourier and Laurent transformation [18, 19], monotone iterative methods [21], global bifurcation theory [12], and degree theory [22, 31].

In recent years, there was much attention focused on the existence, multiplicity and qualitative properties of solutions for p-biharmonic equations under Dirichlet boundary conditions or Navier boundary conditions with Hardy terms [4, 15, 17, 32, 34]. Xie and

Wang [32] studied the following p-biharmonic equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta_p^2 u - \mu \frac{|u|^{p-2}u}{|x|^{2p}} = f(x,u) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\frac{\partial}{\partial n}$ is the outer normal derivative. By using the variational method, the existence of infinitely many solutions with positive energy levels for (1.2) was established. Huang and Liu [15] considered the following p-biharmonic equation with Navier boundary conditions:

$$\begin{cases} \Delta_p^2 u - \mu \frac{|u|^{p-2}u}{|x|^{2p}} = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < p < \frac{N}{2}$. By using invariant sets of gradient flows, the authors proved that (1.3) possesses a sign-changing solution. Furthermore, Yang, Zhang and Liu [34] showed that (1.3) has a positive solution, a negative solution and a sequence of sign-changing solutions when f satisfies appropriate conditions. Bhakta [4] established the qualitative properties of entire solutions for a noncompact problem related to p-biharmonic type equations with Hardy terms.

On the other hand, nonlinear biharmonic equations with negative exponents have been studied expensively [1, 6, 8, 13, 20]. Guerra [13] gave a complete description of entire radially symmetric solutions for the following biharmonic equation:

$$\Delta^2 u = -u^{-q}, \qquad u > 0 \quad \text{in } \mathbb{R}^3,$$

where $q > 1$. Moreover, Cowan et al. [8] dealt with the regularity of the extremal solution of the following fourth order boundary value problems:

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Very recently, Ansari, Vaezpour and Hesaaraki [1] considered fourth order elliptic problem with the combinations of Hardy term and negative exponents,

$$\begin{cases} \Delta^2 u - \lambda M(\|\nabla u\|^2)\Delta u - \frac{\mu}{|x|^4} u = \frac{h(x)}{u^\gamma} + k(x)u^\alpha & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded C^4-domain, λ and μ are positive parameters and $0 < \alpha < 1$, $0 < \gamma < 1$ are constants. Here M, h and k are given continuous functions satisfying suitable hypotheses. By using the Galerkin method and the sharp angle lemma, the authors proved that problem (1.4) has a positive solution for $0 < \mu < (\frac{N(N-4)}{4})^2$.

We say that $u \in W := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a weak solution of (1.1), if for every $\varphi \in W$, there holds

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi \, dx - \int_\Omega \frac{\mu}{|x|^{2p}}|u|^{p-2}u\varphi \, dx = \int_\Omega f(x)u^{-q}\varphi \, dx + \lambda \int_\Omega g(x)u^\gamma \varphi \, dx. \quad (1.5)$$

The following Rellich inequality will be used in this paper:

$$\int_\Omega |\Delta u|^p \, dx \geq \mu_{N,p} \int_\Omega \frac{|u|^p}{|x|^{2p}} \, dx, \quad \forall u \in W,$$

and it is not achieved [9, 24]. For any $u \in W$, and $0 < \mu < \mu_{N,p}$. The energy functional corresponding to (1.1) is defined by

$$\begin{aligned} I_{\lambda,\mu}(u) = {} & \frac{1}{p}\int_\Omega \left(|\Delta u|^p - \frac{\mu}{|x|^{2p}}|u|^p\right) dx - \frac{1}{1-q}\int_\Omega f(x)|u|^{1-q} \, dx \\ & - \frac{\lambda}{\gamma+1}\int_\Omega g(x)|u|^{\gamma+1} \, dx. \end{aligned} \quad (1.6)$$

For $\mu \in [0, \mu_{N,p})$, W is equipped with the following norm:

$$\|u\|_\mu^p = \int_\Omega \left(|\Delta u|^p - \frac{\mu}{|x|^{2p}}|u|^p\right) dx.$$

Negative exponent term u^{-q} implies that $I_{\lambda,\mu}$ is not differential on W, therefore, critical point theory cannot be applied to the problem (1.1) directly. We consider the following manifold:

$$\mathcal{M} = \left\{u \in W : \|u\|_\mu^p = \int_\Omega f(x)|u|^{1-q} \, dx + \lambda \int_\Omega g(x)|u|^{\gamma+1} \, dx\right\},$$

and make the following splitting for $\mathcal{M}$:

$$\mathcal{M}^+ = \left\{u \in \mathcal{M} : (p+q-1)\|u\|_\mu^p > \lambda(\gamma+q)\int_\Omega g(x)|u|^{\gamma+1} \, dx\right\}, \quad (1.7)$$

$$\mathcal{M}^0 = \left\{u \in \mathcal{M} : (p+q-1)\|u\|_\mu^p = \lambda(\gamma+q)\int_\Omega g(x)|u|^{\gamma+1} \, dx\right\}, \quad (1.8)$$

$$\mathcal{M}^- = \left\{u \in \mathcal{M} : (p+q-1)\|u\|_\mu^p < \lambda(\gamma+q)\int_\Omega g(x)|u|^{\gamma+1} \, dx\right\}. \quad (1.9)$$

In this paper, we will study the dependence of problem (1.1) on q, γ, f, g and Ω and evaluate the extremal value of λ related to multiplicity of positive solutions for problem (1.1). Our idea comes from [7, 28, 29]. Our results improve and complement previous ones obtained in [23, 25]. Denote $\|u\|_t^t = \int_\Omega |u|^t \, dx$ and $D^{2,p}(\mathbb{R}^N)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\Delta u|^p \, dx)^{\frac{1}{p}}$.

λ_1 denotes the smallest eigenvalue for

$$\Delta_p^2 u - \frac{\mu}{|x|^{2p}}|u|^{p-2}u = \lambda_1 |u|^{p-2}u, \quad x \in \Omega \setminus \{0\}, u \in W, \quad (1.10)$$

and φ_1 denotes the corresponding eigenfunction with $\varphi_1 > 0$ in Ω [3, 10, 26, 33, 35]. The following minimization problem will be useful in the following discussions:

$$S_\mu = \inf\left\{\int_{\mathbb{R}^N}\left(|\Delta u|^p - \frac{\mu}{|x|^{2p}}|u|^p\right)dx, u \in D^{2,p}(\mathbb{R}^N), \int_{\mathbb{R}^N}|u|^{p^*}\,dx = 1\right\} > 0, \tag{1.11}$$

and S_μ is achieved by a family of functions [4, 11]. Thus, for every $u \in W \setminus \{0\}$, $\|u\|_{p^*} \le \frac{\|u\|_\mu}{\sqrt[p]{S_\mu}}$. Therefore, combining with the Hölder inequality, we deduce that

$$\begin{aligned}\int_\Omega |u|^{\gamma+1}\,dx &\le \left[\int_\Omega |u|^{(\gamma+1)\frac{p^*}{\gamma+1}}\,dx\right]^{\frac{\gamma+1}{p^*}}\left(\int_\Omega 1\,dx\right)^{\frac{p^*-\gamma-1}{p^*}} \\ &= |\Omega|^{\frac{p^*-\gamma-1}{p^*}}\|u\|_{p^*}^{\gamma+1} \\ &\le |\Omega|^{\frac{p^*-\gamma-1}{p^*}}\left(\frac{\|u\|_\mu}{\sqrt[p]{S_\mu}}\right)^{\gamma+1},\end{aligned} \tag{1.12}$$

$$\begin{aligned}\int_\Omega |u|^{1-q}\,dx &\le \left[\int_\Omega |u|^{(1-q)\frac{p^*}{1-q}}\,dx\right]^{\frac{1-q}{p^*}}\left(\int_\Omega 1\,dx\right)^{\frac{p^*-1+q}{p^*}} \\ &= |\Omega|^{\frac{p^*-1+q}{p^*}}\|u\|_{p^*}^{1-q} \\ &\le |\Omega|^{\frac{p^*-1+q}{p^*}}\left(\frac{\|u\|_\mu}{\sqrt[p]{S_\mu}}\right)^{1-q},\end{aligned} \tag{1.13}$$

and

$$\begin{aligned}\int_\Omega |u|^{1-q}\,dx &\le \left[\int_\Omega |u|^{(1-q)\frac{\gamma+1}{1-q}}\,dx\right]^{\frac{1-q}{\gamma+1}}\left(\int_\Omega 1\,dx\right)^{\frac{\gamma+q}{\gamma+1}} \\ &= |\Omega|^{\frac{\gamma+q}{\gamma+1}}\|u\|_{\gamma+1}^{1-q}.\end{aligned} \tag{1.14}$$

Our main results are stated in the following theorems.

Theorem 1.1 *Assume that* $\lambda \in (0, \Lambda)$, *where*

$$\begin{aligned}\Lambda \ge T_\mu &= \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|f\|_\infty}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|g\|_\infty}\right)\left(\frac{S_\mu}{|\Omega|^{\frac{p}{N}}}\right)^{\frac{q+\gamma}{p+q-1}} \\ &> 0.\end{aligned} \tag{1.15}$$

Then problem (1.1) *admits at least two solutions* $u_0 \in \mathcal{M}^+$, $U_0 \in \mathcal{M}^-$, *with* $\|U_0\|_\mu > \|u_0\|_\mu$.

Corollary 1.2 *Let* $U_{\lambda,\mu,\varepsilon} \in \mathcal{M}^-$ *be the solution of problem* (1.1) *with* $\gamma = \varepsilon + p - 1$, *where* $\lambda \in (0, T_\mu)$. *Then*

$$\|U_{\lambda,\mu,\varepsilon}\|_\mu > C_{\mu,\varepsilon}\left(\frac{T_\mu}{\lambda}\right)^{\frac{1}{\varepsilon}}$$

with

$$C_{\mu,\varepsilon} = |\Omega|^{\frac{1}{p}} \left(\|f\|_\infty\right)^{\frac{1}{p+q-1}} \left(1 + \frac{p+q-1}{\varepsilon}\right)^{\frac{1}{p+q-1}} \left(\frac{|\Omega|^{\frac{2}{N}}}{\sqrt[p]{S_\mu}}\right)^{\frac{1-q}{p+q-1}} \to \infty, \quad \text{as } \varepsilon \to 0. \quad (1.16)$$

Theorem 1.3 *There exists* $\lambda^* = \lambda^*(N, \Omega, \mu, q, \gamma) > 0$ *such that problem* (1.1) *with* $f = g = 1$ *admits at least a positive solution for every* $0 < \lambda < \lambda^*$ *and has no solution for every* $\lambda > \lambda^*$.

2 Some lemmas

Lemma 2.1 *Assume that* $\lambda \in (0, T_\mu)$, *where* T_μ *is defined in* (1.15). *Then* $\mathcal{M}^\pm \neq \emptyset$ *and* $\mathcal{M}^0 = \{0\}$.

Proof (i) We can choose $u^* \in \mathcal{M} \setminus \{0\}$ such that $\int_\Omega f(x)|u^*|^{1-q}\,dx > 0$ and $\int_\Omega g(x) \times |u^*|^{\gamma+1}\,dx > 0$ from the conditions imposed on f and g. Denote

$$\begin{aligned}\varphi_\mu(t) &:= \frac{1}{t^\gamma}\frac{d}{dt} I_{\lambda,\mu}(tu^*) \\ &= t^{p-1-\gamma}\|u^*\|_\mu^p - t^{-q-\gamma}\int_\Omega f(x)|u^*|^{1-q}\,dx - \lambda\int_\Omega g(x)|u^*|^{\gamma+1}\,dx, \quad t > 0.\end{aligned}$$

Note that $\varphi'_\mu(t) = (p-1-\gamma)t^{p-2-\gamma}\|u^*\|_\mu^p + (q+\gamma)t^{-1-q-\gamma}\int_\Omega f(x)|u^*|^{1-q}\,dx$. Let $\varphi'_\mu(t) = 0$, we have

$$t := t_{\max} = \left[\frac{(\gamma-p+1)\|u^*\|_\mu^p}{(q+\gamma)\int_\Omega f(x)|u^*|^{1-q}\,dx}\right]^{\frac{1}{1-q-p}}. \quad (2.1)$$

It is easy to check that $\varphi_\mu(t) \to -\infty$ as $t \to 0^+$ and $\varphi_\mu(t) \to -\lambda\int_\Omega g(x)|u^*|^{\gamma+1}\,dx < 0$ as $t \to \infty$. Furthermore, $\varphi_\mu(t)$ attains its maximum at $t_{\max}$. By (1.12) and (1.13), we obtain

$$\begin{aligned}&\varphi_\mu(t_{\max}) \\ &= \left[\frac{(\gamma-p+1)\|u^*\|_\mu^p}{(q+\gamma)\int_\Omega f(x)|u^*|^{1-q}\,dx}\right]^{\frac{p-\gamma-1}{1-q-p}}\|u^*\|_\mu^p \\ &\quad - \left[\frac{(\gamma-p+1)\|u^*\|_\mu^p}{(q+\gamma)\int_\Omega f(x)|u^*|^{1-q}\,dx}\right]^{\frac{-q-\gamma}{1-q-p}}\int_\Omega f(x)|u^*|^{1-q}\,dx \\ &\quad - \lambda\int_\Omega g(x)|u^*|^{\gamma+1}\,dx \\ &= \left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\frac{(\|u^*\|_\mu^p)^{\frac{-\gamma-q}{1-q-p}}}{(\int_\Omega f(x)|u^*|^{1-q}\,dx)^{\frac{p-\gamma-1}{1-q-p}}} \\ &\quad - \left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{-q-\gamma}{1-q-p}}\frac{(\|u^*\|_\mu^p)^{\frac{-\gamma-q}{1-q-p}}}{(\int_\Omega f(x)|u^*|^{1-q}\,dx)^{\frac{p-\gamma-1}{1-q-p}}} \\ &\quad - \lambda\int_\Omega g(x)|u^*|^{\gamma+1}\,dx \\ &= \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\frac{(\|u^*\|_\mu^p)^{\frac{-\gamma-q}{1-q-p}}}{(\int_\Omega f(x)|u^*|^{1-q}\,dx)^{\frac{p-\gamma-1}{1-q-p}}} - \lambda\int_\Omega g(x)|u^*|^{\gamma+1}\,dx\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}} \frac{(\|u^*\|_\mu^p)^{\frac{-\gamma-q}{1-q-p}}}{[\|f\|_\infty|\Omega|^{\frac{p^*-1+q}{p^*}}(\frac{\|u^*\|_\mu}{\sqrt[p]{S_\mu}})^{1-q}]^{\frac{p-\gamma-1}{1-q-p}}} \\
&\quad - \lambda\|g\|_\infty|\Omega|^{\frac{p^*-\gamma-1}{p^*}}\left(\frac{\|u^*\|_\mu}{\sqrt[p]{S_\mu}}\right)^{\gamma+1} \\
&= \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|f\|_\infty}\right)^{\frac{p-\gamma-1}{1-q-p}} \frac{(\sqrt[p]{S_\mu})^{\frac{(1-q)(p-\gamma-1)}{1-q-p}}}{|\Omega|^{\frac{p^*-1+q}{p^*}\frac{p-\gamma-1}{1-q-p}}}\|u^*\|_\mu^{\gamma+1} \\
&\quad - \lambda\|g\|_\infty|\Omega|^{\frac{p^*-\gamma-1}{p^*}}\left(\frac{\|u^*\|_\mu}{\sqrt[p]{S_\mu}}\right)^{\gamma+1} \\
&= \left[\left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|f\|_\infty}\right)^{\frac{p-\gamma-1}{1-q-p}} \frac{(\sqrt[p]{S_\mu})^{\frac{(1-q)(p-\gamma-1)}{1-q-p}}}{|\Omega|^{\frac{p^*-1+q}{p^*}\frac{p-\gamma-1}{1-q-p}}}\right. \\
&\quad \left. - \lambda\|g\|_\infty\frac{|\Omega|^{\frac{p^*-\gamma-1}{p^*}}}{(\sqrt[p]{S_\mu})^{\gamma+1}}\right]\|u^*\|_\mu^{\gamma+1} \\
&:= A(\mu,\lambda)\|u^*\|_\mu^{\gamma+1} \\
&> 0. \qquad (2.2)
\end{aligned}
$$

When $A(\mu,\lambda) = 0$, we get

$$
\begin{aligned}
\lambda &= \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|f\|_\infty}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|g\|_\infty}\right)\frac{(\sqrt[p]{S_\mu})^{\frac{(1-q)(p-\gamma-1)}{1-q-p}+\gamma+1}}{|\Omega|^{\frac{p^*-1+q}{p^*}\frac{p-\gamma-1}{1-q-p}+\frac{p^*-\gamma-1}{p^*}}} \\
&= \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|f\|_\infty}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|g\|_\infty}\right)\left[\frac{S_\mu}{|\Omega|^{\frac{2p}{N}}}\right]^{\frac{q+\gamma}{p+q-1}} = T_\mu,
\end{aligned}
$$

where we use the following two equalities:

$$
\frac{(1-q)(p-\gamma-1)}{1-q-p}+\gamma+1 = \frac{p(q+\gamma)}{q+p-1},
$$

and

$$
\frac{(p^*-1+q)(p-\gamma-1)}{p^*(1-q-p)}+\frac{p^*-\gamma-1}{p^*} = \frac{2p(q+\gamma)}{N(q+p-1)}.
$$

In turn, this is also true. Hence $A(\mu,\lambda) = 0$ if and only if $\lambda = T_\mu$. Thus for $\lambda \in (0, T_\mu)$, we have $A(\mu,\lambda) > 0$. Moreover, by (2.2), we derive that $\varphi_\mu(t_{\max}) > 0$. Consequently, there exist two numbers t_μ^- and t_μ^+ such that $0 < t_\mu^- < t_{\max} < t_\mu^+$, and

$$
\varphi_\mu(t_\mu^-) = 0 = \varphi_\mu(t_\mu^+), \qquad \varphi'_\mu(t_\mu^-) > 0 > \varphi'_\mu(t_\mu^+).
$$

It follows that $t_\mu^- u^* \in \mathcal{M}^+$, and $t_\mu^+ u^* \in \mathcal{M}^-$. In fact, if $\varphi_\mu(t) = 0$, then

$$
\varphi_\mu(t) = t^{p-1-\gamma}\|u\|_\mu^p - t^{-q-\gamma}\int_\Omega f(x)|u|^{1-q}\,dx - \lambda\int_\Omega g(x)|u|^{\gamma+1}\,dx = 0,
$$

namely

$$\|tu\|_\mu^p = \int_\Omega f(x)|tu|^{1-q}\,dx + \lambda \int_\Omega g(x)|tu|^{\gamma+1}\,dx.$$

Hence $tu \in \mathcal{M}$. Furthermore, if $\varphi_\mu'(t) > 0$, then

$$(p-1-\gamma)t^{p-2-\gamma}\|u\|_\mu^p + (q+\gamma)t^{-1-q-\gamma}\int_\Omega f(x)|u|^{1-q}\,dx > 0.$$

That is

$$(p-1-\gamma)\|tu\|_\mu^p + (q+\gamma)\int_\Omega f(x)|tu|^{1-q}\,dx > 0,$$

i.e.,

$$(p-1-\gamma)\|tu\|_\mu^p + (q+\gamma)\left[\|tu\|_\mu^p - \lambda\int_\Omega g(x)|tu|^{\gamma+1}\,dx\right] > 0.$$

Note that $tu \in \mathcal{M}$, we have

$$(p+q-1)\|tu\|_\mu^p - \lambda(q+\gamma)\int_\Omega g(x)|tu|^{\gamma+1}\,dx > 0.$$

Thus $tu \in \mathcal{M}^+$. By a similar argument, if $\varphi_\mu(t) = 0$ and $\varphi_\mu'(t) < 0$, then $tu \in \mathcal{M}^-$. Therefore, both $\mathcal{M}^+$ and $\mathcal{M}^-$ are non-empty sets for every $\lambda \in (0, T_\mu)$.

(ii) We claim that $\mathcal{M}^0 = \{0\}$. Otherwise, we suppose that there exists $u_* \in \mathcal{M}^0$ and $u_* \neq 0$. Since $u_* \in \mathcal{M}^0$, we have

$$(p+q-1)\|u_*\|_\mu^p = \lambda(\gamma+q)\int_\Omega g(x)|u_*|^{\gamma+1}\,dx,$$

moreover

$$\begin{aligned}
0 &= \|u_*\|_\mu^p - \int_\Omega f(x)u_*^{1-q}\,dx - \lambda\int_\Omega g(x)u_*^{\gamma+1}\,dx \\
&= \|u_*\|_\mu^p - \int_\Omega f(x)u_*^{1-q}\,dx - \frac{p+q-1}{\gamma+q}\|u_*\|_\mu^p \\
&= \frac{\gamma-p+1}{\gamma+q}\|u_*\|_\mu^p - \int_\Omega f(x)u_*^{1-q}\,dx.
\end{aligned}$$

For $\lambda \in (0, T_\mu)$ and $u_* \neq 0$, combining with (2.2), we deduce that

$$\begin{aligned}
0 &< A(\mu,\lambda)\|u_*\|_\mu^{\gamma+1} \\
&\leq \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}} \frac{(\|u_*\|_\mu^p)^{\frac{-\gamma-q}{1-q-p}}}{(\frac{\gamma-p+1}{q+\gamma}\|u_*\|_\mu^p)^{\frac{p-\gamma-1}{1-q-p}}} - \left(\frac{q+p-1}{q+\gamma}\right)\|u_*\|_\mu^p = 0,
\end{aligned}$$

which is a contradiction, Thus $u_* = 0$. That is, $\mathcal{M}^0 = \{0\}$. □

The gap structure in $\mathcal{M}$ is embodied in the following lemma.

Lemma 2.2 *Assume that* $\lambda \in (0, T_\mu)$, *then*

$$\|U\|_\mu > M_\mu(\lambda) > M_{\mu,0} > \|u\|_\mu,$$
$$\|U\|_{\gamma+1} > N_\mu(\lambda) > N_{\mu,0} > \|u\|_{\gamma+1}, \quad \forall u \in \mathcal{M}^+, U \in \mathcal{M}^-,$$

where

$$M_{\mu,0} = \left[\frac{\gamma+q}{\gamma-p+1}\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}\right]^{\frac{1}{p+q-1}},$$
$$M_\mu(\lambda) = \left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{(\sqrt[p]{S_\mu})^{\gamma+1}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*}}}\right]^{\frac{1}{\gamma+1-p}},$$
$$N_{\mu,0} = \left[\frac{\gamma+q}{\gamma-p+1}\|f\|_\infty \frac{|\Omega|^{\frac{\gamma+q}{\gamma+1}+\frac{(p^*-1-\gamma)p}{p^*(\gamma+1)}}}{S_\mu}\right]^{\frac{1}{p+q-1}},$$
$$N_\mu(\lambda) = \left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{S_\mu}{|\Omega|^{p(\frac{p^*-1-\gamma}{p^*(\gamma+1)})}}\right]^{\frac{1}{\gamma+1-p}}.$$

Proof If $u \in \mathcal{M}^+ \subset \mathcal{M}$, then

$$\begin{aligned} 0 &< (p+q-1)\|u\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)|u|^{\gamma+1}\,dx \\ &= (p+q-1)\|u\|_\mu^p - (\gamma+q)\left[\|u\|_\mu^p - \int_\Omega f(x)|u|^{1-q}\,dx\right] \\ &= (p-\gamma-1)\|u\|_\mu^p + (\gamma+q)\int_\Omega f(x)|u|^{1-q}\,dx. \end{aligned}$$

We obtain from (1.13) that

$$\begin{aligned} (\gamma-p+1)\|u\|_\mu^p &< (\gamma+q)\int_\Omega f(x)|u|^{1-q}\,dx \\ &\le (\gamma+q)\|f\|_\infty|\Omega|^{\frac{p^*-1+q}{p^*}}\left(\frac{\|u\|_\mu}{\sqrt[p]{S_\mu}}\right)^{1-q}, \end{aligned}$$

which leads to

$$\|u\|_\mu < \left[\frac{\gamma+q}{\gamma-p+1}\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}\right]^{\frac{1}{p+q-1}} = M_{\mu,0}.$$

By (1.12) and (1.14), we have

$$\begin{aligned} &(\gamma-p+1)\|u\|_{\gamma+1}^p \frac{S_\mu}{|\Omega|^{p(\frac{p^*-1-\gamma}{p^*(\gamma+1)})}} \\ &\quad\le (\gamma-p+1)\frac{S_\mu}{|\Omega|^{p\frac{p^*-1-\gamma}{p^*(\gamma+1)}}}\left[|\Omega|^{\frac{p^*-1-\gamma}{p^*}}\left(\frac{\|u\|_\mu}{\sqrt[p]{S_\mu}}\right)^{\gamma+1}\right]^{\frac{p}{\gamma+1}} \end{aligned}$$

$$
\begin{aligned}
&= (\gamma - p + 1)\|u\|_\mu^p \\
&< (\gamma + q)\int_\Omega f(x)|u|^{1-q}\,dx \\
&\le (\gamma + q)\|f\|_\infty |\Omega|^{\frac{\gamma+q}{\gamma+1}}\|u\|_{\gamma+1}^{1-q},
\end{aligned}
$$

which implies that

$$
\|u\|_{\gamma+1} < \left[\frac{\gamma+q}{\gamma-p+1}\|f\|_\infty \frac{|\Omega|^{\frac{\gamma+q}{\gamma+1}+\frac{(p^*-1-\gamma)p}{p^*(\gamma+1)}}}{S_\mu}\right]^{\frac{1}{p+q-1}} = N_{\mu,0}.
$$

If $U \in \mathcal{M}^- \subset \mathcal{M}$, combining with (1.12), we derive that

$$
\begin{aligned}
(p+q-1)\|U\|_\mu^p &< \lambda(\gamma+q)\int_\Omega g(x)|U|^{\gamma+1}\,dx \\
&\le \lambda(\gamma+q)\|g\|_\infty |\Omega|^{\frac{p^*-\gamma-1}{p^*}}\left(\frac{\|U\|_\mu}{\sqrt[p]{S_\mu}}\right)^{\gamma+1},
\end{aligned}
$$

which leads to

$$
\|U\|_\mu > \left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{(\sqrt[p]{S_\mu})^{\gamma+1}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*}}}\right]^{\frac{1}{\gamma+1-p}} = M_\mu(\lambda).
$$

Furthermore

$$
\begin{aligned}
&(p+q-1)\|U\|_{\gamma+1}^p \frac{S_\mu}{|\Omega|^{p(\frac{p^*-1-\gamma}{p^*(\gamma+1)})}} \\
&\le (p+q-1)\frac{S_\mu}{|\Omega|^{p\frac{p^*-1-\gamma}{p^*(\gamma+1)}}}\left[|\Omega|^{\frac{p^*-1-\gamma}{p^*}}\left(\frac{\|U\|_\mu}{\sqrt[p]{S_\mu}}\right)\right]^{\frac{p}{\gamma+1}} \\
&= (p+q-1)\|U\|_\mu^p \\
&< \lambda(\gamma+q)\int_\Omega g(x)|U|^{\gamma+1}\,dx \\
&\le \lambda(\gamma+q)\|g\|_\infty\|U\|_{\gamma+1}^{\gamma+1},
\end{aligned}
$$

which means that

$$
\|U\|_{\gamma+1} > \left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{S_\mu}{|\Omega|^{p(\frac{p^*-1-\gamma}{p^*(\gamma+1)})}}\right]^{\frac{1}{\gamma+1-p}} = N_\mu(\lambda).
$$

Therefore

$$
\lambda = T_\mu = \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|f\|_\infty}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|g\|_\infty}\right)\left(\frac{S_\mu}{|\Omega|^{\frac{2p}{N}}}\right)^{\frac{q+\gamma}{p+q-1}}
$$

$$\begin{aligned}
\Leftrightarrow \quad M_\mu(\lambda) &= \left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{(\sqrt[p]{S_\mu})^{\gamma+1}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*}}}\right]^{\frac{1}{\gamma+1-p}} \\
&= \lambda^{-\frac{1}{\gamma+1-p}}\left[\frac{p+q-1}{\gamma+q}\frac{1}{\|g\|_\infty}\frac{(\sqrt[p]{S_\mu})^{\gamma+1}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*}}}\right]^{\frac{1}{\gamma+1-p}} \\
&= \left(\frac{q+\gamma}{q+p-1}\right)^{\frac{1}{\gamma+1-p}}\left(\frac{q+\gamma}{\gamma-p+1}\right)^{\frac{1}{p+q-1}}\left(\|f\|_\infty\right)^{\frac{1}{p+q-1}}\left(\|g\|_\infty\right)^{\frac{1}{\gamma+1-p}} \\
&\quad\times\frac{|\Omega|^{\frac{2p}{N}\frac{q+\gamma}{(q+p-1)(\gamma+1-p)}}}{(S_\mu)^{\frac{q+\gamma}{(p+q-1)(\gamma+1-p)}}}\left[\frac{p+q-1}{\gamma+q}\frac{1}{\|g\|_\infty}\frac{(\sqrt[p]{S_\mu})^{\gamma+1}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*}}}\right]^{\frac{1}{\gamma+1-p}} \\
&= \left(\frac{q+\gamma}{\gamma-p+1}\right)^{\frac{1}{p+q-1}}\left(\|f\|_\infty\right)^{\frac{1}{p+q-1}}\frac{|\Omega|^{\frac{2p}{N}\frac{q+\gamma}{(\gamma-p+1)(p+q-1)}-\frac{p^*-1-\gamma}{p^*}}}{(\sqrt[p]{S_\mu})^{p\frac{q+\gamma}{(\gamma-p+1)(p+q-1)}-\frac{\gamma+1}{\gamma+1-p}}} \\
&= \left[\frac{\gamma+q}{\gamma-p+1}\|f\|_\infty\frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}\right]^{\frac{1}{p+q-1}} = M_{\mu,0},
\end{aligned}$$

where we have used the following facts:

$$\begin{aligned}
&\frac{2p}{N}\frac{q+\gamma}{(\gamma-p+1)(p+q-1)}-\frac{p^*-1-\gamma}{p^*(\gamma-p+1)} \\
&\quad= \frac{2p(p^*-p)}{2pp^*}\frac{q+\gamma}{(\gamma-p+1)(p+q-1)}-\frac{p^*-1-\gamma}{p^*(\gamma-p+1)} \\
&\quad= \frac{(\gamma-p+1)(p^*+q-1)}{p^*(\gamma-p+1)(p+q-1)},
\end{aligned}$$

and

$$p\frac{q+\gamma}{(\gamma-p+1)(p+q-1)}-\frac{\gamma+1}{\gamma+1-p} = \frac{pq-q\gamma+\gamma-p-q+1}{(\gamma-p+1)(p+q-1)} = \frac{1-q}{p+q-1}.$$

Similarly

$$\lambda = T_\mu = \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|f\|_\infty}\right)^{\frac{p-\gamma-1}{1-q-p}}\left(\frac{1}{\|g\|_\infty}\right)\left[\frac{S_\mu}{|\Omega|^{\frac{2p}{N}}}\right]^{\frac{q+\gamma}{p+q-1}}.$$

$$\begin{aligned}
\Leftrightarrow \quad N_\mu(\lambda) &= \left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{S_\mu}{|\Omega|^{p(\frac{p^*-1-\gamma}{p^*(\gamma+1)})}}\right]^{\frac{1}{\gamma+1-p}} \\
&= \lambda^{-\frac{1}{\gamma+1-p}}\left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{S_\mu}{|\Omega|^{p(\frac{p^*-1-\gamma}{p^*(\gamma+1)})}}\right]^{\frac{1}{\gamma+1-p}} \\
&= \left(\frac{q+\gamma}{q+p-1}\right)^{\frac{1}{\gamma+1-p}}\left(\frac{q+\gamma}{\gamma-p+1}\right)^{\frac{1}{p+q-1}}\left(\|f\|_\infty\right)^{\frac{1}{p+q-1}}\left(\|g\|_\infty\right)^{\frac{1}{\gamma+1-p}} \\
&\quad\times\frac{|\Omega|^{\frac{2p}{N}\frac{q+\gamma}{(q+p-1)(\gamma+1-p)}}}{(S_\mu)^{\frac{q+\gamma}{(p+q-1)(\gamma+1-p)}}}\left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|W\|_\infty}\frac{S_\mu}{|\Omega|^{p(\frac{p^*-1-\gamma}{p^*(\gamma+1)})}}\right]^{\frac{1}{\gamma+1-p}}
\end{aligned}$$

$$= \left(\frac{q+\gamma}{\gamma-p+1}\right)^{\frac{1}{p+q-1}} (\|f\|_\infty)^{\frac{1}{p+q-1}} \frac{|\Omega|^{\frac{2p}{N}\frac{q+\gamma}{(\gamma-p+1)(p+q-1)}-p\frac{p^*-1-\gamma}{p^*(\gamma+1)(\gamma+1-p)}}}{(S_\mu)^{\frac{q+\gamma}{(\gamma-p+1)(p+q-1)}-\frac{1}{\gamma+1-p}}}$$

$$= \left[\frac{\gamma+q}{\gamma-p+1}\|f\|_\infty \frac{|\Omega|^{\frac{\gamma+q}{\gamma+1}+\frac{(p^*-1-\gamma)p}{p^*(\gamma+1)}}}{S_\mu}\right]^{\frac{1}{p+q-1}} = N_{\mu,0},$$

where we have applied the following equalities:

$$\begin{aligned}
&\frac{2p}{N}\frac{q+\gamma}{(\gamma-p+1)(p+q-1)} - p\frac{p^*-1-\gamma}{p^*(\gamma+1)(\gamma+1-p)} \\
&\quad = \frac{2p(p^*-p)}{2pp^*}\frac{q+\gamma}{(\gamma-p+1)(p+q-1)} - \frac{p^*-1-\gamma}{p^*(\gamma-p+1)} \\
&\quad = \frac{\gamma+q}{\gamma+1} + p\frac{p^*-1-\gamma}{p^*(\gamma+1)},
\end{aligned}$$

and

$$\frac{q+\gamma}{(\gamma-p+1)(p+q-1)} - \frac{1}{\gamma+1-p} = \frac{q+\gamma-(p+q-1)}{(\gamma-p+1)(p+q-1)} = \frac{1}{p+q-1}.$$

Consequently, $M_\mu(\lambda) = M_{\mu,0}$ if and only if $\lambda = T_\mu$ and $N_\mu(\lambda) = N_{\mu,0}$ if and only if $\lambda = T_\mu$ respectively. This completes the proof of Lemma 2.2. □

Lemma 2.3 *Assume that $\lambda \in (0, T_\mu)$. Then $\mathcal{M}^-$ is a closed set in W-topology.*

Proof We choose a sequence $\{U_n\}$ such that $\{U_n\} \subset \mathcal{M}^-$ and $U_n \to U_0$ with $U_0 \in W$. Then

$$\begin{aligned}
\|U_0\|_\mu^p &= \lim_{n\to\infty} \|U_n\|_\mu^p \\
&= \lim_{n\to\infty}\left[\int_\Omega f(x)|U_n|^{1-q}\,dx + \lambda\int_\Omega g(x)|U_n|^{\gamma+1}\,dx\right] \\
&= \int_\Omega f(x)|U_0|^{1-q}\,dx + \lambda\int_\Omega g(x)|U_0|^{\gamma+1}\,dx,
\end{aligned}$$

and

$$\begin{aligned}
&(p+q-1)\|U_0\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)|U_0|^{\gamma+1}\,dx \\
&\quad = \lim_{n\to\infty}\left[(p+q-1)\|U_n\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)|U_n|^{\gamma+1}\,dx\right] \le 0.
\end{aligned}$$

Hence $U_0 \in \mathcal{M}^- \cup \mathcal{M}^0$. By Lemma 2.2, we have

$$\|U_0\|_\mu = \lim_{n\to\infty}\|U_n\|_\mu \ge M_{\mu,0} > 0,$$

that is, $U_0 \neq 0$. Combining with Lemma 2.1, we obtain $U_0 \notin \mathcal{M}^0$. Thus $U_0 \in \mathcal{M}^-$. Therefore $\mathcal{M}^-$ is a closed set in W-topology for every $\lambda \in (0, T_\mu)$. □

Lemma 2.4 *For $u \in \mathcal{M}^{\pm}$, there exist a number $\varepsilon > 0$ and a continuous function $\widetilde{g}(h) > 0$ with $h \in W$ and $\|h\| < \varepsilon$ such that*

$$\widetilde{g}(0) = 1, \qquad \widetilde{g}(h)(u + h) \in \mathcal{M}^{\pm}, \quad \forall h \in W, \|h\| < \varepsilon.$$

Proof We only prove the case that $\mathcal{M}^+$. Define a function $\widetilde{F} : W \times \mathbb{R}^+ \to \mathbb{R}$ by:

$$\widetilde{F}(h,s) = s^{p-1+q}\|u+h\|_\mu^p - \int_\Omega f(x)|u+h|^{1-q}\,dx - \lambda s^{\gamma+q}\int_\Omega g(x)|u+h|^{\gamma+1}\,dx.$$

Note that $u \in \mathcal{M}^+$, we obtain

$$\widetilde{F}(0,1) = \|u\|_\mu^p - \int_\Omega f(x)|u|^{1-q}\,dx - \lambda\int_\Omega g(x)|u|^{\gamma+1}\,dx = 0,$$

and

$$\widetilde{F}_s(0,1) = (p-1+q)\|u\|_\mu^p - (q+\gamma)\lambda\int_\Omega g(x)|u|^{\gamma+1}\,dx > 0. \tag{2.3}$$

At $(0,1)$, using the implicit function theorem, we know that there exists $\overline{\varepsilon} > 0$ such that for $h \in W$ and $\|h\| < \overline{\varepsilon}$, the equation $\widetilde{F}(h,s) = 0$ has a unique continuous solution $s = \widetilde{g}(h) > 0$. Hence $\widetilde{g}(0) = 1$ and

$$\begin{aligned} 0 &= \widetilde{g}(h)^{p-1+q}\|u+h\|_\mu^p - \int_\Omega f(x)|u+h|^{1-q}\,dx - \lambda\widetilde{g}(h)^{\gamma+q}\int_\Omega g(x)|u+h|^{\gamma+1}\,dx \\ &= \frac{\|\widetilde{g}(h)(u+h)\|_\mu^p - \int_\Omega f(x)|\widetilde{g}(h)(u+h)|^{1-q}\,dx - \lambda\int_\Omega g(x)|\widetilde{g}(h)(u+h)|^{\gamma+1}\,dx}{\widetilde{g}(h)^{1-q}}, \end{aligned}$$

i.e.,

$$\widetilde{g}(h)(u+h) \in \mathcal{M}, \quad \forall h \in W, \|h\| < \overline{\varepsilon},$$

and

$$\begin{aligned} \widetilde{F}_s\bigl(h,\widetilde{g}(h)\bigr) &= (p-1+q)\widetilde{g}(h)^{p+q-2}\|u+h\|_\mu^p - (q+\gamma)\lambda\widetilde{g}(h)^{\gamma+q-1}\int_\Omega g(x)|u+h|^{\gamma+1}\,dx \\ &= \frac{(p-1+q)\|\widetilde{g}(h)(u+h)\|_\mu^p - (q+\gamma)\lambda\int_\Omega g(x)|\widetilde{g}(h)(u+h)|^{\gamma+1}\,dx}{\widetilde{g}^{2-q}(h)}, \end{aligned}$$

together with (2.3), these imply that we can choose $\varepsilon > 0$ small enough ($\varepsilon < \overline{\varepsilon}$) such that for every $h \in W$ and $\|h\| < \varepsilon$

$$(p-1+q)\bigl\|\widetilde{g}(h)(u+h)\bigr\|_\mu^p - (q+\gamma)\lambda\int_\Omega g(x)\bigl|\widetilde{g}(h)(u+h)\bigr|^{\gamma+1}\,dx > 0,$$

that is,

$$\widetilde{g}(h)(u+h) \in \mathcal{M}^+, \quad \forall h \in W, \|h\| < \varepsilon.$$

This completes the proof of Lemma 2.3. □

3 Proof of Theorem 1.1

For every $u \in \mathcal{M}$, by (1.13), we have

$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|_\mu^p - \frac{1}{1-q}\int_\Omega f(x)|u|^{1-q}\,dx - \frac{\lambda}{\gamma+1}\int_\Omega g(x)|u|^{\gamma+1}\,dx \\ &= \frac{1}{p}\|u\|_\mu^p - \frac{1}{1-q}\int_\Omega f(x)|u|^{1-q}\,dx - \frac{1}{\gamma+1}\left[\|u\|_\mu^p - \int_\Omega f(x)u^{1-q}\,dx\right] \\ &= \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)\|u\|_\mu^p - \left(\frac{1}{1-q} - \frac{1}{\gamma+1}\right)\int_\Omega f(x)u^{1-q}\,dx \\ &\geq \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)\|u\|_\mu^p - \left(\frac{1}{1-q} - \frac{1}{\gamma+1}\right)\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}\|u\|_\mu^{1-q} \\ &:= \mathcal{K}(\|u\|_\mu). \end{aligned} \tag{3.1}$$

Let

$$\mathcal{K}'(\|u\|_\mu) = \left(1 - \frac{p}{\gamma+1}\right)\|u\|_\mu^{p-1} - \left(1 - \frac{1-q}{\gamma+1}\right)\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}\|u\|_\mu^{-q} = 0.$$

We have

$$\|u\|_\mu := (\|u\|_\mu)_{\min} = \left[\frac{(1 - \frac{1-q}{\gamma+1})\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}}{1 - \frac{p}{\gamma+1}}\right]^{\frac{1}{p+q-1}}.$$

Since $\mathcal{K}''(\|u\|_\mu) > 0$ for all $\|u\|_\mu > 0$ with $\mathcal{K}(\|u\|_\mu) \to 0$ as $\|u\|_\mu \to 0$ and $\mathcal{K}(\|u\|_\mu) \to \infty$ as $\|u\|_\mu \to \infty$. Therefore $\mathcal{K}(u)$ attains its minimum at $(\|u\|_\mu)_{\min}$, and

$$\begin{aligned} \mathcal{K}((\|u\|_\mu)_{\min}) &= \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)\left[\frac{(1 - \frac{1-q}{\gamma+1})\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}}{1 - \frac{p}{\gamma+1}}\right]^{\frac{p}{p+q-1}} \\ &\quad - \left(\frac{1}{1-q} - \frac{1}{\gamma+1}\right)\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}\left[\frac{(1 - \frac{1-q}{\gamma+1})\|f\|_\infty \frac{|\Omega|^{\frac{p^*-1+q}{p^*}}}{(\sqrt[p]{S_\mu})^{1-q}}}{1 - \frac{p}{\gamma+1}}\right]^{\frac{1-q}{p+q-1}}. \end{aligned}$$

By (3.1), we deduce that

$$\lim_{\|u\|_\mu\to\infty} I_{\lambda,\mu}(u) \geq \lim_{\|u\|_\mu\to\infty} \mathcal{K}(\|u\|_\mu) = \infty,$$

namely, $I_{\lambda,\mu}(u)$ is coercive on $\mathcal{M}$. Combining with (3.1), we have

$$I_{\lambda,\mu}(u) \geq \mathcal{K}(u) \geq \mathcal{K}((\|u\|_\mu)_{\min}). \tag{3.2}$$

Thus $I_{\lambda,\mu}(u)$ is bounded below on $\mathcal{M}$. According to Lemma 2.3, if $\lambda \in (0, T_\mu)$, then $\mathcal{M}^+ \cup \mathcal{M}^0$ and $\mathcal{M}^-$ are two closed sets in $\mathcal{M}$. Therefore, we apply the Ekeland variational principle [2] to derive a minimizing sequence $\{u_n\} \subset \mathcal{M}^+ \cup \mathcal{M}^0$ satisfying:

$$\text{(i)} \quad I_{\lambda,\mu}(u_n) < \inf_{\mathcal{M}^+\cup\mathcal{M}^0} I_{\lambda,\mu}(u) + \frac{1}{n};$$

$$\text{(ii)} \quad I_{\lambda,\mu}(u) \geq I_{\lambda,\mu}(u_n) - \frac{1}{n}\|u - u_n\|, \quad \forall u \in \mathcal{M}^+ \cup \mathcal{M}^0.$$

Assume that $u_n \geq 0$ on $\Omega \setminus \{0\}$. Note that $I_{\lambda,\mu}(u)$ is bounded below on $\mathcal{M}$. By (3.2), we get

$$\mathcal{K}\big((\|u_n\|_\mu)_{\min}\big) \leq I_{\lambda,\mu}(u_n) < \inf_{\mathcal{M}^+\cup\mathcal{M}^0} I_{\lambda,\mu}(u) + \frac{1}{n} \leq C_1, \tag{3.3}$$

for n large enough and a positive constant C_1. Hence $\{u_n\}$ is bounded in $\mathcal{M}$. Let us, for a subsequence, suppose that

$$\begin{cases} u_n \rightharpoonup u_0 & \text{in } W, \\ u_n(x) \to u_0(x) & \text{a.e. in } \Omega, \\ u_n \to u_0 & \text{in } L^{1-q}(\Omega) \text{ and } L^{\gamma+1}(\Omega). \end{cases}$$

For every $u \in \mathcal{M}^+$, we deduce from $p > 1$ that

$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|_\mu^p - \frac{1}{1-q}\int_\Omega f(x)|u|^{1-q}\,dx - \frac{\lambda}{\gamma+1}\int_\Omega g(x)|u|^{\gamma+1}\,dx \\ &= \frac{1}{p}\|u\|_\mu^p - \frac{1}{1-q}\left[\|u\|_\mu^p - \lambda\int_\Omega g(x)|u|^{\gamma+1}\,dx\right] - \frac{\lambda}{\gamma+1}\int_\Omega g(x)|u|^{\gamma+1}\,dx \\ &= \left(\frac{1}{p} - \frac{1}{1-q}\right)\|u\|_\mu^p + \left(\frac{1}{1-q} - \frac{1}{\gamma+1}\right)\lambda\int_\Omega g(x)|u|^{\gamma+1}\,dx \\ &< \left(\frac{1}{p} - \frac{1}{1-q}\right)\|u\|_\mu^p + \left(\frac{1}{1-q} - \frac{1}{\gamma+1}\right)\frac{p+q-1}{\gamma+q}\|u\|_\mu^p \\ &= \frac{p+q-1}{\gamma+q}\left(\frac{1}{\gamma+1} - \frac{1}{p}\right)\|u\|_\mu^p < 0, \end{aligned}$$

which implies that $\inf_{\mathcal{M}^+} I_{\lambda,\mu}(u) < 0$. For $\lambda \in (0, T_\mu)$, it follows from Lemma 2.1 that $\mathcal{M}^0 = \{0\}$. Thus $u_n \in \mathcal{M}^+$ for n large enough and $\inf_{\mathcal{M}^+\cup\mathcal{M}^0} I_{\lambda,\mu}(u) = \inf_{\mathcal{M}^+} I_{\lambda,\mu}(u) < 0$. Therefore

$$I_{\lambda,\mu}(u_0) \leq \liminf_{n\to\infty} I_{\lambda,\mu}(u_n) = \inf_{\mathcal{M}^+\cup\mathcal{M}^0} I_{\lambda,\mu} < 0,$$

i.e., $u_0 \geq 0$ and $u_0 \not\equiv 0$.

In the following, we prove that, when $\lambda \in (0, T_\mu)$,

$$(p+q-1)\int_\Omega f(x)u_0^{1-q}\,dx > \lambda(\gamma-q+1)\int_\Omega g(x)u_0^{\gamma+1}\,dx. \tag{3.4}$$

For $\{u_n\} \subset \mathcal{M}^+$, we have

$$(p+q-1)\int_\Omega f(x)u_0^{1-q}\,dx - \lambda(\gamma-p+1)\int_\Omega g(x)u_0^{\gamma+1}\,dx$$

$$= \lim_{n\to\infty}\left[(p+q-1)\int_\Omega f(x)u_n^{1-q}\,dx - \lambda(\gamma-p+1)\int_\Omega g(x)u_n^{\gamma+1}\,dx\right]$$
$$= \lim_{n\to\infty}\left\{(p+q-1)\left[\|u_n\|_\mu^p - \lambda\int_\Omega g(x)u_n^{\gamma+1}\,dx\right] - \lambda(\gamma-p+1)\int_\Omega g(x)u_n^{\gamma+1}\,dx\right\}$$
$$= \lim_{n\to\infty}\left[(p+q-1)\|u_n\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)u_n^{\gamma+1}\,dx\right] \geq 0.$$

We suppose that

$$(p+q-1)\int_\Omega f(x)u_0^{1-q}\,dx - \lambda(\gamma-p+1)\int_\Omega g(x)u_0^{\gamma+1}\,dx = 0. \tag{3.5}$$

It follows from $u_n \in \mathcal{M}$, the weak lower semi-continuity of the norm and (3.5) that

$$0 = \lim_{n\to\infty}\left[\|u_n\|_\mu^p - \int_\Omega f(x)u_n^{1-q}\,dx - \lambda\int_\Omega g(x)u_n^{\gamma+1}\,dx\right]$$
$$\geq \|u_0\|_\mu^p - \int_\Omega f(x)u_0^{1-q}\,dx - \lambda\int_\Omega g(x)u_0^{\gamma+1}\,dx$$
$$= \begin{cases} \|u_0\|_\mu^p - \lambda\frac{\gamma+q}{p+q-1}\int_\Omega g(x)u_0^{\gamma+1}\,dx, \\ \|u_0\|_\mu^p - \lambda\frac{\gamma+q}{\gamma-p+1}\int_\Omega f(x)u_0^{1-q}\,dx. \end{cases}$$

Hence, for every $\lambda \in (0, T_\mu)$ and $u_0 \neq 0$, combining with (2.2), we obtain

$$0 < A(\mu,\lambda)\|u_0\|_\mu^{\gamma+1}$$
$$\leq \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\frac{(\|u_0\|_\mu^p)^{\frac{-\gamma-q}{1-q-p}}}{(\int_\Omega f(x)|u_0|^{1-q}\,dx)^{\frac{p-\gamma-1}{1-q-p}}} - \lambda\int_\Omega g(x)|u_0|^{\gamma+1}\,dx$$
$$\leq \left(\frac{q+p-1}{q+\gamma}\right)\left(\frac{\gamma-p+1}{q+\gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\frac{(\|u_0\|_\mu^p)^{\frac{-\gamma-q}{1-q-p}}}{(\frac{\gamma-p+1}{q+\gamma}\|u_0\|_\mu^p)^{\frac{p-\gamma-1}{1-q-p}}} - \frac{p+q-1}{\gamma+q}\|u_0\|_\mu^p = 0,$$

which is a contradiction. In view of (3.4), we get

$$(p+q-1)\int_\Omega f(x)u_n^{1-q}\,dx - \lambda(\gamma-p+1)\int_\Omega g(x)u_n^{\gamma+1}\,dx \geq C_2 \tag{3.6}$$

for n large enough and some positive constant C_2. Since $u_n \in \mathcal{M}$, we have

$$(p+q-1)\|u_n\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)u_n^{\gamma+1}\,dx \geq C_2 > 0. \tag{3.7}$$

Set $\phi \in \mathcal{M}$ with $\phi \geq 0$. Using Lemma 2.4, there exists $\widetilde{g}_n(t)$ such that $\widetilde{g}_n(0) = 1$ and $\widetilde{g}_n(t)(u_n + t\phi) \in \mathcal{M}^+$. Thus

$$\|u_n\|_\mu^p - \int_\Omega f(x)u_n^{1-q}\,dx - \lambda\int_\Omega g(x)u_n^{\gamma+1}\,dx = 0$$

and

$$\widetilde{g}_n^p(t)\|u_n + t\phi\|_\mu^p - \widetilde{g}_n^{1-q}(t)\int_\Omega f(x)(u_n+t\phi)^{1-q}\,dx - \lambda\widetilde{g}_n^{\gamma+1}(t)\int_\Omega g(x)(u_n+t\phi)^{\gamma+1}\,dx = 0.$$

Therefore

$$\begin{aligned}
0 &= \big[\widetilde{g}_n^p(t) - 1\big]\|u_n + t\phi\|_\mu^p + \big(\|u_n + t\phi\|_\mu^p - \|u_n\|_\mu^p\big) \\
&\quad - \big[\widetilde{g}_n^{1-q}(t) - 1\big]\int_\Omega f(x)(u_n + t\phi)^{1-q}\,dx \\
&\quad - \int_\Omega f(x)\big[(u_n + t\phi)^{1-q} - u_n^{1-q}\big]\,dx - \lambda\big[\widetilde{g}_n^{\gamma+1}(t) - 1\big]\int_\Omega g(x)(u_n + t\phi)^{\gamma+1}\,dx \\
&\quad - \lambda\int_\Omega g(x)\big[(u_n + t\phi)^{\gamma+1} - u_n^{\gamma+1}\big]\,dx \\
&\leq \big[\widetilde{g}_n^p(t) - 1\big]\|u_n + t\phi\|_\mu^p + \big(\|u_n + t\phi\|_\mu^p - \|u_n\|_\mu^p\big) \\
&\quad - \big[\widetilde{g}_n^{1-q}(t) - 1\big]\int_\Omega f(x)(u_n + t\phi)^{1-q}\,dx \\
&\quad - \lambda\big[\widetilde{g}_n^{\gamma+1}(t) - 1\big]\int_\Omega g(x)(u_n + t\phi)^{\gamma+1}\,dx - \lambda\int_\Omega g(x)\big[(u_n + t\phi)^{\gamma+1} - u_n^{\gamma+1}\big]\,dx.
\end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0$, we have

$$\begin{aligned}
0 &\leq p\widetilde{g}_n'(0)\|u_n\|_\mu^p + p\int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n \Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx \\
&\quad - (1-q)\widetilde{g}_n'(0)\int_\Omega f(x)u_n^{1-q}\,dx \\
&\quad - \lambda(\gamma+1)\widetilde{g}_n'(0)\int_\Omega g(x)u_n^{\gamma+1}\,dx - \lambda(\gamma+1)\int_\Omega g(x)u_n^\gamma\phi\,dx \\
&= \widetilde{g}_n'(0)\left[p\|u_n\|_\mu^p - (1-q)\int_\Omega g(x)u_n^{\gamma+1}\,dx\right] \\
&\quad + p\int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n \Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx \\
&\quad - \lambda(\gamma+1)\int_\Omega g(x)u_n^\gamma\phi\,dx \\
&= \widetilde{g}_n'(0)\left[(p+q-1)\|u_n\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)u_n^{\gamma+1}\,dx\right] \\
&\quad + p\int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n \Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx - \lambda(\gamma+1)\int_\Omega g(x)u_n^\gamma\phi\,dx, \qquad (3.8)
\end{aligned}$$

where $\widetilde{g}_n'(0)$ denotes the right derivative of $\widetilde{g}_n(t)$ at zero. If it does not exist, $\widetilde{g}_n'(0)$ should be replaced by $\lim_{k\to\infty}\frac{\widetilde{g}_n(t_k)-\widetilde{g}_n(0)}{t_k}$ for some sequence $\{t_k\}_{k=1}^\infty$ with $\lim_{k\to\infty} t_k = 0$ and $t_k > 0$.

Combining with (3.7) and (3.8), we have $\widetilde{g}_n'(0) \neq -\infty$. Now we prove that $\widetilde{g}_n'(0) \neq +\infty$. Otherwise, we suppose that $\widetilde{g}_n'(0) = +\infty$. Note that $\widetilde{g}_n(t) > \widetilde{g}_n(0) = 1$ for n large enough, and

$$\begin{aligned}
\big|\widetilde{g}_n(t) - 1\big| \cdot \|u_n\| + t\widetilde{g}_n(t)\|\phi\| &\geq \big\|\big[\widetilde{g}_n(t) - 1\big]u_n + t\widetilde{g}_n(t)\phi\big\| \\
&= \big\|\widetilde{g}_n(t)(u_n + t\phi) - u_n\big\|. \qquad (3.9)
\end{aligned}$$

Using condition (ii) with $u = \widetilde{g}_n(t)(u_n + t\phi) \in \mathcal{M}^+$, we deduce that

$$\big[\widetilde{g}_n(t) - 1\big] \cdot \frac{\|u_n\|}{n} + t\widetilde{g}_n(t)\frac{\|\phi\|}{n}$$

$$
\begin{aligned}
&\geq \frac{1}{n}\left\|\widetilde{g}_n(t)(u_n+t\phi)-u_n\right\| \\
&\geq I_{\lambda,\mu}(u_n)-I_{\lambda,\mu}\big(\widetilde{g}_n(t)(u_n+t\phi)\big) \\
&= \frac{1}{p}\|u_n\|_\mu^p-\frac{1}{1-q}\int_\Omega f(x)|u_n|^{1-q}\,dx-\frac{\lambda}{\gamma+1}\int_\Omega g(x)|u_n|^{\gamma+1}\,dx-\frac{1}{p}\widetilde{g}_n^p(t)\|u_n+t\phi\|_\mu^p \\
&\quad+\frac{1}{1-q}\int_\Omega f(x)\big|\widetilde{g}_n(u_n+t\phi)\big|^{1-q}\,dx+\frac{\lambda}{\gamma+1}\int_\Omega g(x)\big|\widetilde{g}_n(u_n+t\phi)\big|^{\gamma+1}\,dx \\
&= \frac{1}{p}\|u_n\|_\mu^p-\frac{1}{1-q}\left[\|u_n\|_\mu^p-\lambda\int_\Omega g(x)|u_n|^{\gamma+1}\,dx\right]-\frac{\lambda}{\gamma+1}\int_\Omega g(x)|u_n|^{\gamma+1}\,dx \\
&\quad-\frac{1}{p}\widetilde{g}_n^p(t)\|u_n+t\phi\|_\mu^p+\frac{1}{1-q}\left[\widetilde{g}_n^p(t)\|u_n+t\phi\|_\mu^p-\lambda\int_\Omega g(x)|u_n+t\phi|^{\gamma+1}\,dx\right] \\
&\quad+\frac{\lambda}{\gamma+1}\widetilde{g}_n^{\gamma+1}(t)\int_\Omega g(x)|u_n+t\phi|^{\gamma+1}\,dx \\
&= \left(\frac{1}{p}-\frac{1}{1-q}\right)\|u_n\|_\mu^p+\left(\frac{1}{1-q}-\frac{1}{\gamma+1}\right)\lambda\int_\Omega g(x)|u_n|^{\gamma+1}\,dx \\
&\quad+\left(\frac{1}{1-q}-\frac{1}{p}\right)\widetilde{g}_n^p(t)\|u_n+t\phi\|_\mu^p \\
&\quad-\left(\frac{1}{1-q}-\frac{1}{\gamma+1}\right)\lambda\widetilde{g}_n^{\gamma+1}(t)\int_\Omega g(x)|u_n+t\phi|^{\gamma+1}\,dx \\
&= \left(\frac{1}{1-q}-\frac{1}{p}\right)\big(\|u_n+t\phi\|_\mu^p-\|u_n\|_\mu^p\big)+\left(\frac{1}{1-q}-\frac{1}{p}\right)\big[\widetilde{g}_n^p(t)-1\big]\|u_n+t\phi\|_\mu^p \\
&\quad-\left(\frac{1}{1-q}-\frac{1}{\gamma+1}\right)\lambda\widetilde{g}_n^{\gamma+1}(t)\int_\Omega g(x)\big[(u_n+t\phi)^{\gamma+1}-u_n^{\gamma+1}\big]\,dx \\
&\quad-\left(\frac{1}{1-q}-\frac{1}{\gamma+1}\right)\lambda\big[\widetilde{g}_n^{\gamma+1}(t)-1\big]\int_\Omega g(x)u_n^{\gamma+1}\,dx.
\end{aligned}
$$

Dividing by $t>0$ and letting $t\to 0$, we obtain

$$
\begin{aligned}
&\widetilde{g}_n'(0)\cdot\frac{\|u_n\|}{n}+\frac{\|\phi\|}{n} \\
&\quad\geq \left(\frac{1}{1-q}-\frac{1}{p}\right)\cdot p\int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi-\mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx \\
&\qquad+\left(\frac{1}{1-q}-\frac{1}{p}\right)\cdot p\widetilde{g}_n'(0)\|u_n\|_\mu^p \\
&\qquad-\lambda\left(\frac{1}{1-q}-\frac{1}{\gamma+1}\right)(\gamma+1)\int_\Omega g(x)u_n^\gamma\phi\,dx \\
&\qquad-\lambda\left(\frac{1}{1-q}-\frac{1}{\gamma+1}\right)(\gamma+1)\widetilde{g}_n'(0)\int_\Omega g(x)u_n^{\gamma+1}\,dx \\
&\quad= \frac{p-1+q}{1-q}\int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi-\mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx+\frac{p-1+q}{1-q}\widetilde{g}_n'(0)\|u_n\|_\mu^p \\
&\qquad-\lambda\frac{\gamma+q}{1-q}\int_\Omega g(x)u_n^\gamma\phi\,dx-\lambda\frac{\gamma+q}{1-q}\widetilde{g}_n'(0)\int_\Omega g(x)u_n^{\gamma+1}\,dx \\
&\quad= \frac{\widetilde{g}_n'(0)}{1-q}\left[(p-1+q)\|u_n\|_\mu^p-\lambda(\gamma+q)\int_\Omega g(x)u_n^{\gamma+1}\,dx\right] \\
&\qquad+\frac{p-1+q}{1-q}\int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi-\mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx-\lambda\frac{\gamma+q}{1-q}\int_\Omega g(x)u_n^\gamma\phi\,dx,
\end{aligned}
$$

that is,

$$\begin{aligned}\frac{\|\phi\|}{n} &\geq \frac{\widetilde{g}_n'(0)}{1-q}\left[(p-1+q)\|u_n\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)u_n^{\gamma+1}\,dx - \frac{(1-q)\|u_n\|}{n}\right] \\ &\quad + \frac{p-1+q}{1-q}\int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx \\ &\quad - \lambda\frac{\gamma+q}{1-q}\int_\Omega g(x)u_n^\gamma\phi\,dx, \end{aligned}\tag{3.10}$$

which is not true since $\widetilde{g}_n'(0) = +\infty$ and

$$(p-1+q)\|u_n\|_\mu^p - \lambda(\gamma+q)\int_\Omega g(x)u_n^{\gamma+1}\,dx - \frac{(1-q)\|u_n\|}{n} \geq C_2 - \frac{(1-q)C_3}{n} > 0.$$

It follows from (3.7), (3.8) and (3.10) that

$$\left|\widetilde{g}_n'(0)\right| \leq C_4$$

for n sufficiently large and a suitable positive constant C_4.

In the following, we prove that $u_0 \in \mathcal{M}^+$ is a solution of problem (1.1). By (3.9) and condition (ii) again, we have

$$\begin{aligned}&\frac{1}{n}\left[\left|\widetilde{g}_n(t)-1\right|\cdot\|u_n\| + t\widetilde{g}_n(t)\|\phi\|\right] \\ &\quad\geq \frac{1}{n}\left\|\widetilde{g}_n(t)(u_n+t\phi) - u_n\right\| \\ &\quad\geq I_{\lambda,\mu}(u_n) - I_{\lambda,\mu}\left(\widetilde{g}_n(t)(u_n+t\phi)\right) \\ &\quad= \frac{1}{p}\|u_n\|_\mu^p - \frac{1}{1-q}\int_\Omega f(x)|u_n|^{1-q}\,dx - \frac{\lambda}{\gamma+1}\int_\Omega g(x)|u_n|^{\gamma+1}\,dx - \frac{1}{p}\widetilde{g}_n^p(t)\|u_n+t\phi\|_\mu^p \\ &\qquad + \frac{1}{1-q}\int_\Omega f(x)\left|\widetilde{g}_n(u_n+t\phi)\right|^{1-q}dx + \frac{\lambda}{\gamma+1}\int_\Omega g(x)\left|\widetilde{g}_n(u_n+t\phi)\right|^{\gamma+1}dx \\ &\quad= -\frac{\widetilde{g}_n^p(t)-1}{p}\|u_n\|_\mu^p - \frac{\widetilde{g}_n^p(t)}{p}\left(\|u_n+t\phi\|_\mu^p - \|u_n\|_\mu^p\right) \\ &\qquad + \frac{\widetilde{g}_n^{1-q}(t)-1}{1-q}\int_\Omega f(x)(u_n+t\phi)^{1-q}\,dx \\ &\qquad + \frac{1}{1-q}\int_\Omega f(x)\left[(u_n+t\phi)^{1-q} - u_n^{1-q}\right]dx + \frac{\lambda(\widetilde{g}_n^{\gamma+1}(t)-1)}{\gamma+1}\int_\Omega g(x)(u_n+t\phi)^{\gamma+1}\,dx \\ &\qquad + \frac{\lambda}{\gamma+1}\int_\Omega g(x)\left[(u_n+t\phi)^{\gamma+1} - u_n^{\gamma+1}\right]dx.\end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0^+$, we derive that

$$\begin{aligned}&\frac{1}{n}\left[\left|\widetilde{g}_n'(0)\right|\cdot\|u_n\| + \|\phi\|\right] \\ &\quad\geq -\widetilde{g}_n'(0)\|u_n\|_\mu^p - \int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx + \widetilde{g}_n'(0)\int_\Omega f(x)u_n^{1-q}\,dx \\ &\qquad + \lambda\widetilde{g}_n'(0)\int_\Omega g(x)u_n^{\gamma+1}\,dx + \lambda\int_\Omega g(x)u_n^\gamma\phi\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \liminf_{t\to 0^+} \frac{1}{1-q} \int_\Omega \frac{f(x)[(u_n+t\phi)^{1-q} - u_n^{1-q}]}{t}\,dx \\
&= -\widetilde{g}_n'(0)\left[\|u_n\|_\mu^p - \int_\Omega f(x)u_n^{1-q}\,dx - \lambda\int_\Omega g(x)u_n^{\gamma+1}\,dx\right] \\
&\quad - \int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n \Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx + \lambda\int_\Omega g(x)u_n^\gamma\phi\,dx \\
&\quad + \liminf_{t\to 0^+} \frac{1}{1-q} \int_\Omega \frac{f(x)[(u_n+t\phi)^{1-q} - u_n^{1-q}]}{t}\,dx.
\end{aligned}
$$

Noting $f(x)[(u_n + t\phi)^{1-q} - u_n^{1-q}] \geq 0$, for every $x \in \Omega$ and $t > 0$, together with the Fatou lemma, we find that

$$
\liminf_{t\to 0^+}\left[\frac{f(x)[(u_n+t\phi)^{1-q} - u_n^{1-q}]}{t}\right]
$$

is integrable, and

$$
\begin{aligned}
&\int_\Omega f(x)u_n^{-q}\phi\,dx \\
&\quad\leq \liminf_{t\to 0^+}\frac{1}{1-q}\int_\Omega \frac{f(x)[(u_n+t\phi)^{1-q} - u_n^{1-q}]}{t}\,dx \\
&\quad\leq \frac{|\widetilde{g}_n'(0)|\|u_n\| + \|\phi\|}{n} + \int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx \\
&\qquad - \lambda\int_\Omega g(x)u_n^\gamma\phi\,dx \\
&\quad\leq \frac{C_3C_4 + \|\phi\|}{n} + \int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx - \lambda\int_\Omega g(x)u_n^\gamma\phi\,dx.
\end{aligned}
$$

Applying the Fatou lemma again, we have

$$
\begin{aligned}
&\int_\Omega f(x)u_0^{-q}\phi\,dx \\
&\quad= \int_\Omega\left[\liminf_{n\to\infty} f(x)u_n^{-q}\phi\right]dx \leq \liminf_{n\to\infty}\int_\Omega f(x)u_n^{-q}\phi\,dx \\
&\quad\leq \liminf_{n\to\infty}\left[\frac{C_3C_4 + \|\phi\|}{n} + \int_\Omega\left(|\Delta u_n|^{p-2}\Delta u_n\Delta\phi - \mu\frac{|u_n|^{p-2}u_n\phi}{|x|^{2p}}\right)dx\right. \\
&\qquad\left. - \lambda\int_\Omega g(x)u_n^\gamma\phi\,dx\right] \\
&\quad= \int_\Omega\left(|\Delta u_0|^{p-2}\Delta u_0\Delta\phi - \mu\frac{|u_0|^{p-2}u_0\phi}{|x|^{2p}}\right)dx - \lambda\int_\Omega g(x)u_0^\gamma\phi\,dx.
\end{aligned}
$$

Since $\int_\Omega u_0^{-q}\varphi_1\,dx < \infty$, we have $u_0 > 0$ a.e. in Ω. For every $\phi \in \mathcal{M}$ and $\phi \geq 0$, we have

$$
\begin{aligned}
&\int_\Omega\left(|\Delta u_0|^{p-2}\Delta u_0\Delta\phi - \mu\frac{|u_0|^{p-2}u_0\phi}{|x|^{2p}}\right)dx - \int_\Omega f(x)u_0^{-q}\phi\,dx \\
&\quad - \lambda\int_\Omega g(x)u_0^\gamma\phi\,dx \geq 0.
\end{aligned}
\tag{3.11}
$$

Set $\phi = u_0$ in (3.11), we derive that

$$\|u_0\|_\mu^p = \int_\Omega \left(|\Delta u_0|^p - \mu \frac{|u_0|^p}{|x|^{2p}} \right) dx \geq \int_\Omega f(x) u_0^{1-q}\, dx + \lambda \int_\Omega g(x) u_0^{\gamma+1}\, dx.$$

Furthermore

$$\begin{aligned} \|u_0\|_\mu^p &\leq \liminf_{n\to\infty} \|u_n\|_\mu^p \leq \limsup_{n\to\infty} \|u_n\|_\mu^p \\ &= \limsup_{n\to\infty} \left[\int_\Omega f(x) u_n^{1-q}\, dx + \lambda \int_\Omega g(x) u_n^{\gamma+1}\, dx \right] \\ &= \int_\Omega f(x) u_0^{1-q}\, dx + \lambda \int_\Omega g(x) u_0^{\gamma+1}\, dx. \end{aligned} \tag{3.12}$$

Hence

$$\|u_0\|_\mu^p = \int_\Omega f(x) u_0^{1-q}\, dx + \lambda \int_\Omega g(x) u_0^{\gamma+1}\, dx. \tag{3.13}$$

Therefore $u_n \to u_0$ in $\mathcal{M}$ and $u_0 \in \mathcal{M}$. By (3.4), we have

$$\begin{aligned} &(p+q-1)\|u_0\|_\mu^p - \lambda(\gamma+q) \int_\Omega g(x) u_0^{\gamma+1}\, dx \\ &\quad = (p+q-1)\left[\int_\Omega f(x) u_0^{1-q}\, dx + \lambda \int_\Omega g(x) u_0^{\gamma+1}\, dx \right] - \lambda(\gamma+q) \int_\Omega g(x) u_0^{\gamma+1}\, dx \\ &\quad = (p+q-1) \int_\Omega f(x) u_0^{1-q}\, dx - \lambda(\gamma-1) \int_\Omega g(x) u_0^{\gamma+1}\, dx > 0, \end{aligned}$$

i.e., $u_0 \in \mathcal{M}^+$.

Next, we only need to show that u_0 is a positive weak solution of problem (1.1). Define

$$\Phi = (u_0 + \varepsilon\phi)^+, \quad \phi \in W, \varepsilon > 0.$$

Substituting Φ into (3.11), combining with (3.12), we deduce that

$$\begin{aligned} 0 &\leq \int_\Omega \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta \Phi - \mu \frac{|u_0|^{p-2} u_0 \Phi}{|x|^{2p}} - f(x) u_0^{-q} \Phi - \lambda g(x) u_0^\gamma \Phi \right] dx \\ &= \int_{\Omega_1} \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta \Phi - \mu \frac{|u_0|^{p-2} u_0 \Phi}{|x|^{2p}} - f(x) u_0^{-q} \Phi - \lambda g(x) u_0^\gamma \Phi \right] dx \\ &\quad + \int_{\Omega_2} \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta \Phi - \mu \frac{|u_0|^{p-2} u_0 \Phi}{|x|^{2p}} - f(x) u_0^{-q} \Phi - \lambda g(x) u_0^\gamma \Phi \right] dx \\ &= \int_\Omega \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta (u_0 + \varepsilon\phi) - \mu \frac{|u_0|^{p-2} u_0 (u_0 + \varepsilon\phi)}{|x|^{2p}} - f(x) u_0^{-q} (u_0 + \varepsilon\phi) \right. \\ &\quad \left. - \lambda g(x) u_0^\gamma (u_0 + \varepsilon\phi) \right] dx \\ &\quad - \int_{\Omega_2} \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta (u_0 + \varepsilon\phi) - \mu \frac{|u_0|^{p-2} u_0 (u_0 + \varepsilon\phi)}{|x|^{2p}} - f(x) u_0^{-q} (u_0 + \varepsilon\phi) \right. \\ &\quad \left. - \lambda g(x) u_0^\gamma (u_0 + \varepsilon\phi) \right] dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega} \left[|\Delta u_0|^p - \mu \frac{|u_0|^p}{|x|^{2p}} - f(x) u_0^{1-q} - \lambda g(x) u_0^{\gamma+1} \right] dx \\
&\quad + \varepsilon \int_{\Omega} \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta \phi - \mu \frac{|u_0|^{p-2} u_0 \phi}{|x|^{2p}} - f(x) u_0^{-q} \phi - \lambda g(x) u_0^{\gamma} \phi \right] dx \\
&\quad - \int_{\Omega_2} \left[|\Delta u_0|^p + \varepsilon |\Delta u_0|^{p-2} \Delta u_0 \Delta \phi - \mu \frac{|u_0|^{p-2} u_0 (u_0 + \varepsilon \phi)}{|x|^{2p}} \right] dx \\
&\quad - \int_{\Omega_2} \left[-f(x) u_0^{-q} (u_0 + \varepsilon \phi) - \lambda g(x) u_0^{\gamma+1} - \varepsilon \lambda g(x) u_0^{\gamma} \phi \right] dx \\
&\leq \varepsilon \int_{\Omega} \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta \phi - \mu \frac{|u_0|^{p-2} u_0 \phi}{|x|^{2p}} - f(x) u_0^{-q} \phi - \lambda g(x) u_0^{\gamma} \phi \right] dx \\
&\quad - \varepsilon \int_{\Omega_2} |\Delta u_0|^{p-2} \Delta u_0 \Delta \phi \, dx + \lambda \|g\|_\infty \int_{\Omega_2} |\varepsilon \phi|^{\gamma+1} \, dx + \varepsilon \lambda \int_{\Omega_2} g(x) u_0^{\gamma} \phi \, dx \\
&= \varepsilon \int_{\Omega} \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta \phi - \mu \frac{|u_0|^{p-2} u_0 \phi}{|x|^{2p}} - f(x) u_0^{-q} \phi - \lambda g(x) u_0^{\gamma} \phi \right] dx \\
&\quad - \varepsilon \int_{\Omega_2} |\Delta u_0|^{p-2} \Delta u_0 \Delta \phi \, dx + \varepsilon \lambda \varepsilon^{\gamma} \|g\|_\infty \int_{\Omega_2} |\phi|^{\gamma+1} \, dx + \varepsilon \lambda \int_{\Omega_2} g(x) u_0^{\gamma} \phi \, dx,
\end{aligned}
$$

where $\Omega_1 = \{x | u_0(x) + \varepsilon\phi(x) > 0, x \in \Omega\}$ and $\Omega_2 = \{x | u_0(x) + \varepsilon\phi(x) \leq 0, x \in \Omega\}$. Since the measure of Ω_2 tends to zero as $\varepsilon \to 0$, we have $\int_{\Omega_2} |\Delta u_0|^{p-2} \Delta u_0 \Delta\phi \, dx \to 0$ as $\varepsilon \to 0$. By the same arguments, we have $\lambda\varepsilon^{\gamma} \|g\|_\infty \int_{\Omega_2} |\phi|^{\gamma+1} \, dx \longrightarrow 0$ and $\lambda \int_{\Omega_2} g(x) u_0^{\gamma} \phi \, dx \longrightarrow 0$ as $\varepsilon \to 0$. Dividing by ε and taking the limit for $\varepsilon \to 0$, we deduce that

$$
\int_{\Omega} \left[|\Delta u_0|^{p-2} \Delta u_0 \Delta \phi - \mu \frac{|u_0|^{p-2} u_0 \phi}{|x|^{2p}} - f(x) u_0^{-q} \phi - \lambda g(x) u_0^{\gamma} \phi \right] dx \geq 0.
$$

Therefore u_0 is a positive weak solution of problem (1.1).

We adopt the Ekeland variational principle again to derive a minimizing sequence $U_n \subset \mathcal{M}^-$ for the minimization problem $\inf_{\mathcal{M}^-} I_{\lambda,\mu}$ such that for $U_n \in \mathcal{M}$, $U_n \rightharpoonup U_0$ weakly in $\mathcal{M}$ and pointwise a.e. in Ω. By similar arguments to those in (3.4) and (3.6), for $\lambda \in (0, T_\mu)$, we have

$$
(p + q - 1) \int_{\Omega} f(x) |U_0|^{1-q} \, dx - \lambda(\gamma - p + 1) \int_{\Omega} g(x) |U_0|^{\gamma+1} \, dx < 0, \tag{3.14}
$$

which leads to

$$
(p + q - 1) \int_{\Omega} f(x) |U_n|^{1-q} \, dx - \lambda(\gamma - p + 1) \int_{\Omega} g(x) |U_n|^{\gamma+1} \, dx \leq -C_5,
$$

for n large enough and a positive constant C_5. Therefore $U_0 > 0$ is the positive weak solution of problem (1.1). Furthermore $U_0 \in \mathcal{M}$. By (3.14), we obtain

$$
\begin{aligned}
&(p + q - 1) \|U_0\|_\mu^p - (q + \gamma) \lambda \int_{\Omega} g(x) U_0^{\gamma+1} \, dx \\
&\quad = (p + q - 1) \left[\int_{\Omega} f(x) U_0^{1-q} \, dx + \lambda \int_{\Omega} g(x) U_0^{\gamma+1} \, dx \right] - \lambda(\gamma + q) \int_{\Omega} g(x) U_0^{\gamma+1} \, dx \\
&\quad = (p + q - 1) \int_{\Omega} f(x) U_0^{1-q} \, dx - \lambda(\gamma - p + 1) \int_{\Omega} g(x) U_0^{\gamma+1} \, dx < 0,
\end{aligned}
$$

i.e., $U_0 \in \mathcal{M}^-$. According to Lemma 2.2, we know that problem (1.1) has at least two positive weak solutions $u_0 \in \mathcal{M}^+$ and $U_0 \in \mathcal{M}^-$ with $\|U_0\|_\mu > \|u_0\|_\mu$ for every $\lambda \in (0, T_\mu)$. This completes the proof of Theorem 1.1.

4 Proof of Corollary 1.2

For every $U \in \mathcal{M}^-$, by Lemma 2.2, we deduce that

$$\begin{aligned}
\|U\|_\mu &> M_\mu(\lambda) \\
&= \left[\frac{p+q-1}{\lambda(\gamma+q)}\frac{1}{\|g\|_\infty}\frac{(\sqrt[p]{S_\mu})^{\gamma+1}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*}}}\right]^{\frac{1}{\gamma+1-p}} \\
&= \left(\frac{1}{\lambda}\right)^{\frac{1}{\gamma+1-p}}\left(\frac{p+q-1}{\gamma+q}\right)^{\frac{1}{\gamma+1-p}}\left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{\gamma+1-p}}\frac{(\sqrt[p]{S_\mu})^{\frac{\gamma+1}{\gamma+1-p}}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*(\gamma+1-p)}}} \\
&= (T_\mu)^{-\frac{1}{\gamma+1-p}}\left(\frac{p+q-1}{\gamma+q}\right)^{\frac{1}{\gamma+1-p}}\left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{\gamma+1-p}}\frac{(\sqrt[p]{S_\mu})^{\frac{\gamma+1}{\gamma+1-p}}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*(\gamma+1-p)}}}\left(\frac{T_\mu}{\lambda}\right)^{\frac{1}{\gamma+1-p}}.
\end{aligned}$$

Combining with the definition of T_μ, we have

$$\begin{aligned}
\|U\|_\mu &> \left(\frac{q+\gamma}{q+p-1}\right)^{\frac{1}{\gamma+1-p}}\left(\frac{q+\gamma}{\gamma-p+1}\right)^{\frac{1}{p+q-1}}\left(\|f\|_\infty\right)^{\frac{1}{p+q-1}}\left(\|g\|_\infty\right)^{\gamma-p+1}\frac{|\Omega|^{\frac{2p}{N}\frac{q+\gamma}{p+q-1}\frac{1}{\gamma+1-p}}}{S_\mu^{\frac{q+\gamma}{p+q-1}\frac{1}{\gamma+1-p}}} \\
&\quad\times\left(\frac{p+q-1}{\gamma+q}\right)^{\frac{1}{\gamma+1-p}}\left(\frac{1}{\|g\|_\infty}\right)^{\frac{1}{\gamma+1-p}}\frac{(\sqrt[p]{S_\mu})^{\frac{\gamma+1}{\gamma+1-p}}}{|\Omega|^{\frac{p^*-1-\gamma}{p^*(\gamma+1-p)}}}\left(\frac{T_\mu}{\lambda}\right)^{\frac{1}{\gamma+1-p}} \\
&= \left(\frac{q+\gamma}{\gamma-p+1}\right)^{\frac{1}{p+q-1}}\left(\|f\|_\infty\right)^{\frac{1}{p+q-1}}\left(\frac{|\Omega|^{\frac{2p}{N}\frac{q+\gamma}{p+q-1}\frac{1}{\gamma+1-p}-\frac{p^*-1-\gamma}{p^*(\gamma+1-p)}}}{(\sqrt[p]{S_\mu})^{p\cdot\frac{q+\gamma}{p+q-1}\frac{1}{\gamma+1-p}-\frac{\gamma+1}{\gamma+1-p}}}\right)\left(\frac{T_\mu}{\lambda}\right)^{\frac{1}{\gamma+1-p}} \\
&= |\Omega|^{\frac{1}{p}}\left(\frac{q+\gamma}{\gamma-p+1}\right)^{\frac{1}{p+q-1}}\left(\|f\|_\infty\right)^{\frac{1}{p+q-1}}\left(\frac{|\Omega|^{\frac{2}{N}}}{\sqrt[p]{S_\mu}}\right)^{\frac{1-q}{p+q-1}}\left(\frac{T_\mu}{\lambda}\right)^{\frac{1}{\gamma+1-p}} \\
&= |\Omega|^{\frac{1}{p}}\left(\|f\|_\infty\right)^{\frac{1}{p+q-1}}\left(1+\frac{p+q-1}{\gamma-p+1}\right)^{\frac{1}{p+q-1}}\left(\frac{|\Omega|^{\frac{2}{N}}}{\sqrt[p]{S_\mu}}\right)^{\frac{1-q}{p+q-1}}\left(\frac{T_\mu}{\lambda}\right)^{\frac{1}{\gamma+1-p}},
\end{aligned}$$

where we adopted the following facts:

$$\begin{aligned}
&\frac{2p}{N}\frac{q+\gamma}{p+q-1}\frac{1}{\gamma+1-p}-\frac{p^*-1-\gamma}{p^*(\gamma+1-p)} \\
&\quad= \frac{p^*-1+q}{p^*(p+q-1)} = \frac{\frac{Np}{N-2p}+q-1}{\frac{Np}{N-2p}(p+q-1)} \\
&\quad= \frac{N(p+q-1)+2p(1-q)}{Np(p+q-1)} = \frac{1}{p}+\frac{2}{N}\cdot\frac{1-q}{p+q-1}, \\
&p\cdot\frac{q+\gamma}{p+q-1}\frac{1}{\gamma+1-p}-\frac{\gamma+1}{\gamma+1-p} = \frac{(1-q)(\gamma+1-p)}{(p+q-1)(\gamma+1-p)} = \frac{1-q}{p+q-1}.
\end{aligned}$$

Let $U_{\lambda,\mu,\varepsilon} \in \mathcal{M}^-$ be the solution of problem (1.1) with $\gamma = \varepsilon + p - 1$, where $\lambda \in (0, T_\mu)$. Then

$$\|U_{\lambda,\mu,\varepsilon}\|_\mu > C_{\mu,\varepsilon}\left(\frac{T_\mu}{\lambda}\right)^{\frac{1}{\varepsilon}},$$

where $C_{\mu,\varepsilon}$ is given in (1.16). This completes the proof of Corollary 1.2.

5 Proof of Theorem 1.3

For simplicity, we consider problem (1.1) with $f = g = 1$,

$$\begin{cases} \Delta_p^2 u - \mu\frac{|u|^{p-2}u}{|x|^{2p}} = u^{-q} + \lambda u^\gamma & \text{in } \Omega \setminus \{0\}, \\ u(x) > 0 & \text{in } \Omega \setminus \{0\}, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Let us define

$$\lambda^* = \lambda^*(N, \Omega, \mu, q, \gamma) = \sup\bigl\{\lambda > 0 : \text{problem (5.1) has a positive solution}\bigr\}.$$

Using Theorem 1.1, we provide uniform estimates for $\lambda^*(N, \Omega, \mu, q, \gamma)$.

Lemma 5.1 *For $1 < p < \frac{N}{2}$, $0 < \mu < \mu_{N,p}$, $0 < q < 1 < \gamma < p^* - 1$ and $\Omega \in \mathbb{U}$, where $\mathbb{U} = \{\Omega \in \mathbb{R}^N : \Omega$ is an open and bounded domain$\}$, we have*

$$0 < \lambda^- \le \lambda^* \le \lambda^+ < \infty,$$

where

$$\lambda^- = \left(\frac{q + p - 1}{q + \gamma}\right)\left(\frac{\gamma - p + 1}{q + \gamma}\right)^{\frac{p-\gamma-1}{1-q-p}}\left[\frac{S_\mu}{|\Omega|^{\frac{2p}{N}}}\right]^{\frac{q+\gamma}{p+q-1}}$$

and

$$\lambda^+ = \lambda_1^{\frac{\gamma+q}{q-1+p}}\left(\frac{\gamma - p + 1}{\gamma + q}\right)^{\frac{\gamma-p+1}{q+p-1}}\frac{-1 + p + q}{\gamma + q} + \frac{1}{2}.$$

Proof (1) Assume that $\lambda \in (0, \lambda^-)$, then problem (5.1) has at least two solutions. By the definition of λ^*, we have $\lambda^* \ge \lambda^- > 0$.

(2) Assume that (5.1) has a positive solution u. Integrating over Ω by multiplying (5.1) by φ_1, we obtain

$$\lambda_1 \int_\Omega |u|^{p-2}u\varphi_1\,dx = \int_\Omega \left(\Delta_p^2 u - \mu\frac{|u|^{p-2}u}{|x|^{2p}}\right)\varphi_1\,dx = \int_\Omega u^{-q}\varphi_1\,dx + \lambda\int_\Omega u^\gamma\varphi_1\,dx. \tag{5.2}$$

We claim that there exists $\lambda^+ > 0$ such that

$$t^{-q} + \lambda^+ t^\gamma > \lambda_1 t^{p-1}, \quad \forall t > 0. \tag{5.3}$$

In fact, letting

$$F_\lambda(t) = t^{-q} + \lambda t^{\gamma} - \lambda_1 t^{p-1} = t^{\gamma}\left(t^{-q-\gamma} + \lambda - \lambda_1 t^{-\gamma+p-1}\right) := t^{\gamma}\cdot G_\lambda(t), \quad t > 0. \tag{5.4}$$

We have

$$G'_\lambda(t) = (-\gamma - q)t^{-\gamma-q-1} + \lambda_1(\gamma - p + 1)t^{-\gamma+p-2} = 0,$$

i.e.,

$$t := t_{\min} = \left(\frac{\gamma + q}{\lambda_1(\gamma - p + 1)}\right)^{\frac{1}{q-1+p}}.$$

Then $G_\lambda(t)$ attains minimum at $t_{\min}$, and

$$G_\lambda(t_{\min}) = \lambda + \lambda_1^{\frac{\gamma+q}{q-1+p}}\left(\frac{\gamma - p + 1}{\gamma + q}\right)^{\frac{\gamma-p+1}{q+p-1}}\frac{1 - p - q}{\gamma + q}.$$

We may choose $\lambda = \lambda_1^{\frac{\gamma+q}{q-1+p}}(\frac{\gamma-p+1}{\gamma+q})^{\frac{\gamma-p+1}{q+p-1}}\frac{-1+p+q}{\gamma+q} + \frac{1}{2} = \lambda^+ > 0$ such that

$$G_{\lambda^+}(t) \geq G_{\lambda^+}(t_{\min}) = \frac{1}{2} > 0, \quad \text{for } t > 0.$$

Therefore

$$F_{\lambda^+}(t) = t^{\gamma}\cdot G_{\lambda^+}(t) > 0 \quad \text{for } t > 0.$$

Using (5.3) with $t = u$, we have

$$\int_\Omega u^{-q}\varphi_1\,dx + \lambda^+\int_\Omega u^{\gamma}\varphi_1\,dx \geq \lambda_1\int_\Omega |u|^{p-2}u\varphi_1\,dx. \tag{5.5}$$

Combining with (5.2) and (5.5), we obtain $\lambda \leq \lambda^+$. Since λ is arbitrary, we have $\lambda^* \leq \lambda^+ < \infty$. □

Proof of Theorem 1.3 We only prove the case that $0 < \lambda < \lambda^*$. By the definition of λ^*, there exists $\overline{\lambda} \in (\lambda, \lambda^*)$ such that the problem

$$\Delta_p^2 u - \mu\frac{|u|^{p-2}u}{|x|^{2p}} = u^{-q} + \overline{\lambda}u^{\gamma}$$

has a positive solution, denoted by $u_{\overline{\lambda}}$. It follows that

$$\Delta_p^2 u_{\overline{\lambda}} - \mu\frac{|u_{\overline{\lambda}}|^{p-2}u_{\overline{\lambda}}}{|x|^{2p}} = u_{\overline{\lambda}}^{-q} + \overline{\lambda}u_{\overline{\lambda}}^{\gamma} \geq u_{\overline{\lambda}}^{-q} + \lambda u_{\overline{\lambda}}^{\gamma}.$$

Hence $u_{\overline{\lambda}}$ is an upper solution of (5.1). Note that $\lim_{t\to 0^+} G_\lambda(t) = \infty$, we can take $\varepsilon > 0$ small enough with $\varepsilon\varphi_1 < u_{\overline{\lambda}}$ and $G_\lambda(\varepsilon\varphi_1) \geq 0$. Thus

$$F_\lambda(\varepsilon\varphi_1) = (\varepsilon\varphi_1)^{\gamma}G_\lambda(\varepsilon\varphi_1) \geq 0, \quad \text{for all } \lambda > 0,$$

i.e.,

$$\lambda_1(\varepsilon\varphi_1)^{p-1} \leq (\varepsilon\varphi_1)^{-q} + \lambda(\varepsilon\varphi_1)^{\gamma}, \quad \text{for all } \lambda > 0. \tag{5.6}$$

Combining with (1.10) and (5.6), we obtain

$$\begin{aligned}\Delta_p^2(\varepsilon\varphi_1) - \mu\frac{|(\varepsilon\varphi_1)|^{p-2}(\varepsilon\varphi_1)}{|x|^{2p}} &= \varepsilon^{p-1}\left(\Delta_p^2\varphi_1 - \mu\frac{|\varphi_1|^{p-2}\varphi_1}{|x|^{2p}}\right) \\ &= \varepsilon^{p-1}\lambda_1|\varphi_1|^{p-1} = \lambda_1(\varepsilon\varphi_1)^{p-1} \leq (\varepsilon\varphi_1)^{-q} + \lambda(\varepsilon\varphi_1)^{\gamma},\end{aligned}$$

namely, $\varepsilon\varphi_1$ is a lower solution of (5.1). Note that $\Delta_p^2 - \frac{\mu}{|x|^{2p}}$ is monotone, then problem (5.1) has a positive solution u_λ with $\varepsilon\varphi_1 \leq u_\lambda \leq u_{\bar{\lambda}}$. □

6 Conclusions

In this paper, we study a class of p-biharmonic equations with Hardy potential and negative exponents. We establish the dependence of the above problem on q, γ, f, g and Ω and evaluate the extremal value of λ related to the multiplicity of positive solutions for this problem.

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