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Discrete majorization type inequalities for convex functions on rectangles

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Abstract
In this paper, we present several discrete majorization type inequalities for the convex functions defined on rectangles.

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1 Introduction and preliminaries

We start this section by giving some brief introduction about convex function and related results.

Definition 1.1 Let $\mathbf{I} \subseteq \mathbb{R}^n$ be a convex set. Then the function $\phi : \mathbf{I} \mapsto \mathbb{R}$ is said to be convex if the inequality

$$\phi\big(\zeta\mathbf{x} + (1-\zeta)\mathbf{y}\big) \le \zeta\phi(\mathbf{x}) + (1-\zeta)\phi(\mathbf{y})$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbf{I}$ and $\zeta \in [0,1]$.

It is well-known that a convex function may not be differentiable. If the function ϕ is convex, then the support line inequality

$$\phi(\mathbf{x}) - \phi(\mathbf{y}) \ge \nabla_+\phi(\mathbf{y})(\mathbf{x}-\mathbf{y})$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbf{I}$, where

$$\nabla_+\phi(\mathbf{y})(\mathbf{x}-\mathbf{y}) = \left\langle \frac{\partial\phi_+(\mathbf{y})}{\partial\mathbf{y}}, (\mathbf{x}-\mathbf{y}) \right\rangle, \qquad \frac{\partial\phi_+(\mathbf{y})}{\partial\mathbf{y}} = \left(\frac{\partial\phi_+(\mathbf{y})}{\partial y_1}, \frac{\partial\phi_+(\mathbf{y})}{\partial y_2}, \dots, \frac{\partial\phi_+(\mathbf{y})}{\partial y_n} \right)$$

for $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbf{I}$ and $\langle\cdot,\cdot\rangle$ is the ordinary inner product in $\mathbb{R}^n$.

Convex functions have many important applications in mathematics, physics, statistics and engineering [1–20]. Currently, many refinements, variants, generalizations and extensions for the convexity can be found in the literature [21–50].

In [51], Dragomir introduced the definition of the coordinate convex functions as follows:

Definition 1.2 (See [51]) Let $[a_1, a_2]$ and $[b_1, b_2]$ be two intervals in $\mathbb{R}$ and $S = [a_1, a_2] \times [b_1, b_2]$. A function $\phi : S \mapsto \mathbb{R}$ is said to be coordinate convex on S if the partial functions $\phi_y : [a_1, a_2] \mapsto \mathbb{R}$ and $\phi_x : [b_1, b_2] \mapsto \mathbb{R}$ defined by

$$\phi_y(u) = \phi(u, y), \qquad \phi_x(v) = \phi(x, v)$$

are convex.

Lemma 1.3 (See [51]) *Every convex function defined on a rectangle is coordinate convex, but the converse is not true, in general.*

In the remaining part of this section, we give a comprehensive introduction about majorization theory.

Let $n \geq 2$, $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be two n-tuples of real numbers, and

$$a_{[1]} \geq a_{[2]} \geq \cdots \geq a_{[n]}, \qquad b_{[1]} \geq b_{[2]} \geq \cdots \geq b_{[n]}$$

be their ordered arrangement.

Definition 1.4 The n-tuple $\mathbf{b}$ is said to be majorized by the n-tuple $\mathbf{a}$, or $\mathbf{a}$ majorizes $\mathbf{b}$, in symbols $\mathbf{a} \succ \mathbf{b}$, if

$$\sum_{i=1}^{k} a_{[i]} \geq \sum_{i=1}^{k} b_{[i]} \quad (k = 1, 2, \dots, n-1),$$
$$\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i.$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be two vectors such that $\mathbf{a}$ majorizes $\mathbf{b}$. Then from the basic knowledge of linear algebra we clearly see that there exist a set of probabilities $(q_1, q_2, \dots, q_n)$ with $\sum_{i=1}^{n} q_i = 1$ and a set of permutations $(P_1, P_2, \dots, P_n)$ such that $\mathbf{a} = \sum_{i=1}^{n} P_i q_i \mathbf{b}$. Alternatively, it can be shown that there exists a doubly stochastic matrix D such that $\mathbf{a} = D\mathbf{b}$. In fact, the latter characterization of majorization relation implies that the set of vectors $\mathbf{a}$ that satisfy $\mathbf{a} \succ \mathbf{b}$ is the convex hull spanned by the $n!$ points formed from the permutations of the elements of $\mathbf{b}$.

Let S and T be two Hermitian operators. Then we say that the Hermitian operator S majorizes the Hermitian operator T if the set of eigenvalues of S majorizes the set of eigenvalues values of T.

Majorization is a partial order relation between the vectors, which precisely defines the vague notion that the components of one vector are "less spread out" or "more nearly equal" than the components of another vector. And the functions that preserve the majorization order are called Schur convex functions. Many problems arising in signal processing and communications involve comparing vector-valued strategies or solving optimization problems with vector- or matrix-valued variables. Majorization theory is a key tool that allows us to solve or simplify these problems.

The following Theorem 1.5 is well-known in the literature as the majorization theorem and for its proof we refer to Marshall and Olkin [52]. This result is due to Hardy, Littlewood and Pólya [53] and it can also be found in [54].

Theorem 1.5 *Let $\mathbf{I}$ be an interval in $\mathbb{R}$, and $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two n-tuples such that $a_i, b_i \in \mathbf{I}$ $(i = 1, 2, \ldots, n)$. Then the inequality*

$$\sum_{i=1}^{n} \phi(a_i) \geq \sum_{i=1}^{n} \phi(b_i)$$

holds for every continuous convex function $\phi : \mathbf{I} \mapsto \mathbb{R}$ if and only if $\boldsymbol{a} \succ \boldsymbol{b}$.

The following Theorem 1.6 is a weighted version of Theorem 1.5 and is given by Fuchs [55].

Theorem 1.6 *Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two decreasing n-tuples such that $a_i, b_i \in \mathbf{I}$ $(i = 1, 2, \ldots, n)$, and $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ be a real n-tuple with*

$$\sum_{i=1}^{k} p_i a_i \geq \sum_{i=1}^{k} p_i b_i \quad (k = 1, 2, \ldots, n-1),$$
$$\sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i b_i.$$

Then the inequality

$$\sum_{i=1}^{n} p_i \phi(a_i) \geq \sum_{i=1}^{n} p_i \phi(b_i) \tag{1.1}$$

holds for each continuous convex function $\phi : \mathbf{I} \mapsto \mathbb{R}$.

Another result similar to that above with some relaxed conditions on **a**, **b** and stricter condition on function ϕ was obtained by Bullen, Vasić and Stanković [56].

Theorem 1.7 *Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two decreasing n-tuples, and $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ be a real n-tuple. If the inequality*

$$\sum_{i=1}^{k} p_i a_i \geq \sum_{i=1}^{k} p_i b_i \tag{1.2}$$

holds for $k = 1, 2, \ldots, n$, then inequality (1.1) holds for each continuous increasing convex function $\phi : \mathbf{I} \mapsto \mathbb{R}$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are increasing n-tuples and the reverse inequality in (1.2) holds for $k = 1, 2, \ldots, n$, then inequality (1.1) holds for each continuous decreasing convex function $\phi : \mathbf{I} \mapsto \mathbb{R}$.

Dragomir [57] presented another majorization result, which has been obtained by using support line and Chebyshev's inequalities.

Theorem 1.8 *Let $\mathbf{I}$ be an interval in $\mathbb{R}$, $\phi : \mathbf{I} \mapsto \mathbb{R}$ be a convex function, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two real n-tuples such that $a_i, b_i \in \mathbf{I}$ $(i = 1, 2, \ldots, n)$, and $\boldsymbol{p} =$*

$(p_1, p_2, \dots, p_n)$ be a non-negative real n-tuple with $P_n = \sum_{i=1}^{n} p_i > 0$. If $\boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$ are monotonic in the same sense with $\sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i ab_i$, then one has

$$\sum_{i=1}^{n} p_i \phi(a_i) \geq \sum_{i=1}^{n} p_i \phi(b_i). \tag{1.3}$$

If ϕ is strictly convex on $\boldsymbol{I}$ and $p_i > 0$ $(i = 1, 2, \dots, n)$, then equality holds in (1.3) if and only if $a_i = b_i$ for all $i = 1, 2, \dots, n$.

In this paper, our focus is on the majorization type results for the convex functions defined on rectangles. We shall extend classical majorization inequality for majorized tuples and establish weighted versions of majorization inequalities for certain tuples, for example, monotonic tuples in the same sense, monotonic tuples in mean, etc. For obtaining these results, we use Chebyshev's inequality, Abel transformation, support line inequality of convex function and the fact that every convex function defined on rectangles is coordinate convex. At the end of the paper, we provide Favard's type inequalities by using the generalized majorization results.

2 Main results

We start by giving a majorization inequality for the convex functions defined on rectangles by using majorized tuples.

Theorem 2.1 *Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be any two intervals in $\mathbb{R}$, $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ be two n-tuples such that $a_i, b_i \in \mathbf{I}_1$ $(i = 1, 2, \dots, n)$, and $\boldsymbol{c} = (c_1, c_2, \dots, c_m)$ and $\boldsymbol{d} = (d_1, d_2, \dots, d_m)$ be two m-tuples such that $c_j, d_j \in \mathbf{I}_2$ $(j = 1, 2, \dots, m)$. If $\boldsymbol{a} \succ \boldsymbol{b}$ and $\boldsymbol{c} \succ \boldsymbol{d}$, then the inequality*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \phi(a_i, c_j) \geq \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(b_i, d_j) \tag{2.1}$$

holds for each convex function $\phi : \boldsymbol{I}_1 \times \boldsymbol{I}_2 \mapsto \mathbb{R}$.

Proof Without loss of generality, we assume that the tuples $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are in decreasing order and $a_i \neq b_i$, $c_j \neq d_j$ for all $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$. Let

$$A_k = \sum_{i=1}^{k} a_i, \qquad B_k = \sum_{i=1}^{k} b_i \quad (k = 1, 2, \dots, n),$$
$$C_l = \sum_{j=1}^{l} c_j, \qquad D_l = \sum_{j=1}^{l} d_j \quad (l = 1, 2, \dots, m),$$
$$A_0 = B_0 = C_0 = D_0 = 0.$$

Then it follows from the definition of majorization that

$$A_n = B_n, \qquad C_m = D_m.$$

Let $t_{i,j}$ and $s_{i,j}$ be defined by

$$t_{i,j} := \nabla\phi(a_i, b_i; c_j) = \frac{\phi(a_i, c_j) - \phi(b_i, c_j)}{a_i - b_i},$$
$$s_{i,j} := \nabla\phi(b_i; c_j, d_j) = \frac{\phi(b_i, c_j) - \phi(b_i, d_j)}{c_j - d_j}.$$

Then we clearly see that

$$\begin{aligned}\phi(a_i, c_j) - \phi(b_i, d_j) &= \phi(a_i, c_j) - \phi(b_i, c_j) + \phi(b_i, c_j) - \phi(b_i, d_j)\\ &= \frac{\phi(a_i, c_j) - \phi(b_i, c_j)}{a_i - b_i}(a_i - b_i) + \frac{\phi(b_i, c_j) - \phi(b_i, d_j)}{c_j - d_j}(c_j - d_j)\\ &= t_{i,j}(A_i - A_{i-1} - B_i + B_{i-1}) + s_{i,j}(C_j - C_{j-1} - D_j + D_{j-1}).\end{aligned}$$

Summing over all i and j gives

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m}\phi(b_i, d_j)\\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j}(A_i - A_{i-1} - B_i + B_{i-1})\\
&\qquad+ \sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j}(C_j - C_{j-1} - D_j + D_{j-1})\\
&\quad= \sum_{j=1}^{m}\left[\sum_{i=1}^{n} t_{i,j}(A_i - B_i) - \sum_{i=1}^{n} t_{i,j}(A_{i-1} - B_{i-1})\right]\\
&\qquad+ \sum_{i=1}^{n}\left[\sum_{j=1}^{m} s_{i,j}(C_j - D_j) - \sum_{j=1}^{m} s_{i,j}(C_{j-1} - D_{j-1})\right]\\
&\quad= \sum_{j=1}^{m}\left[\sum_{i=1}^{n-1} t_{i,j}(A_i - B_i) - \sum_{i=2}^{n} t_{i,j}(A_{i-1} - B_{i-1})\right]\\
&\qquad+ \sum_{i=1}^{n}\left[\sum_{j=1}^{m-1} s_{i,j}(C_j - D_j) - \sum_{j=2}^{m} s_{i,j}(C_{j-1} - D_{j-1})\right]\\
&\quad= \sum_{j=1}^{m}\left[\sum_{i=1}^{n-1} t_{i,j}(A_i - B_i) - \sum_{i=1}^{n-1} t_{i+1,j}(A_i - B_i)\right]\\
&\qquad+ \sum_{i=1}^{n}\left[\sum_{j=1}^{m-1} s_{i,j}(C_j - D_j) - \sum_{j=1}^{m-1} s_{i,j+1}(C_j - D_j)\right]\\
&\quad= \sum_{j=1}^{m}\left[\sum_{i=1}^{n-1}(t_{i,j} - t_{i+1,j})(A_i - B_i)\right]\\
&\qquad+ \sum_{i=1}^{n}\left[\sum_{j=1}^{m-1}(s_{i,j} - s_{i,j+1})(C_j - D_j)\right]. \qquad (2.2)\end{aligned}$$

Since ϕ is a convex function on $\mathbf{I}_1 \times \mathbf{I}_2$, ϕ is a coordinate convex function on $\mathbf{I}_1 \times \mathbf{I}_2$. Thus, $t_{i,j}$ is decreasing with respect to i for each fixed j and $s_{i,j}$ is decreasing with respect to j for each fixed i. Hence $t_{i,j} - t_{i+1,j} \geq 0$ for all $i \in \{1, 2, \dots, n-1\}$ and $s_{i,j} - s_{i,j+1} \geq 0$ for all $j \in \{1, 2, \dots, m-1\}$. From the definition of majorization we get $A_i - B_i \geq 0$ for all $i \in \{1, 2, \dots, n-1\}$ and $C_j - D_j \geq 0$ for all $j \in \{1, 2, \dots, m-1\}$. Therefore, the right-hand side of (2.2) is non-negative, and hence we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m}\phi(b_i, d_j) \geq 0,$$

which is equivalent to (2.1). □

In the following Theorem 2.2, we prove a general inequality for the convex functions defined on rectangles, which implies majorization inequality for certain tuples.

Theorem 2.2 *Let* $\mathbf{I}_1$ *and* $\mathbf{I}_2$ *be any two intervals in* $\mathbb{R}$, $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ *and* $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ *be two* n*-tuples such that* $a_i, b_i \in \mathbf{I}_1$ $(i = 1, 2, \dots, n)$, $\boldsymbol{c} = (c_1, c_2, \dots, c_m)$ *and* $\boldsymbol{d} = (d_1, d_2, \dots, d_m)$ *be two* m*-tuples such that* $c_j, d_j \in \mathbf{I}_2$ $(j = 1, 2, \dots, m)$ *and* $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ *and* $\boldsymbol{w} = (w_1, w_2, \dots, w_m)$ *be any positive real* n*- and* m*-tuples, respectively. If* $\phi : \mathbf{I}_1 \times \mathbf{I}_2 \mapsto \mathbb{R}$ *is a convex function, then one has*

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j \phi(b_i, d_j) \\ \geq \sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j t_i(a_i - b_i) + \sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j s_j(c_j - d_j),\end{aligned} \tag{2.3}$$

where t_i *is the positive partial derivative of* ϕ *with respect to the first variable at* b_i $(i = 1, 2, \dots, n)$ *and* s_j *is the positive partial derivative of* ϕ *with respect to the second variable at* d_j $(j = 1, 2, \dots, m)$.

Proof It follows from the convexity of the function $\phi : \mathbf{I}_1 \times \mathbf{I}_2 \mapsto \mathbb{R}$ that

$$\phi(x, y) - \phi(w, z) \geq \langle \nabla\phi(w, z), (x - w, y - z)\rangle$$

for all $(x, y), (w, z) \in \mathbf{I}_1 \times \mathbf{I}_2$. That is,

$$\phi(x, y) - \phi(w, z) \geq \frac{\partial\phi}{\partial w}(w, z)(x - w) + \frac{\partial\phi}{\partial z}(w, z)(y - z). \tag{2.4}$$

Now, applying (2.4) and by choosing $x \to a_i$, $y \to c_i$, $w \to b_i$ and $z \to d_j$, we get

$$\phi(a_i, c_j) - \phi(b_i, d_j) \geq t_i(a_i - b_i) + s_j(c_j - d_j). \tag{2.5}$$

Multiplying both sides of (2.5) by $p_i w_j$ and summing over the indices, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) \\ \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j t_i(a_i - b_i) + \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j(c_j - d_j).\end{aligned}$$

□

If $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_m)$ are two tuples, then throughout this paper P_k and W_j are defined by $P_k = \sum_{i=1}^{k} p_i$ and $W_j = \sum_{i=1}^{j} w_i$, $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Some majorization type results, which are obtained from the above Theorem 2.2, are given in the form of the following Propositions 2.3 and 2.5.

Proposition 2.3 *Assume that all the hypotheses of Theorem 2.2 hold. Additionally, **b** and **a** – **b** are monotonic in the same sense, and **c** and **c** – **d** are monotonic in the same sense. If*

$$\sum_{i=1}^{n} a_i p_i = \sum_{i=1}^{n} b_i p_i \tag{2.6}$$

and

$$\sum_{j=1}^{m} c_j w_j = \sum_{j=1}^{m} d_j w_j, \tag{2.7}$$

then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j). \tag{2.8}$$

Proof Since ϕ is a convex function on $\mathbf{I}_1 \times \mathbf{I}_2$, ϕ is a coordinate convex function on $\mathbf{I}_1 \times \mathbf{I}_2$. If **b** is an increasing n-tuple, then $(t_1, t_2, \dots, t_n)$ is an increasing n-tuple, where t_i is the positive partial derivative of ϕ with respect to the first variable at b_i ($i = 1, 2, \dots, n$). If **b** and **a** – **b** are increasing n-tuples, then, applying Chebyshev's inequality to the first term on right-hand side of (2.3) and using (2.6), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j t_i(a_i - b_i) &= \sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n} p_i t_i(a_i - b_i)\right] \\ &\geq \sum_{j=1}^{m} w_j \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i t_i \sum_{i=1}^{n} p_i(a_i - b_i)\right] = 0.\end{aligned} \tag{2.9}$$

Similarly, since ϕ is a coordinate convex function on $\mathbf{I}_1 \times \mathbf{I}_2$, if **d** is an increasing m-tuple, then $(s_1, s_2, \dots, s_m)$ is an increasing m-tuple, where s_j is the positive partial derivative of ϕ with respect to the second variable at d_j ($j = 1, 2, \dots, m$). If **d** and **c** – **d** are increasing m-tuples, then, applying Chebyshev's inequality to the second term on right-hand side of

(2.3) and using (2.7), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}&p_i w_j s_j(c_j-d_j)\\&=\sum_{i=1}^{n}p_i\left[\sum_{j=1}^{m}w_j s_j(c_j-d_j)\right]\\&\geq\sum_{i=1}^{n}p_i\left[\frac{1}{W_m}\sum_{j=1}^{m}w_j s_j\sum_{j=1}^{m}w_j(c_j-d_j)\right]=0.\end{aligned}\tag{2.10}$$

Using (2.9) and (2.10) in (2.3), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j s_j(a_i-b_i)-\sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j s_j(c_j-d_j)\geq 0,$$

which is equivalent to (2.8).

Similarly, we can prove inequality (2.8) in the remaining cases. □

Remark 2.4 In what follows, a convex function is said to be monotonic increasing if it is monotonic increasing with respect to each of its variables.

Proposition 2.5 *Let all the assumptions of Theorem 2.2 hold. If* $\phi:\mathbf{I}_1\times\mathbf{I}_2\mapsto\mathbb{R}$ *is an increasing convex function,* $\boldsymbol{b}$ *and* $\boldsymbol{a}-\boldsymbol{b}$ *are monotonic in the same sense,* $\boldsymbol{d}$ *and* $\boldsymbol{c}-\boldsymbol{d}$ *are monotonic in the same sense, and*

$$\sum_{i=1}^{n}a_i p_i\geq\sum_{i=1}^{n}b_i p_i\tag{2.11}$$

and

$$\sum_{j=1}^{m}c_j w_j\geq\sum_{j=1}^{m}d_j w_j,$$

then inequality (2.8) *holds.*

Proof Since ϕ is an increasing function on $\mathbf{I}_1\times\mathbf{I}_2$, we have $t_i\geq 0$ $(i=1,2,\dots,n)$, where t_i is the positive partial derivative of ϕ with respect to the first variable at b_i $(i=1,2,\dots,n)$, thus

$$\sum_{i=1}^{n}p_i t_i\geq 0.\tag{2.12}$$

Using (2.11) and (2.12) in the right-hand side of (2.9), we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j t_i(a_i-b_i)\geq 0.\tag{2.13}$$

Similarly, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j (c_j - d_j) \geq 0. \tag{2.14}$$

Using (2.13) and (2.14) in (2.3), we get (2.8).

Similarly, we can prove inequality (2.8) in the remaining cases. □

The following Theorem 2.6 is another weighted discrete version of majorization theorem.

Theorem 2.6 *Let $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ be two intervals in $\mathbb{R}$, $\phi : \mathbf{I}_1 \times \mathbf{I}_2 \mapsto \mathbb{R}$ be a convex function, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two n-tuples such that $a_i, b_i \in \mathbf{I}_1$ $(i = 1, 2, \ldots, n)$, $\boldsymbol{c} = (c_1, c_2, \ldots, c_m)$ and $\boldsymbol{d} = (d_1, d_2, \ldots, d_m)$ be two m-tuples such that $c_j, d_j \in \mathbf{I}_2$ $(j = 1, 2, \ldots, m)$, $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ be two any positive real n- and m-tuples, respectively, and*

$$\sum_{i=1}^{k} b_i p_i \leq \sum_{i=1}^{k} a_i p_i \quad (k = 1, 2, \ldots, n-1), \tag{2.15}$$

$$\sum_{i=1}^{k} d_j w_j \leq \sum_{i=1}^{k} c_j w_j \quad (k = 1, 2, \ldots, m-1), \tag{2.16}$$

$$\sum_{i=1}^{n} b_i p_i = \sum_{i=1}^{n} a_i p_i, \tag{2.17}$$

$$\sum_{i=1}^{m} d_j w_j = \sum_{i=1}^{m} c_j w_j. \tag{2.18}$$

Then the following statements are true:

(i) *If $\boldsymbol{b}$ and $\boldsymbol{d}$ are decreasing n- and m-tuples, respectively, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j). \tag{2.19}$$

(ii) *If $\boldsymbol{a}$ and $\mathbf{c}$ are increasing n- and m-tuples, respectively, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j). \tag{2.20}$$

Proof For part (i), we use Abel's transformation to prove part (i). Let

$$A_0 = B_0 = 0, \qquad A_k = \sum_{i=1}^{k} p_i a_i, \qquad B_k = \sum_{i=1}^{k} p_i b_i \quad (k = 1, 2, \ldots, n)$$

and

$$C_0 = D_0 = 0, \qquad C_k = \sum_{j=1}^{k} w_j c_j, \qquad D_k = \sum_{j=1}^{k} w_j d_j \quad (k = 1, 2, \ldots, m).$$

Then from (2.17) and (2.18) we have

$$A_n = B_n, \qquad C_m = D_m.$$

Since ϕ is a convex function on $\mathbf{I}_1 \times \mathbf{I}_2$, ϕ is a coordinate convex function on $\mathbf{I}_1 \times \mathbf{I}_2$. If $\mathbf{b}$ and $\mathbf{d}$ are decreasing n- and m-tuples, respectively, then $(t_1, t_2, \ldots, t_n)$ and $(s_1, s_2, \ldots, s_m)$ are decreasing n- and m-tuples, respectively, where t_i is the positive partial derivative of ϕ with respect to the first variable at b_i ($i = 1, 2, \ldots, n$) and s_j is the positive partial derivative of ϕ with respect to the second variable at d_j ($j = 1, 2, \ldots, m$). It follows from (2.3) that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) \\
&\quad \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j t_i (a_i - b_i) + \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j (c_j - d_j) \\
&\quad = \sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n} t_i (p_i a_i - p_i b_i)\right] + \sum_{i=1}^{n} p_i \left[\sum_{j=1}^{m} s_j (w_j c_j - w_j d_j)\right] \\
&\quad = \sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n} t_i (A_i - A_{i-1} - B_i + B_{i-1})\right] \\
&\qquad + \sum_{i=1}^{n} p_i \left[\sum_{j=1}^{m} s_j (C_i - C_{i-1} - D_i + D_{i-1})\right] \\
&\quad = \sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n} t_i (A_i - B_i) - \sum_{i=1}^{n} t_i (A_{i-1} - B_{i-1})\right] \\
&\qquad + \sum_{i=1}^{n} p_i \left[\sum_{j=1}^{m} s_j (C_i - D_i) - \sum_{j=1}^{m} s_j (C_{i-1} - D_{i-1})\right] \\
&\quad = \sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n-1} (t_i - t_{i+1})(A_i - B_i)\right] + \sum_{i=1}^{n} p_i \left[\sum_{j=1}^{m-1} (s_j - s_{j+1})(C_i - D_i)\right].
\end{aligned} \tag{2.21}$$

Since $(t_1, t_2, \ldots, t_n)$ and $(s_1, s_2, \ldots, s_m)$ are decreasing n- and m-tuples, respectively, $t_i - t_{i+1} \geq 0$ ($i = 1, 2, \ldots, n-1$) and $s_j - s_{j+1} \geq 0$ ($j = 1, 2, \ldots, m-1$). Also from the assumptions (2.15) and (2.16) we know that $A_i - B_i \geq 0$ ($i = 1, 2, \ldots, n-1$) and $C_j - D_j \geq 0$ ($j = 1, 2, \ldots, m-1$). Thus

$$\sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n-1} (t_i - t_{i+1})(A_i - B_i)\right] + \sum_{i=1}^{n} p_i \left[\sum_{j=1}^{m-1} (s_j - s_{j+1})(C_i - D_i)\right] \geq 0. \tag{2.22}$$

Using (2.22) in (2.21), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) \geq 0,$$

which is equivalent to (2.19).

Similarly, we can prove inequality (2.20) for the remaining cases. □

Definition 2.7 Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be a positive real n-tuple. Then the real n-tuple $\mathbf{a} = (a_1, a_2, \dots, a_n)$ is said to be monotonic increasing in mean relative to $\mathbf{p} = (p_1, p_2, \dots, p_n)$ if

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i a_i \leq \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} a_i p_i \quad (k = 1, 2, \dots, n-1),$$

and decreasing in mean relative to $\mathbf{p} = (p_1, p_2, \dots, p_n)$ if

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i a_i \geq \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} a_i p_i \quad (k = 1, 2, \dots, n-1).$$

The following Lemma 2.8 is due to Biernacki [58] (for a generalization, see Burkill and Mirsky [59]).

Lemma 2.8 *Let* $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ *and* $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ *be any two real n-tuples, which are monotonic in mean relative to positive real n-tuple* $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ *in the same sense, that is,*

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i a_i \lesseqqgtr \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} a_i p_i \quad (k = 1, 2, \dots, n-1)$$

and

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i b_i \lesseqqgtr \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} b_i p_i \quad (k = 1, 2, \dots, n-1).$$

Then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i \geq \frac{1}{P_n}\sum_{i=1}^{n} a_i p_i \frac{1}{P_n}\sum_{i=1}^{n} b_i p_i. \tag{2.23}$$

If one tuple is decreasing in mean and the other one is increasing in mean, then the reverse inequality holds in (2.23).

Now, we state another result for convex functions and for arbitrary monotonic tuples in mean.

Theorem 2.9 *Let all the assumptions of Theorem* 2.2 *hold. Additionally, if* $\boldsymbol{b}$ *and* $\boldsymbol{a} - \boldsymbol{b}$ *are monotonic n-tuples in mean relative to* $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ *in the same sense, and* $\boldsymbol{c}$ *and* $\boldsymbol{c} - \boldsymbol{d}$ *are monotonic m-tuples in mean relative to* $\boldsymbol{w} = (w_1, w_2, \dots, w_m)$ *in the same sense, then*

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) \\
&\quad \geq \frac{1}{P_n}\sum_{j=1}^{m}\sum_{i=1}^{n} w_j p_i t_i \sum_{i=1}^{n} p_i (a_i - b_i) \\
&\qquad + \frac{1}{W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j \sum_{j=1}^{m} w_j (c_j - d_j),
\end{aligned} \tag{2.24}$$

where t_i is the positive partial derivative of ϕ with respect to the first variable at b_i ($i = 1, 2, \ldots, n$) and s_j is the partial positive derivative of ϕ with respect to the second variable at d_j ($j = 1, 2, \ldots, m$).

Proof It follows from the proof of Proposition 2.3 that $(t_1, t_2, \ldots, t_n)$ is an increasing n-tuple. Now if **b** and **a** – **b** are monotonic increasing in mean relative to **p**, then, applying Chebyshev's inequality to first term on the right-hand side of (2.3), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j t_i (a_i - b_i) &= \sum_{j=1}^{m} w_j \left[\sum_{i=1}^{n} p_i t_i (a_i - b_i)\right] \\ &\geq \sum_{j=1}^{m} w_j \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i t_i \sum_{i=1}^{n} p_i (a_i - b_i)\right] \\ &= \frac{1}{P_n}\sum_{j=1}^{m}\sum_{i=1}^{n} w_j p_i t_i \sum_{i=1}^{n} p_i (a_i - b_i). \end{aligned} \tag{2.25}$$

Similarly, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j (c_j - d_j) \geq \frac{1}{W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j \sum_{j=1}^{m} w_j (c_j - d_j). \tag{2.26}$$

Using (2.25) and (2.26) in (2.3), we get (2.24).

Similarly, we can prove inequality (2.24) in the remaining cases. □

Corollary 2.10 *Assume that all the hypotheses of Theorem 2.9 hold. Additionally, if*

$$\sum_{i=1}^{n} a_i p_i = \sum_{i=1}^{n} b_i p_i \tag{2.27}$$

and

$$\sum_{j=1}^{m} c_j w_j = \sum_{j=1}^{m} d_j w_j, \tag{2.28}$$

then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j). \tag{2.29}$$

Proof Using (2.27) and (2.28) on the right-hand side of (2.24), we get (2.29). □

In the following Corollary 2.11, we obtain a majorization inequality by using an increasing convex function.

Corollary 2.11 *Let all the assumptions of Theorem 2.9 hold. If $\phi : \mathbf{I}_1 \times \mathbf{I}_2 \mapsto \mathbb{R}$ is an increasing convex function, and*

$$\sum_{i=1}^{n} a_i p_i \geq \sum_{i=1}^{n} b_i p_i \tag{2.30}$$

and

$$\sum_{j=1}^{m} c_j w_j \geq \sum_{j=1}^{m} d_j w_j, \tag{2.31}$$

then inequality (2.29) holds.

Proof Since ϕ is an increasing function on $\mathbf{I}_1 \times \mathbf{I}_2$, we get that $t_i \geq 0, s_j \geq 0$ ($i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$), where t_i is the positive partial derivative of ϕ with respect to the first variable at b_i ($i = 1, 2, \ldots, n$) and s_j is the partial positive derivative of ϕ with respect to the second variable at d_j ($j = 1, 2, \ldots, m$), thus

$$\sum_{i=1}^{n} p_i t_i \geq 0, \tag{2.32}$$

$$\sum_{j=1}^{m} w_j s_j \geq 0. \tag{2.33}$$

Hence using (2.30), (2.31), (2.32) and (2.33) on the right-hand side of (2.24), we obtain inequality (2.29). □

The following Lemma 2.12 can be found in the literature [60].

Lemma 2.12 *Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be a positive real n-tuple and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be an increasing real n-tuple. Then the inequality*

$$\sum_{i=1}^{k} x_i v_i \sum_{i=1}^{n} v_i \leq \sum_{i=1}^{n} x_i v_i \sum_{i=1}^{k} v_i \tag{2.34}$$

holds for $k = 1, 2, \ldots, n$. If $\boldsymbol{x}$ is a decreasing real n-tuple, then the reverse inequality holds in (2.32).

If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ with $b_i \neq 0$ for $i = 1, 2, \ldots, n$, then $\frac{\mathbf{a}}{\mathbf{b}} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n})$.

Theorem 2.13 *Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be any two intervals in $\mathbb{R}$, $\phi : \mathbf{I}_1 \times \mathbf{I}_2 \mapsto \mathbb{R}$ be a convex function, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two positive n-tuples such that $a_i, b_i \in \mathbf{I}_1$ ($i = 1, 2, \ldots, n$), $\boldsymbol{c} = (c_1, c_2, \ldots, c_m)$ and $\boldsymbol{d} = (d_1, d_2, \ldots, d_m)$ be two positive m-tuples such that $c_j, d_j \in \mathbf{I}_2$ ($j = 1, 2, \ldots, m$), $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)$ be any positive real n- and m-tuples, respectively, and $\boldsymbol{a}/\boldsymbol{b}$ and $\mathbf{c}/\boldsymbol{d}$ are decreasing n- and m-tuples, respectively. Then the following statements are true*:

(i) *If* $\mathbf{a}$ *is an increasing* n*-tuple and* $\mathbf{c}$ *is an increasing* m*-tuple, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi\left(\frac{a_i}{\sum_{i=1}^{n}p_ia_i},\frac{c_j}{\sum_{j=1}^{m}w_jc_j}\right)\\ \leq\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi\left(\frac{b_i}{\sum_{i=1}^{n}b_ip_i},\frac{d_j}{\sum_{j=1}^{m}w_jd_j}\right). \tag{2.35}$$

(ii) *If* $\mathbf{b}$ *is a decreasing* n*-tuple and* $\mathbf{d}$ *is a decreasing* m*-tuple, then the reverse inequality holds in* (2.33).

If $\mathbf{a}/\mathbf{b}$ *and* $\mathbf{c}/\mathbf{d}$ *are increasing* n *and* m*-tuples, respectively, then we have the following statements*:

(iii) *If* $\mathbf{b}$ *is an increasing* n*-tuple and* $\mathbf{d}$ *is an increasing* m*-tuple, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi\left(\frac{b_i}{\sum_{i=1}^{n}b_ip_i},\frac{d_j}{\sum_{j=1}^{m}w_jd_j}\right)\\ \leq\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi\left(\frac{a_i}{\sum_{i=1}^{n}p_ia_i},\frac{c_j}{\sum_{j=1}^{m}w_jc_j}\right). \tag{2.36}$$

(iv) *If* $\mathbf{a}$ *is a decreasing* n*-tuple and* $\mathbf{c}$ *is a decreasing* m*-tuple, then the reverse inequality holds in* (2.34).

Proof (i) Let $\mathbf{a}/\mathbf{b}$ and $\mathbf{c}/\mathbf{d}$ are decreasing n- and m-tuples, respectively. Then using Lemma 2.12 with $\mathbf{x}=\mathbf{a}/\mathbf{b}$ and $\mathbf{v}=\mathbf{pb}$, we obtain

$$\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{k}p_ib_i\leq\sum_{i=1}^{k}a_ip_i\sum_{i=1}^{n}p_ib_i\quad(k=1,2,\ldots,n).$$

That is,

$$\sum_{i=1}^{k}p_i\left(\frac{b_i}{\sum_{i=1}^{n}p_ib_i}\right)\leq\sum_{i=1}^{k}p_i\left(\frac{a_i}{\sum_{i=1}^{n}p_ia_i}\right)\quad(k=1,2,\ldots,n). \tag{2.37}$$

Similarly, using Lemma 2.12 with $\mathbf{x}=\mathbf{c}/\mathbf{d}$ and $\mathbf{v}=\mathbf{dw}$, we get

$$\sum_{i=1}^{n}w_jc_j\sum_{i=1}^{k}w_jd_j\leq\sum_{i=1}^{k}c_jw_j\sum_{i=1}^{n}d_jw_j\quad(k=1,2,\ldots,m).$$

That is,

$$\sum_{j=1}^{k}w_j\left(\frac{d_j}{\sum_{j=1}^{m}w_jd_j}\right)\leq\sum_{j=1}^{k}w_j\left(\frac{c_j}{\sum_{j=1}^{m}w_jc_j}\right)\quad(k=1,2,\ldots,m). \tag{2.38}$$

Also, it is obvious that

$$\sum_{i=1}^{n}p_i\left(\frac{b_i}{\sum_{i=1}^{n}p_ib_i}\right)=\sum_{i=1}^{n}p_i\left(\frac{a_i}{\sum_{i=1}^{n}p_ia_i}\right) \tag{2.39}$$

and

$$\sum_{j=1}^{n} w_j\left(\frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right) = \sum_{j=1}^{n} w_j\left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right). \tag{2.40}$$

If **a** and **c** are increasing, then using Theorem 2.6(ii) and (2.35)–(2.38), we have

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right)\\ &\quad\le \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{b_i}{\sum_{i=1}^{n} b_i p_i}, \frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right).\end{aligned}$$

Similarly, we can prove the remaining cases. □

Definition 2.14 (See [61]) An n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is said to be concave if

$$2a_k \ge a_{k+1} + a_{k-1}$$

for all $k = 2, 3, \ldots, n-1$.

Definition 2.15 (See [61]) An n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is said to be convex if

$$2a_k \le a_{k+1} + a_{k-1}$$

for all $k = 2, 3, \ldots, n-1$.

Corollary 2.16 *Let $\phi : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a convex function, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ be a positive n-tuple, $\boldsymbol{c} = (c_1, c_2, \ldots, c_m)$ be a positive m-tuple, and $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)$ be any positive real n- and m-tuples, respectively. Then the following statements are true*:

(i) *If **a** is an increasing concave n-tuple and **c** is an increasing concave m-tuple, then*

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{i-1}{\sum_{i=1}^{n} p_i (i-1)}, \frac{j-1}{\sum_{j=1}^{m} w_j (j-1)}\right)\\ &\quad\ge \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right).\end{aligned} \tag{2.41}$$

(ii) *If **a** is an increasing convex n-tuple with $a_1 = 0$ and **c** is an increasing convex m-tuple with $c_1 = 0$, then the reverse inequality holds in* (2.39).

(iii) *If **a** is a decreasing concave n-tuple and **c** is a decreasing concave m-tuple, then*

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{n-i}{\sum_{i=1}^{n} p_i (n-i)}, \frac{m-j}{\sum_{j=1}^{m} w_j (m-j)}\right)\\ &\quad\ge \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right).\end{aligned} \tag{2.42}$$

(iv) *If **a** is a decreasing convex n-tuple with $a_n = 0$ and **c** is a decreasing convex m-tuple with $c_m = 0$, then the reverse inequality holds in* (2.40).

Proof (i) Let $\mathbf{b} = (b_1, b_2, \dots, b_n)$ and $\mathbf{d} = (d_1, d_2, \dots, d_m)$ be respectively the n- and m-tuples such that $b_1 = \epsilon < a_1/a_2$, $d_1 = \delta < c_1/c_2$, $b_i = i-1$ for $i = 2, 3, \dots, n$, and $d_j = j-1$ for $j = 2, 3, \dots, m$. Then $\mathbf{a}/\mathbf{b}$ and $\mathbf{c}/\mathbf{d}$ are decreasing n- and m-tuples, respectively. It follows from Theorem 2.13 that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right) \\
&\quad\le p_1 w_1 \phi\left(\frac{\epsilon}{\epsilon p_1 + \sum_{i=2}^{n}(i-1)p_i}, \frac{\delta}{w_1\delta + \sum_{j=2}^{m}(j-1)w_j}\right) \\
&\qquad + \sum_{i=2}^{n}\sum_{j=2}^{m} p_i w_j \phi\left(\frac{i-1}{\epsilon p_1 + \sum_{i=2}^{n}(i-1)p_i}, \frac{j-1}{w_1\delta + \sum_{j=2}^{m}(j-1)w_j}\right).
\end{aligned}$$

Taking $\epsilon \to 0$ and $\delta \to 0$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right) \\
&\quad\le p_1 w_1 \phi(0,0) + \sum_{i=2}^{n}\sum_{j=2}^{m} p_i w_j \phi\left(\frac{i-1}{\epsilon p_1 + \sum_{i=2}^{n}(i-1)p_i}, \frac{j-1}{w_1\delta + \sum_{j=2}^{m}(j-1)w_j}\right) \\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{i-1}{\epsilon p_1 + \sum_{i=2}^{n}(i-1)p_i}, \frac{j-1}{w_1\delta + \sum_{j=2}^{m}(j-1)w_j}\right).
\end{aligned}$$

This proves (2.39).

Similarly, we can use other parts of Theorem 2.13 to prove the required results for the remaining cases. □

Remark 2.17 For some related integral majorization inequalities for the convex functions defined on rectangles, see [62]; and for some other recent results related to majorization, see [63–65].

3 Results and discussion

In the article, we have generalized the classical majorization inequality for majorized tuples, established several weighted version of majorization inequalities for certain tuples and provided Favard's type inequalities by the use of Chebyshev's inequality, Abel transformation and support line inequality.

4 Conclusions

In this paper, we have extended some discrete majorization type inequalities of convex functions from intervals to rectangles. The given results are generalizations of the previously known results. Our approach may have further applications in the theory of majorization.

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