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Self-adaptive subgradient extragradient method with inertial modification for solving monotone variational inequality problems and quasi-nonexpansive fixed point problems

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Abstract

In this paper, we introduce a new algorithm with self-adaptive method for finding a solution of the variational inequality problem involving monotone operator and the fixed point problem of a quasi-nonexpansive mapping with a demiclosedness property in a real Hilbert space. The algorithm is based on the subgradient extragradient method and inertial method. At the same time, it can be considered as an improvement of the inertial extragradient method over each computational step which was previously known. The weak convergence of the algorithm is studied under standard assumptions. It is worth emphasizing that the algorithm that we propose does not require one to know the Lipschitz constant of the operator. Finally, we provide some numerical experiments to verify the effectiveness and advantage of the proposed algorithm.

Keywords: Variational inequality problem; Fixed point problem; Extragradient method; Subgradient extragradient method; Inertial method; Self-adaptive method

1 Introduction

Throughout this paper, let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively.

The variational inequality problem (VIP) is the problem to find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

The solution set of VIP is denoted by $VI(C,A)$. The variational inequality problem is an important branch of the nonlinear problem and it has received a lot of attentions by many authors in recent years (see [1–3] and the references therein). Under appropriate conditions, there are two general approaches for solving the variational inequality problem, one is the regularized method and the other is the projection method. Now, we mainly study the projection method.

For every point $x \in H$, there exists a unique point in C such that

$$\|x - P_C x\| = \inf\bigl\{\|x - y\| : y \in C\bigr\},$$

where $P_C : H \to C$ is called the metric projection from H into C.

We know that the variational inequality problem can be turned into the fixed point problem, which means (1) is equivalent to

$$x^* = P_C(I - \lambda A)x^*, \tag{2}$$

where $P_C : H \to C$ is the metric projection and $\lambda > 0$. Thus we generate $\{x_n\}$ in the following manner:

$$x_{n+1} = P_C(I - \lambda A)x_n. \tag{3}$$

This simple algorithm is an extension of the projection gradient method. However, the convergence of this method requires a slight assumption that the operator $A : H \to H$ is strongly monotone or inverse strongly monotone.

To avoid this strong assumption, Korpelevich [4] proposed an algorithm which was called the extragradient method:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{4}$$

for each $n = 1, 2, \dots$, where $\lambda \in (0, 1/L)$. At the beginning, this method was used to solve saddle point problems. Soon, this method was extended to Euclidean spaces and Hilbert spaces. In particular, this method only requires that the operator A is monotone and L-Lipschitz continuous in a Hilbert space. If $VI(C,A) \neq \emptyset$, the sequence $\{x_n\}$ generated by (4) converges weakly to an element of $VI(C,A)$.

However, the extragradient method needs to calculate two projections from H onto the closed convex set C and it is applicable to the case that P_C has a closed form which means that P_C has an explicit expression. In fact, in some cases, the projection onto the nonempty closed convex subset C might be difficult to calculate. To overcome this drawback, it has received great attentions by many authors who had improved it in various ways.

To our knowledge, there were four kinds of methods to overcome this drawback. The first one was the modification of the extragradient method by Tseng [5] who proposed it in 2000 with the following remarkable scheme:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n), \end{cases} \tag{5}$$

where A is monotone, L-Lipschitz continuous and $\lambda \in (0, 1/L)$. From (5), we find using this method one only needs to calculate one projection, which is simpler than (4). The second one was the subgradient extragradient method which was proposed by Censor et al. [6] in

2011:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H \mid \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{6}$$

where A is monotone, L-Lipschitz continuous and $\lambda \in (0, 1/L)$. The key operation of the subgradient extragradient method replaces the second projection onto C of the extragradient method by a projection onto a special constructible half-space, which significantly reduces the difficulty of calculations.

Before explaining the third method, let us take a look at the inertial method. In 2001, Alvarez and Attouch [7] applied the inertial technique to obtain an inertial proximal method to solve the problem of finding zero of a maximal monotone operator, which works as follows:

$$\text{find } x_{n+1} \in H, \text{ such that } 0 \in \lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}),$$

where $x_{n-1}, x_n \in H$, $\theta_n \in [0, 1)$ and $\lambda_n > 0$. It also can be written in the following form:

$$x_{n+1} = J^A_{\lambda_n}\big(x_n + \theta_n(x_n - x_{n-1})\big), \tag{7}$$

where $J^A_{\lambda_n}$ is the resolvent of A with parameter λ_n and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$. Recently, considerable interest has been shown in studying the inertial method by many authors. They constructed fast iterative algorithms by using inertial method. The third method which was studied by Q.L. Dong et al. [8] in 2017:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda A w_n), \\ d(w_n, y_n) = (w_n - y_n) - \lambda(A w_n - A y_n), \\ x_{n+1} = w_n - \gamma \beta_n d(w_n, y_n), \end{cases} \tag{8}$$

for each $k \geq 1$, where $\gamma \in (0, 2)$, $\lambda > 0$,

$$\beta_n := \begin{cases} \varphi(w_n, y_n)/\|d(w_n, y_n)\|^2, & \text{if } d(w_n, y_n) \neq 0, \\ 0, & \text{if } d(w_n, y_n) = 0, \end{cases}$$

$$\varphi(w_n, y_n) = \big\langle w_n - y_n, d(w_n, y_n)\big\rangle.$$

This algorithm incorporate the inertial terms in the projection and contraction algorithm, which does not need the summability condition for the sequence. The fourth one was a self-adaptive algorithm which was based on Tseng's extragradient method [9] and it was proposed by Duong Viet Thong and Dang Van Hieu [9] in 2017. The algorithm is described as follows.

It is worth mentioning that the Algorithm 1 does not require one to know the Lipschitz constant of the operator A, which is different from the other three algorithms. If $VI(C, A) \neq$

Algorithm 1 A SA-Tseng's EGM for monotone variational inequality problem

Step 1: Choose $x_0 \in H$, $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$.

Step 2: Given the current iterate x_n, compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ satisfying

$$\lambda \|Ax_n - Ay_n\| \le \mu \|x_n - y_n\|.$$

If $y_n = x_n$, then stop and x_n is the solution of the variational inequality problem. Otherwise,

Step 3: Compute the new iterate x_{n+1} via the following iterate formula:

$$x_{n+1} = y_n - \lambda_n (Ay_n - Ax_n).$$

Set $n := n + 1$ and return to Step 2.

∅, the sequences $\{x_n\}$ generated by (5), (6), (8) and Algorithm 1 all converge weakly to an element of $VI(C,A)$. For Algorithm 1, it does not require to know the Lipschitz constant, but the step size may involve computation of additional projections.

In 2016, Mainge and Gobinddass [10] got x_{n+1} by the following algorithm:

$$\begin{cases} y_n = x_n + \theta_n (x_n - x_{n-1}), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases} \tag{9}$$

where $\theta_n = \frac{\lambda_n}{\delta \lambda_{n-1}}$, $\lambda_n k_n \le \varepsilon \delta (\sqrt{2} - 1)$, $\lambda_n \le k \lambda_{n-1} (\delta + \frac{\lambda_{n-1}}{\lambda_{n-2}})^{\frac{1}{2}}$, $\{\lambda_n\} \subset [\overline{\mu}, \overline{\nu}]$ and

$$k_n := \begin{cases} \frac{\|Ay_n - Ay_{n-1}\|}{\|y_n - y_{n-1}\|}, & \text{if } y_n - y_{n-1} \ne 0, \\ 0, & \text{if } y_n - y_{n-1} = 0. \end{cases}$$

In this iterative algorithm, it does not require either additional projection for the determination of the step-sizes or the knowledge of the Lipschitz constant of the operator.

The fixed point problem is the problem to find $x^* \in H$ such that

$$Tx^* = x^*, \tag{10}$$

where x^* is called a fixed point of $T : H \to H$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. Recently, many iterative methods have been proposed (see [6, 11–21] and the references therein) for finding a common element of $\mathrm{Fix}(T)$ and $VI(C,A)$ in a real Hilbert space.

In this paper, motivated and inspired by the above results, we introduce a new algorithm with self-adaptive subgradient extragradient method and inertial modification for finding a solution of the variational inequality problem involving monotone operator and the fixed point problem of a quasi-nonexpansive mapping with a demiclosedness property in a real Hilbert space. Then the weak convergence theorem will be proved in Sect. 3.

This paper is organized as follows. In Sect. 2, we list some lemmas which will be used for further proof. In Sect. 3, we proposed a new algorithm, then the weak convergence theorem is analyzed. In Sect. 4, we give some numerical examples to illustrate the efficiency and advantage of our algorithm.

2 Preliminaries

In this section, we introduce some lemmas which will be used in this paper. Assume H is a real Hilbert space and C is a nonempty closed convex subset of H. In the following of the paper, we use the symbol $x_n \to x$ to denote the strong convergence of the sequence $\{x_n\}$ to x as $n \to \infty$ and use the symbol $x_n \rightharpoonup x$ to denote the weak convergence of the sequence $\{x_n\}$ to x as $n \to \infty$. If there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to a point z, then z is called a weak cluster point of $\{x_n\}$ and the set of all weak cluster points of $\{x_n\}$ is denoted by $\omega_w(x_n)$.

Lemma 2.1 ([22]) *Let H be a real Hilbert space, for each $x, y \in H$ and $\lambda \in \mathbb{R}$, we have*
(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;
(ii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

In the following, we gather some characteristic properties of P_C.

Lemma 2.2 ([23]) *Let H be a real Hilbert space and C be a nonempty closed subset of H. Then*
(i) $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle$, $\forall x, y \in H$;
(ii) $\|x - P_C x\|^2 + \|y - P_C x\|^2 \le \|x - y\|^2$, $\forall x \in H, y \in C$.

Lemma 2.3 *Let H be a real Hilbert space and C be a nonempty closed subset of H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there hold the inequality $\langle x - z, y - z\rangle \le 0$, $\forall y \in C$.*

Next, we present some concepts of an operator.

Definition 2.4 ([24]) An operator $A : H \to H$ is said to be:
(i) monotone, if

$$\langle x - y, Ax - Ay\rangle \ge 0, \quad \forall x, y \in H;$$

(ii) L-Lipschitz continuous with $L > 0$, if

$$\|Ax - Ay\| \le L\|x - y\|, \quad \forall x, y \in H;$$

(iii) nonexpansive, if

$$\|Ax - Ay\| \le \|x - y\|, \quad \forall x, y \in H;$$

(iv) quasi-nonexpansive, if

$$\|Ax - p\| \leq \|x - p\|, \quad \forall x \in H, p \in Fix(A),$$

where $Fix(A) \neq \emptyset$.

Remark 2.5 ([25]) It is well that every nonexpansive mapping with a nonempty set of fixed point is quasi-nonexpansive. However, a quasi-nonexpansive mapping may not be a nonexpansive mapping.

Lemma 2.6 ([23]) *Assume that $T : H \to H$ is a nonlinear operator with* $\mathrm{Fix}(T) \neq \emptyset$. *Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds*:

$$x_n \rightharpoonup x \quad \textit{and} \quad (I - T)x_n \to 0 \Rightarrow x \in \mathrm{Fix}(T).$$

Remark 2.7 We know that the Lemma 2.6 is clearly established when the operator T is nonexpansive. However, there exists a quasi-nonexpansive mapping T but $I - T$ is not demiclosed at zero. Therefore, in this paper, we need to emphasize that $T : H \to H$ is a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero.

Example 1 Let H be the line real and $C = [0, \frac{3}{2}]$. Define the operator T on C by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1], \\ x \cos 2\pi x, & \text{if } x \in (1, \frac{3}{2}]. \end{cases}$$

Indeed, it is easy to see that $\mathrm{Fix}(T) = \{0\}$.

On the one hand, for any $x \in [0,1]$, we have

$$|Tx - 0| = \left| \frac{x}{2} - 0 \right| \leq |x - 0|.$$

On the other hand, for any $x \in (1, \frac{3}{2}]$, we have

$$|Tx - 0| = |x \cos 2\pi x - 0| = |x \cos 2\pi x - 0| \leq |x| = |x - 0|.$$

Thus, the operator T is quasi-nonexpansive.

By taking $\{x_n\} \subset (1, \frac{3}{2}]$ and $x_n \to 1$ as $n \to \infty$, we have

$$\left|(I - T)x_n\right| = |x_n - Tx_n| = |x_n - x_n \cos 2\pi x_n| = |x_n| \cdot |1 - \cos 2\pi x_n| \to 0 \quad (n \to \infty).$$

But $1 \notin \mathrm{Fix}(T)$, so $I - T$ is not demiclosed at zero.

Lemma 2.8 ([7]) *Let $\{\varphi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in* $[0, +\infty)$ *such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n, \quad \forall n \geq 1, \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(i) $\sum_{n=1}^{+\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$;
(ii) *there exists* $\varphi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \varphi_n = \varphi^*$.

Lemma 2.9 ([26]) *Let* $A : H \to H$ *be a monotone and* L*-Lipschitz continuous mapping on* C. *Let* $S = P_C(I - \mu A)$, *where* $\mu > 0$. *If* $\{x_n\}$ *is a sequence in* H *satisfying* $x_n \rightharpoonup q$ *and* $x_n - Sx_n \to 0$, *then* $q \in VI(C, A) = \mathrm{Fix}(S)$.

Lemma 2.10 ([27]) *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H *and* $\{x_n\}$ *be a sequence in* H. *The following two properties hold*:
(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each* $x \in C$;
(ii) $\omega_w(x_n) \subset C$.
Then the sequence $\{x_n\}$ *converges weakly to a point in* C.

3 Main results

In this section, we propose a new iterative algorithm with self-adaptive method for solving monotone variational inequality problems and quasi-nonexpansive fixed point problems in a Hilbert space. Meanwhile, we combine subgradient extragradient method and inertial modification for the algorithm. Under the assumption $\mathrm{Fix}(T) \cap VI(C, A) \neq \emptyset$, we prove the weak convergence theorem. Let H be a real Hilbert space. Let C be a nonempty closed convex subset in H. Let $A : H \to H$ be a monotone and L-Lipschitz continuous operator. In particular, the information of the Lipschitz constant L does not require to be known. Let $T : H \to H$ be a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. The algorithm is described as follows.

Before giving the theorem and its proof, we propose several useful lemmas firstly.

Lemma 3.1 *The sequence* $\{\lambda_n\}$ *generated by Algorithm* 2 *is a monotonically decreasing sequence, and its lower bound is* $\min\{\frac{\mu}{L}, \lambda_0\}$.

Proof It is obvious that the sequence $\{\lambda_n\}$ is a monotonically decreasing sequence.

Since A is L-Lipschitz continuous with $L > 0$, we have

$$\|Ax_n - Ay_n\| \le L\|x_n - y_n\|.$$

In the case of $Ax_n - Ay_n \neq 0$, we have

$$\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|} \ge \frac{\mu}{L}.$$

Clearly, the lower bound of the sequence $\{\lambda_n\}$ is $\min\{\frac{\mu}{L}, \lambda_0\}$. □

Lemma 3.2 *If* $w_n = y_n = x_{n+1}$, *then* $w_n \in \mathrm{Fix}(T) \cap VI(C, A)$.

Proof If $w_n = y_n$, we have $w_n \in VI(C, A)$.

Besides, since $w_n = y_n$, $y_n = P_C(w_n - \lambda_n Aw_n)$, according to Lemma 2.3, we have $\langle w_n - \lambda_n Aw_n - y_n, x - y_n\rangle \le 0$, $\forall x \in C$. Since $w_n = y_n$, $z_n = P_{T_n}(w_n - \lambda_n Ay_n)$, where $T_n = \{x \in H \mid \langle w_n - \lambda_n Aw_n - y_n, x - y_n\rangle \le 0\}$, we have $y_n = z_n$.

Algorithm 2 A ISA-SEGM for monotone variational inequality problem

Step 1: Choose $x_0, x_1 \in H$, $\mu \in (0,1)$, $\lambda_0 > 0$.

Step 2: Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$, compute

$$y_n = P_C(w_n - \lambda_n A w_n).$$

Step 3: Compute

$$z_n = P_{T_n}(w_n - \lambda_n A y_n),$$

where $T_n = \{x \in H | \langle w_n - \lambda_n A w_n - y_n, x - y_n\rangle \le 0\}$.

Step 4: Compute

$$x_{n+1} = (1-\beta_n)w_n + \beta_n T z_n$$

and

$$\lambda_{n+1} := \begin{cases} \min\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n\}, & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

If $w_n = y_n = x_{n+1}$, then $w_n \in \mathrm{Fix}(T) \cap VI(C,A)$.

Set $n := n+1$ and return to Step 2.

On the other hand, if $w_n = y_n = x_{n+1}$, by $x_{n+1} = (1-\beta_n)w_n + \beta_n T z_n$, we have

$$w_n = (1-\beta_n)w_n + \beta_n T w_n,$$

through deformation, we can get $Tw_n = w_n$, which means $w_n \in \mathrm{Fix}(T)$.

Therefore, $w_n \in \mathrm{Fix}(T) \cap VI(C,A)$. □

Lemma 3.3 *Let $\{z_n\}$ be a sequence generated by Algorithm 2, then, for all $p \in VI(C,A)$, and for n sufficiently large, we have*

$$\begin{aligned} \|z_n - p\|^2 \le{}& \|w_n - p\|^2 - (1-\mu\gamma)\|y_n - w_n\|^2 \\ & - (1-\mu\gamma)\|z_n - y_n\|^2 - 2\lambda_n\langle Ap, y_n - p\rangle. \end{aligned} \tag{11}$$

Proof Since $p \in VI(C,A)$ and $VI(C,A) \subset C \subset T_n$, we have

$$\begin{aligned} \|z_n - p\|^2 &= \|P_{T_n}(w_n - \lambda_n A y_n) - p\|^2 \\ &\le \langle P_{T_n}(w_n - \lambda_n A y_n) - P_{T_n} p, w_n - \lambda_n A y_n - p\rangle \\ &= \langle z_n - p, w_n - \lambda_n A y_n - p\rangle \\ &= \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}\|w_n - \lambda_n A y_n - p\|^2 - \frac{1}{2}\|z_n - w_n + \lambda_n A y_n\|^2 \\ &= \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 + \frac{1}{2}\lambda_n^2\|Ay_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&- \langle w_n - p, \lambda_n A y_n\rangle - \frac{1}{2}\|z_n - w_n\|^2 - \frac{1}{2}\lambda_n^2\|Ay_n\|^2 \\
&- \langle z_n - w_n, \lambda_n A y_n\rangle \\
=\ & \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|z_n - w_n\|^2 - \langle z_n - p, \lambda_n A y_n\rangle.
\end{aligned}
$$

It implies that

$$
\|z_n - p\|^2 \le \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n A y_n\rangle.
$$

Because the operator A is monotone and $\lambda_n > 0$, we have

$$
2\lambda_n\langle Ay_n - Ap, y_n - p\rangle \ge 0.
$$

So

$$
\begin{aligned}
\|z_n - p\|^2 \le\ & \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n A y_n\rangle \\
& + 2\lambda_n\langle Ay_n - Ap, y_n - p\rangle \\
=\ & \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\lambda_n\bigl(\langle Ay_n - Ap, y_n - p\rangle \\
& - \langle z_n - p, Ay_n\rangle\bigr) \\
=\ & \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\lambda_n\bigl(\langle Ay_n, y_n - p\rangle - \langle Ap, y_n - p\rangle \\
& - \langle z_n - p, Ay_n\rangle\bigr) \\
=\ & \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\lambda_n\bigl(\langle Ay_n, y_n - z_n\rangle - \langle Ap, y_n - p\rangle\bigr) \\
=\ & \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\lambda_n\langle Ay_n - Aw_n, y_n - z_n\rangle \\
& + 2\lambda_n\langle Aw_n, y_n - z_n\rangle - 2\lambda_n\langle Ap, y_n - p\rangle.
\end{aligned}
$$

Since $\{\lambda_n\}$ is a monotonically decreasing sequence, the limit of $\{\lambda_n\}$ exists and $\frac{\lambda_n}{\lambda_{n+1}} \ge 1$. We denote $\lambda = \lim_{n\to\infty}\lambda_n$. Therefore, we have $\lim_{n\to\infty} = \frac{\lambda_n}{\lambda_{n+1}} = 1$. We have $\mu \in (0,1)$, $\frac{1}{\mu} > 1$, let $\gamma = \frac{1+\frac{1}{\mu}}{2}$, $1 < \gamma < \frac{1}{\mu}$. Therefore, $\exists N \in \mathbb{N}$, $\forall n > N$, $\frac{\lambda_n}{\lambda_{n+1}} < \gamma$. Therefore, $1 - \mu\gamma < 1$.

$$
\begin{aligned}
2\lambda_n\langle Ay_n - Aw_n, y_n - z_n\rangle &\le 2\lambda_n\|y_n - z_n\| \cdot \|Ay_n - Aw_n\| \\
&\le 2\lambda_n\|y_n - z_n\| \cdot \frac{\mu}{\lambda_{n+1}}\|y_n - w_n\| \\
&\le 2\mu\gamma\|y_n - z_n\| \cdot \|y_n - w_n\| \\
&\le \mu\gamma\|y_n - z_n\|^2 + \mu\gamma\|y_n - w_n\|^2.
\end{aligned}
$$

Since $z_n \in T_n$, we have

$$
\langle w_n - \lambda_n A w_n - y_n, z_n - y_n\rangle \le 0.
$$

So

$$
\begin{aligned}
2\lambda_n\langle Aw_n, y_n - z_n\rangle &\le 2\langle y_n - w_n, z_n - y_n\rangle \\
&= \|z_n - w_n\|^2 - \|y_n - w_n\|^2 - \|z_n - y_n\|^2.
\end{aligned}
$$

Therefore

$$\begin{aligned}
\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\lambda_n\langle Ay_n - Aw_n, y_n - z_n\rangle \\
&\quad + 2\lambda_n\langle Aw_n, y_n - z_n\rangle - 2\lambda_n\langle Ap, y_n - p\rangle \\
&\le \|w_n - p\|^2 - \|z_n - w_n\|^2 + \mu\gamma\|y_n - z_n\|^2 + \mu\gamma\|y_n - w_n\|^2 \\
&\quad + \|z_n - w_n\|^2 - \|y_n - w_n\|^2 - \|z_n - y_n\|^2 - 2\lambda_n\langle Ap, y_n - p\rangle \\
&= \|w_n - p\|^2 - (1-\mu\gamma)\|y_n - w_n\|^2 - (1-\mu\gamma)\|z_n - y_n\|^2 \\
&\quad - 2\lambda_n\langle Ap, y_n - p\rangle.
\end{aligned}$$

□

Theorem 3.4 *Assume that the sequence $\{\alpha_n\}$ is non-decreasing such that $0 \le \alpha_n \le \alpha \le \frac{1}{4}$ and the sequence $\{\beta_n\}$ is a sequence of real numbers such that $0 < \beta \le \beta_n \le \frac{1}{2}$. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to an element of* $\mathrm{Fix}(T) \cap VI(C, A)$.

Proof Let $p \in \mathrm{Fix}(T) \cap VI(C, A)$.

From Lemma 3.3, we have $\exists N \ge 0$, $\forall n > N$, $\|z_n - p\| \le \|w_n - p\|$.

Since T is quasi-nonexpansive, by Lemma 2.1, we have $\forall n > N$

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|(1-\beta_n)w_n + \beta_n Tz_n - p\big\|^2 \\
&= \big\|(1-\beta_n)(w_n - p) + \beta_n(Tz_n - p)\big\|^2 \\
&= (1-\beta_n)\|w_n - p\|^2 + \beta_n\|Tz_n - p\|^2 - \beta_n(1-\beta_n)\|Tz_n - w_n\|^2 \\
&\le (1-\beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|Tz_n - w_n\|^2 \\
&\le (1-\beta_n)\|w_n - p\|^2 + \beta_n\|w_n - p\|^2 - \beta_n(1-\beta_n)\|Tz_n - w_n\|^2 \\
&= \|w_n - p\|^2 - \beta_n(1-\beta_n)\|Tz_n - w_n\|^2.
\end{aligned} \tag{12}$$

Since $x_{n+1} = (1-\beta_n)w_n + \beta_n Tz_n$, we can write it as

$$Tz_n - w_n = \frac{1}{\beta_n}(x_{n+1} - w_n). \tag{13}$$

Combining (12) and (13), with $\beta_n \le \frac{1}{2}$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|w_n - p\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - w_n\|^2 \\
&\le \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2.
\end{aligned} \tag{14}$$

Besides,

$$\begin{aligned}
\|w_n - p\|^2 &= \big\|x_n + \alpha_n(x_n - x_{n-1}) - p\big\|^2 \\
&= \big\|(1+\alpha_n)(x_n - p) - \alpha_n(x_{n-1} - p)\big\|^2 \\
&= (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 \\
&\quad + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2
\end{aligned} \tag{15}$$

and

$$\begin{aligned}\|x_{n+1}-w_n\|^2 &= \left\|x_{n+1}-\left(x_n+\alpha_n(x_n-x_{n-1})\right)\right\|^2\\ &= \|x_{n+1}-x_n\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2-2\alpha_n\langle x_{n+1}-x_n,x_n-x_{n-1}\rangle\\ &\geq \|x_{n+1}-x_n\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2\\ &\quad -2\alpha_n\|x_{n+1}-x_n\|\cdot\|x_n-x_{n-1}\|\\ &\geq \|x_{n+1}-x_n\|^2+\alpha_n^2\|x_n-x_{n-1}\|^2-\alpha_n\|x_{n+1}-x_n\|^2\\ &\quad -\alpha_n\|x_n-x_{n-1}\|^2\\ &= (1-\alpha_n)\|x_{n+1}-x_n\|^2+\left(\alpha_n^2-\alpha_n\right)\|x_n-x_{n-1}\|^2. \end{aligned}\tag{16}$$

Combining (14), (15), (16), and $\{\alpha_n\}$ being non-decreasing, we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2\\ &\quad -(1-\alpha_n)\|x_{n+1}-x_n\|^2-\left(\alpha_n^2-\alpha_n\right)\|x_n-x_{n-1}\|^2\\ &= (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +\left[\alpha_n(1+\alpha_n)-\left(\alpha_n^2-\alpha_n\right)\right]\|x_n-x_{n-1}\|^2\\ &= (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +2\alpha_n\|x_n-x_{n-1}\|^2\\ &\leq (1+\alpha_{n+1})\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +2\alpha_n\|x_n-x_{n-1}\|^2. \end{aligned}\tag{17}$$

Therefore,

$$\begin{aligned}&\|x_{n+1}-p\|^2-\alpha_{n+1}\|x_n-p\|^2+2\alpha_{n+1}\|x_{n+1}-x_n\|^2\\ &\quad\leq \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2\\ &\qquad +2\alpha_{n+1}\|x_{n+1}-x_n\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad= \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2\\ &\qquad +(2\alpha_{n+1}-1+\alpha_n)\|x_{n+1}-x_n\|^2. \end{aligned}\tag{18}$$

Put $\Gamma_n := \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2$.
From (18), we obtain

$$\Gamma_{n+1}-\Gamma_n \leq (2\alpha_{n+1}-1+\alpha_n)\|x_{n+1}-x_n\|^2.\tag{19}$$

We have $0\leq\alpha_n\leq\alpha\leq\frac{1}{4}$, $-(2\alpha_{n+1}-1+\alpha_n)\geq\frac{1}{4}$.
So $\Gamma_{n+1}-\Gamma_n\leq-\delta\|x_{n+1}-x_n\|^2\leq 0$, where $\delta=\frac{1}{4}$, which implies that the sequence $\{\Gamma_n\}$ is non-increasing.

Besides,

$$\begin{aligned} \Gamma_n &= \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2 \\ &\geq \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 \end{aligned} \tag{20}$$

and

$$\begin{aligned} \Gamma_{n+1} &= \|x_{n+1} - p\|^2 - \alpha_{n+1}\|x_n - p\|^2 + 2\alpha_{n+1}\|x_{n+1} - x_n\|^2 \\ &\geq -\alpha_{n+1}\|x_n - p\|^2. \end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned} \|x_n - p\|^2 &\leq \alpha_n\|x_{n-1} - p\|^2 + \Gamma_n \\ &\leq \alpha\|x_{n-1} - p\|^2 + \Gamma_1 \\ &\leq \cdots \\ &\leq \alpha^n\|x_0 - p\|^2 + \left(1 + \cdots + \alpha^{n-1}\right)\Gamma_1 \\ &\leq \alpha^n\|x_0 - p\|^2 + \frac{\Gamma_1}{1-\alpha}. \end{aligned} \tag{22}$$

This implies that the sequence $\{x_n\}$ is bounded.
Combining (21) and (22), we have

$$\begin{aligned} -\Gamma_{n+1} &\leq \alpha_{n+1}\|x_n - p\|^2 \\ &\leq \alpha\|x_n - p\|^2 \\ &\leq \alpha^{n+1}\|x_0 - p\|^2 + \frac{\alpha\Gamma_1}{1-\alpha}. \end{aligned} \tag{23}$$

From (19), we have

$$\begin{aligned} \delta\sum_{n=1}^{k}\|x_{n+1} - x_n\|^2 &\leq \Gamma_1 - \Gamma_{k+1} \\ &\leq \Gamma_1 + \alpha^{k+1}\|x_0 - p\|^2 + \frac{\alpha\Gamma_1}{1-\alpha} \\ &= \alpha^{k+1}\|x_0 - p\|^2 + \frac{\Gamma_1}{1-\alpha} \\ &\leq \|x_0 - p\|^2 + \frac{\Gamma_1}{1-\alpha}, \end{aligned} \tag{24}$$

which means

$$\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 < +\infty \tag{25}$$

and

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{26}$$

Besides, since $\alpha_n \le \alpha$, we have

$$\begin{aligned} \|x_{n+1} - w_n\| &= \|x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|x_n - x_{n-1}\| \\ &\le \|x_{n+1} - x_n\| + \alpha\|x_n - x_{n-1}\|. \end{aligned} \tag{27}$$

Therefore, by (26) and (27), we can obtain

$$\|x_{n+1} - w_n\| \to 0 \quad (n \to \infty). \tag{28}$$

We have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 - (1 - \alpha_n)\|x_{n+1} - x_n\|^2 \\ &\quad + 2\alpha_n\|x_n - x_{n-1}\|^2 \\ &\le (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{29}$$

Therefore, by (29) and Lemma 2.8, we have

$$\lim_{n\to\infty} \|x_n - p\|^2 = l \tag{30}$$

by (15), we have

$$\lim_{n\to\infty} \|w_n - p\|^2 = l \tag{31}$$

and we also have

$$\lim_{n\to\infty} \|x_n - w_n\|^2 = 0. \tag{32}$$

We have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le (1 - \beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1 - \beta_n)\|Tz_n - w_n\|^2 \\ &\le (1 - \beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2, \end{aligned}$$

which means

$$\|z_n - p\|^2 \ge \frac{\|x_{n+1} - p\|^2 - \|w_n - p\|^2}{\beta_n} + \|w_n - p\|^2. \tag{33}$$

Combining (30), (31) and (33), the sequence $\{\beta_n\}$ being bounded, we have

$$\liminf_{n\to\infty} \|z_n - p\|^2 \ge \lim_{n\to\infty} \|w_n - p\|^2 = l.$$

By (11), we have

$$\limsup_{n\to\infty} \|z_n - p\|^2 \le \lim_{n\to\infty} \|w_n - p\|^2 = l.$$

Therefore, we have

$$\lim_{n\to\infty} \|z_n - p\|^2 = l. \tag{34}$$

From Lemma 3.3, we can obtain $\forall n > N$

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1 - \mu\gamma)\|y_n - w_n\|^2$$

and

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1 - \mu\gamma)\|z_n - y_n\|^2.$$

By (31) and (34), we have

$$\lim_{n\to\infty} \|y_n - w_n\| = 0 \tag{35}$$

and

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{36}$$

So

$$\lim_{n\to\infty} \|z_n - w_n\| \le \lim_{n\to\infty} \bigl(\|z_n - y_n\| + \|y_n - w_n\|\bigr) = 0,$$

which implies

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{37}$$

Therefore, by (12), (30) and (31), we have

$$\lim_{n\to\infty} \|Tz_n - w_n\| = 0. \tag{38}$$

So

$$\lim_{n\to\infty} \|Tz_n - z\| \le \lim_{n\to\infty} \bigl(\|Tz_n - w_n\| + \|w_n - z_n\|\bigr) = 0,$$

which implies

$$\lim_{n\to\infty} \|Tz_n - z_n\| = 0. \tag{39}$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in H$ such that $x_{n_k} \rightharpoonup q$.

So, by (32) we have $\omega_{n_k} \rightharpoonup q$ and by (37) we have $z_{n_k} \rightharpoonup q$.

Since $z_{n_k} \rightharpoonup q$ and $I - T$ is demiclosed at zero, by Lemma 2.6, we have $q \in \operatorname{Fix}(T)$.

On the other hand, for all $n \in \mathbb{R}$, we have $\lambda_n > \frac{\mu}{L}$. We have $\omega_{n_k} \rightharpoonup q$ and

$$\|w_n - y_n\| = \bigl\|w_n - P_C(I - \mu A)w_n\bigr\| \to 0.$$

By Lemma 2.9, we have $q \in VI(C,A)$.

Therefore, $q \in \mathrm{Fix}(T) \cap VI(C,A)$.

By Lemma 2.10, we get the conclusion that the sequence $\{x_n\}$ converges weakly to an element of $\mathrm{Fix}(T) \cap VI(C,A)$.

This completes the proof. □

4 Numerical experiments

In this section, we give some numerical examples to illustrate the efficiency and advantage of our algorithm in comparisons with the well-known algorithm. We compare Algorithm 2 with the weakly convergent Algorithm 1 [19].

We choose $\alpha_n = \frac{1}{4}$, $\beta_n = \frac{1}{2}$, $\mu = \frac{1}{2}$, $\lambda_0 = \frac{1}{7}$. The starting point is $x_0 = x_1 = (1,1,\ldots,1) \in \mathfrak{R}^m$. In order to show the converges of the algorithm, we illustrate the behavior of the sequence $D_n = \|x_n - x^*\|^2$, $n = 0,1,2,\ldots$, when the execution time in second elapses where x^* is the solution of the problem and $\{x_n\}$ is the sequence generated by the algorithms. Now we introduce the examples in detail.

Example 2 Let A be a Lipschitz continuous and monotone mapping. Let T be a quasi-nonexpansive mapping. Assume $\mathrm{Fix}(T) \cap VI(C,A) \neq \emptyset$ and $C = [-2,5]$, $H = \mathbb{R}$. Let A and T be given by

$$\begin{aligned} &Ax := x + \sin x, \\ &Tx := \frac{x}{2}\sin x. \end{aligned}$$

In the following, let us verify if A and T meet the requirements of the topic.

First, for all $x, y \in H$, we have

$$\begin{aligned} &\|Ax - Ay\| = \|x + \sin x - y - \sin y\| \leq \|x - y\| + \|\sin x - \sin y\| \leq 2\|x - y\|, \\ &\langle Ax - Ay, x - y\rangle = (x + \sin x - y - \sin y)(x - y) = (x - y)^2 + (\sin x - \sin y)(x - y) \geq 0. \end{aligned}$$

Therefore, $\|Ax - Ay\| \leq L\|x - y\|$, where $L = 2$ and $\langle Ax - Ay, x - y\rangle \geq 0$. Therefore, A is L-Lipschitz continuous and monotone.

Second, for $Tx = \frac{x}{2}\sin x$, if $x \neq 0$ and $Tx = x$, then we have $x = \frac{x}{2}\sin x$, and $\sin x = 2$, which is impossible. Therefore, we obtain $x = 0$, which means $\mathrm{Fix}(T) = \{0\}$.

For all $x \in H$,

$$\|Tx - 0\| = \left\|\frac{x}{2}\sin x\right\| \leq \left\|\frac{x}{2}\right\| < \|x\| = \|x - 0\|,$$

which means T is quasi-nonexpansive.

Besides, take $x = 2\pi$ and $y = \frac{3\pi}{2}$, we have

$$\|Tx - Ty\| = \left\|\frac{2\pi}{2}\sin 2\pi - \frac{3\pi}{4}\sin\frac{3\pi}{2}\right\| = \frac{3\pi}{4} > \left\|2\pi - \frac{3\pi}{2}\right\| = \frac{\pi}{2},$$

which means T is not a nonexpansive mapping.

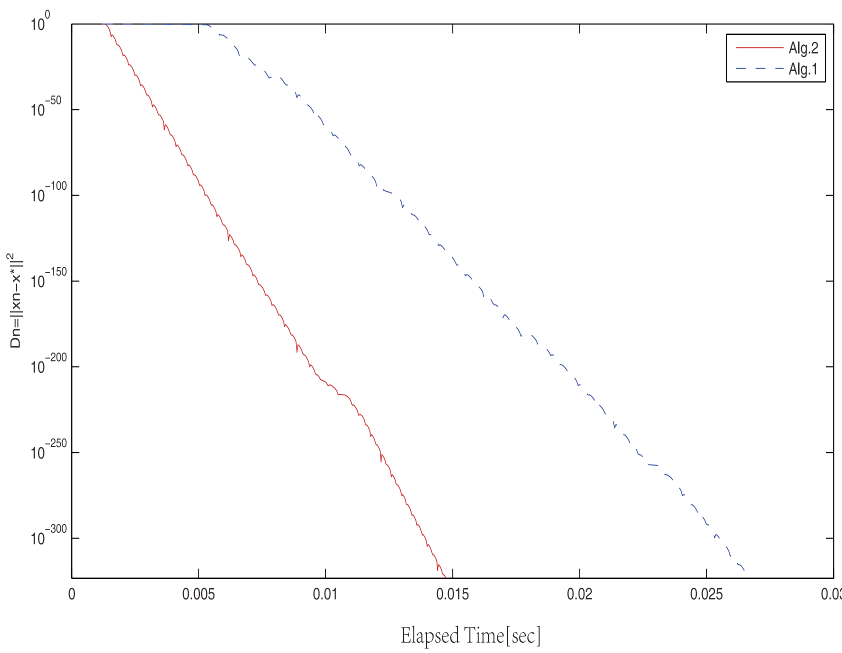

Figure 1 Experiment for Example 2

Therefore, A and T meet the requirements of the topic. The numerical results for the example are shown in Fig. 1.

From Fig. 1, we can see that the Algorithm 2 converges for a shorter time than the previously studied Algorithm 1 [19].

Example 3 We consider the operator $T : H \to H$ with $Tx = -\frac{1}{2}x$ and a linear operator $A : \Re^m \to \Re^m$ in the form $A(x) = Mx + q$ [28, 29], where

$$M = NN^T + S + D,$$

N is a $m \times m$ matrix, S is a $m \times m$ skew-symmetric matrix, D is a $m \times m$ diagonal matrix which its diagonal entries are nonnegative, and $q \in \Re^m$ is a vector, therefore M is positive definite. The feasible set is

$$C = \{x = (x_1, \ldots, x_m \in \Re^m\} : -2 \leq x_i \leq 5, i = 1, 2, \ldots, m\}.$$

It is obvious that A is monotone and Lipschitz continuous. For experiments, q is equal to zero vector, all the entries of N, S are generated randomly and uniformly in $[-2, 2]$, and the diagonal entries of D are in $(0, 2)$.

We can easily see that the solution of the algorithm in this case is $x^* = 0$. In order to illustrate the effectiveness of the algorithm, we show the behavior of D_n when execution time elapses(in second) by Figs. 2, 3, 4 in $\Re^{20}$, $\Re^{50}$, $\Re^{100}$ respectively.

According to Figs. 2, 3, and 4, we have confirmed that the proposed algorithm have the competitive advantages over the existing Algorithm 1 [19].

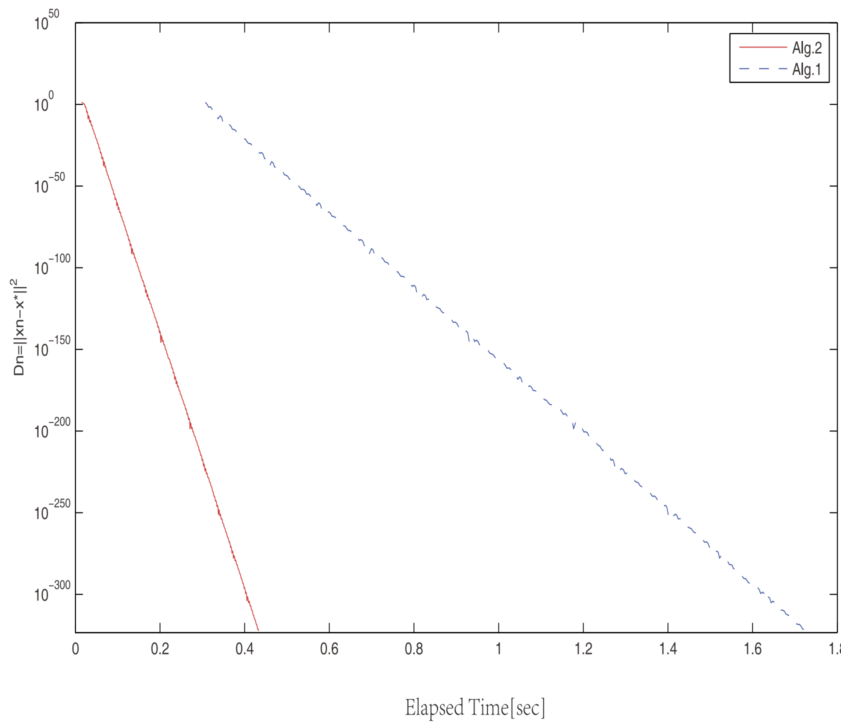

Figure 2 Experiment with $m = 20$ for Example 3

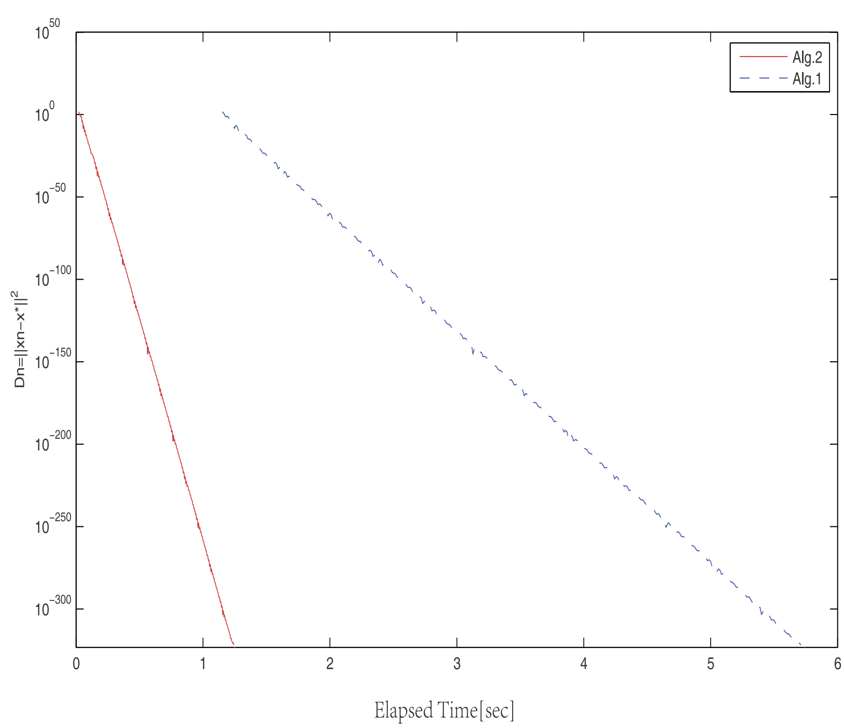

Figure 3 Experiment with $m = 50$ for Example 3

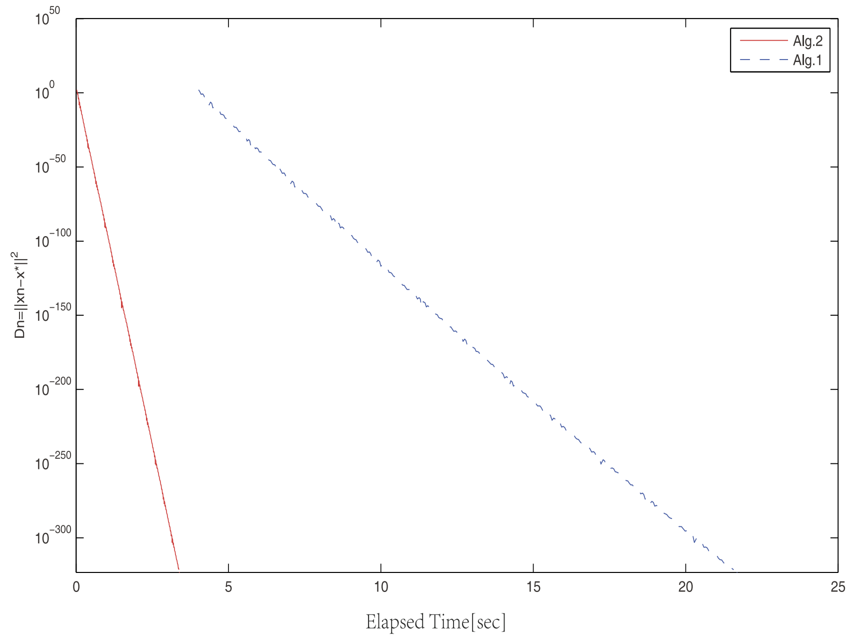

Figure 4 Experiment with $m = 100$ for Example 3

5 Conclusion

In this paper, we introduce a new algorithm with self-adaptive method for finding a solution of the variational inequality problem involving monotone operator and the fixed point problem of a quasi-nonexpansive mapping with a demiclosedness property in a real Hilbert space. We combine a subgradient extragradient method and inertial modification for the algorithm. Under some suitable conditions, we have proved the weak convergence of the algorithm. In particular, it is worth emphasizing that the algorithm that we propose does not need any additional projections of the Lipschitz constant. Finally, some numerical experiments are performed to verify the convergence of the algorithm and compared with previously known Algorithm 1 [19].

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