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On the degenerate (h,q)-Changhee numbers and polynomials

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Abstract

The Changhee numbers and polynomials are introduced by Kim, Kim and Seo (Adv. Stud. Theor. Phys. 7(20):993–1003, 2013), and the generalizations of those polynomials are characterized. In this paper, we investigate a new q-analog of the higher order degenerate Changhee polynomials and numbers. We derive some new interesting identities related to the degenerate (h,q)-Changhee polynomials and numbers.

Keywords: (h,q)-Euler polynomials; Degenerate (h,q)-Changhee polynomials; Fermionic p-adic q-integral on $\mathbb{Z}_p$

1 Introduction

For a fixed odd prime number p, we make use of the following notation. $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is defined $|p|_p = p^{-1}$ (see [14, 15, 17, 19, 30]).

When one says q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$, $|x|_p \le 1$.

The q-analog of number x is defined as

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \to \mathbb{R} \text{ is uniformly differentiable}\}$. For $f \in UD(\mathbb{Z}_p)$, the *fermionic p-adic q-integral on $\mathbb{Z}_p$* is defined by Kim as follows (see [9, 14, 15, 17, 19, 20]):

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1}$$

If we put f_1 to the translation of f with $f_1(x) = f(x+1)$, then, by (1), we get

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0) \quad \text{(see [1, 17–20, 23, 24, 28, 31])}. \tag{2}$$

As is well known, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$\left(e^t-1\right)^m = m!\sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!} \quad \text{(see [3–5, 11])}. \tag{4}$$

By (3), we have

$$\left(\log(1+x)\right)^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!} \quad (n\geq 0). \tag{5}$$

The unsigned Stirling numbers of the first kind are given by

$$x^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n}\left|S_1(n,l)\right|x^l \quad \text{(see [3, 5])}. \tag{6}$$

Note that if we replace x to $-x$ in (3), then

$$\begin{aligned}(-x)_n = (-1)^n x^{(n)} &= \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l \\ &= (-1)^n\sum_{l=0}^{n}\left|S_1(n,l)\right|x^l \end{aligned} \tag{7}$$

(see [3, 5, 28, 31]). Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

In [16], Kim firstly constructed the new (h,q)-extension of the Bernoulli numbers and polynomials with the aid of q-Volkenborn integration, and Simsek gave the Witt-formula for (h,q)-Bernoulli numbers in [27, 34]. Ozden and Simsek defined (h,q)-extension of Euler numbers and polynomials withe the aid of fermionic integral of the function $f(x) = q^{hx}e^{xt}$ in [29], and found recurrence identities for (h,q)-Euler polynomials and the alternating sums of powers of consecutive (h,q)-integers in [35]. In Chapter 6 of [27], the author discusses several generalizations of Bernoulli numbers and associated polynomials with interpolation at negative integers.

Kim et al. introduced the *Changhee polynomials of the first kind of order r*, defined by the generating function to be

$$\left(\frac{2}{2+t}\right)^r(1+t)^x = \sum_{n=0}^{\infty}\mathrm{Ch}_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [12, 13])}, \tag{8}$$

and Moon et al. defined the *q-Changhee polynomials of order r* as follows:

$$\left(\frac{1+q}{q(1+t)+1}\right)^r(1+t)^x = \sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [28, 31])}.$$

By (2), we note that

$$\left(\frac{1+q}{q(1+t)+1}\right)^r(1+t)^x = \int_{\mathbb{Z}_p}(1+t)^{x+y}\,d\mu_{-q}(y),$$

and thus we see that $\sum_{n=0}^{\infty}\mathrm{Ch}_n^{(r)}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}(1+t)^{x+y}\,d\mu_{-q}(y)$.

In [31], the authors defined the generalization of the q-Changhee polynomials which are called by (h,q)-*Changhee polynomials of the first kind* and (h,q)-*Changhee polynomials of the second kind*, respectively, defined by the fermionic p-adic q-integral on $\mathbb{Z}_p$ to be

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Ch}_{n,h,q}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^h y(1+t)^{x+y}\,d\mu_{-q}(y)\\ &= \frac{1+q}{q^{h+1}(1+t)+1}(1+t)^x,\end{aligned} \tag{9}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,h,q}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{hy}(1+t)^{-x-y}\,d\mu_{-q}(y)\\ &= \frac{1+q}{q^{h+1}+1+t}(1+t)^{r-x}.\end{aligned} \tag{10}$$

As is well known, the *Euler polynomials* are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \quad \text{(see [2, 7, 8, 17, 19, 20, 33, 36, 37])}.$$

In [4], Carlitz first introduced the concept of degenerate numbers and polynomials which are related to Euler polynomials as follows:

$$\sum_{n=0}^{\infty}E_n(x|\lambda)\frac{t^n}{n!} = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}, \tag{11}$$

where $\lambda \in \mathbb{R}$. Note that, by (11), we see that

$$\begin{aligned}\lim_{\lambda\to 0}\sum_{n=0}^{\infty}E_n(x|\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\\ &= \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\end{aligned}$$

and thus we get

$$\lim_{\lambda\to 0}E_n(x|\lambda) = E_n(x).$$

In the recent years, the degenerate of some special polynomials are investigated by many authors (see [4, 10, 11, 21–24, 26]). In particular, the *degenerate Changhee polynomials* which are defined by the generating function to be

$$\sum_{n=0}^{\infty}\mathrm{Ch}_{n,\lambda}^{*}(x)\frac{t^n}{n!} = \frac{2\lambda}{2\lambda+\log(1+\lambda t)}\left(1+\log(1+\lambda)^{\frac{1}{\lambda t}}\right)^x \quad \text{(see [26])}, \tag{12}$$

and Kim et al. defined the *degenerate q-Changhee polynomials* as follows:

$$\sum_{n=0}^{\infty} \mathrm{Ch}_{n,\lambda,q}(x)\frac{t^n}{n!} = \frac{q\lambda + \lambda}{q\log(1+\lambda t) + q\lambda + \lambda}\big(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\big)^x \quad \text{(see [22, 24])}. \tag{13}$$

In the past decade, many researchers have investigated the various generalization of Changhee polynomials (see [1, 6, 12, 13, 24–26, 28, 31]), and in [1, 31], the authors gave new q-analog of Changhee numbers and polynomials.

In this paper, we introduce a new q-analog of degenerate Changhee numbers and polynomials of the first kind and the second kind of order r, and derive some new interesting identities related to the degenerate q-Changhee polynomials of order r.

2 q-Analog of degenerate Changhee polynomials

Let assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t| < p^{-\frac{1}{p-1}}$. By (2), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} q^{hy}\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)^{x+y} d\mu_{-q}(y) \\ &\quad = \frac{1+q}{q^{h+1}(1+\frac{1}{\lambda}\log(1+\lambda t)) + 1}\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)^x, \end{aligned} \tag{14}$$

where $h \in \mathbb{Z}$. By (14), we define the *q-analog of degenerate Changhee polynomials* by the generating function to be

$$\sum_{n=0}^{\infty} \mathrm{Ch}_{n,h,q}(x|\lambda)\frac{t^n}{n!} = \frac{1+q}{q^{h+1}(1+\frac{1}{\lambda}\log(1+\lambda t)) + 1}\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)^x. \tag{15}$$

In the special case $x = 0$, $\mathrm{Ch}_{n,h,q}(\lambda) = \mathrm{Ch}_{n,h,q}(0|\lambda)$ are called the *q-analog of degenerate Changhee numbers*.

Note that

$$\begin{aligned} \lim_{\lambda\to 0}\sum_{n=0}^{\infty} \mathrm{Ch}_{n,h,q}(x|\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{1+q}{q^{h+1}(1+\frac{1}{\lambda}\log(1+\lambda t)) + 1}\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)^x \\ &= \frac{q+1}{q^{h+1}(1+t)+1}(1+t)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_{n,h,q}(x)\frac{t^n}{n!}, \end{aligned}$$

and so we see that

$$\lim_{\lambda\to 0} \mathrm{Ch}_{n,h,q}(x|\lambda) = \mathrm{Ch}_{n,h,q}(x), \tag{16}$$

and, if we put $h = 0$, then

$$\lim_{\lambda\to 0} \mathrm{Ch}^{(0)}_{n,q}(x|\lambda) = \mathrm{Ch}_{n,q}(x) \quad \text{and} \quad \mathrm{Ch}_{n,0,q}(x|\lambda) = \mathrm{Ch}_{n,\lambda,q}(x). \tag{17}$$

By (16) and (17), we see that q-analog of degenerate Changhee polynomials are closely related to the q-Changhee polynomials and degenerate q-Changhee polynomials.

By using (7) and (14), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{hy}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x+y} d\mu_{-q}(y)\\
&= \int_{\mathbb{Z}_p} q^{hy}\sum_{n=0}^{\infty}\binom{x+y}{n}\lambda^{-n}\bigl(\log(1+\lambda t)\bigr)^n \, d\mu_{-q}(y)\\
&= \int_{\mathbb{Z}_p} q^{hy}\sum_{n=0}^{\infty}\binom{x+y}{n}\lambda^{-n} n! \sum_{l=n}^{\infty} S_1(l,n)\frac{(\lambda t)^l}{l!}\, d\mu_{-q}(y)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\lambda^{n-m} m! S_1(n,m)\int_{\mathbb{Z}_p} q^{hy}\binom{x+y}{m}\, d\mu_{-q}(y)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m} S_1(n,m)\int_{\mathbb{Z}_p} q^{hy}(x+y)_m\, d\mu_{-q}(y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{18}$$

By (14) and (18), we have

$$\mathrm{Ch}_{n,h,q}(x|\lambda) = \sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p} q^{hy}(x+y)_m\, d\mu_{-q}(y). \tag{19}$$

By (3), we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{hy}(x+y)_m\, d\mu_{-q}(y) &= \sum_{l=0}^{m} S_1(m,l)\int_{\mathbb{Z}_p} q^{hy}(x+y)^l\, d\mu_{-q}(y)\\
&= \sum_{l=0}^{m} S_1(m,l)E_l(x|h,q),
\end{aligned} \tag{20}$$

where $E_n(x|h,q)$ is the nth (h,q)-*Euler polynomials* which are defined by the generating function to be

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n(x|h,q)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{hy}e^{t(x+y)}\, d\mu_{-q}(y)\\
&= \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{hy}(x+y)^n\, d\mu_{-q}(y)\right)\frac{t^n}{n!}\\
&= \frac{q+1}{q^{h+1}e^t+1}e^{xt} \quad \text{(see [32])}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\frac{q+1}{q^{h+1}(1+\frac{1}{\lambda}\log(1+\lambda t))+1}\\
&= (q+1)\sum_{m=0}^{\infty}(-1)^m q^{m(h+1)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^m\\
&= (q+1)\sum_{m=0}^{\infty}(1-)^m q^{m(h+1)}\sum_{l=0}^{m}\lambda^{-l}\binom{m}{l}\bigl(\log(1+\lambda t)\bigr)^l
\end{aligned}$$

$$= (q+1)\sum_{m=0}^{\infty} q^{m(h+1)}(-1)^m \sum_{l=0}^{m} \lambda^{-l}\binom{m}{l} l! \sum_{r=l}^{\infty} S_1(r,l)\lambda^r \frac{t^r}{r!}$$

$$= (q+1)\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} q^{(n+l)(h+1)}(-1)^{n+l}\binom{n+l}{l}\sum_{r=0}^{\infty}\lambda^r S_1(r+l,l)\frac{t^{r+l}}{(r+l)!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{l=0}^{n} q^{(m+l)(h+1)}(-1)^{m+l}\binom{m+l}{l}\lambda^{n-l}S_1(n,l)\right)\frac{t^n}{n!}. \tag{21}$$

By (19), (20) and (21), we obtain the following theorem.

Theorem 2.1 *For each nonnegative integer n, we have*

$$\begin{aligned}\mathrm{Ch}_{n,h,q}(x|\lambda) &= \sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p} q^{hy}(x+y)_m\,d\mu_{-q}(y)\\ &= \sum_{m=0}^{n}\sum_{l=0}^{m}\lambda^{n-m}S_1(n,m)S_1(m,l)E_l(x|h,q)\end{aligned}$$

and

$$\mathrm{Ch}_{n,h,q}(\lambda) = \sum_{m=0}^{\infty}\sum_{l=0}^{n} q^{(m+l)(h+1)}(-1)^{m+l}\binom{m+l}{l}\lambda^{n-l}S_1(n,l).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (15) and by using (4), we have

$$\begin{aligned}\frac{q+1}{q^{h+1}(1+t)+1}(1+t)^x &= \sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(h)}(x|\lambda)\frac{1}{n!}\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^n\\ &= \sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(h)}(x|\lambda)\lambda^{-n}\frac{(e^t-1)^n}{\lambda}\\ &= \left(\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(h)}(x|\lambda)\lambda^{-n}\frac{1}{n!}\right)\left(n!\sum_{l=n}^{\infty}S_2(l,n)\frac{(\lambda t)^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}\,\mathrm{Ch}_{m,q}^{(h)}(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{22}$$

and, thus, by (9) and (22), we have the following corollary.

Corollary 2.2 *For each nonnegative integer n, we have*

$$\mathrm{Ch}_{n,h,q}(x) = \sum_{m=0}^{n}\lambda^{n-m}S_2(n,m)\,\mathrm{Ch}_{m,h,q}(x|\lambda).$$

From (1) and (14), we note that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(q^{h+1}\,\mathrm{Ch}_{n,h,q}(x+1|\lambda)+\mathrm{Ch}_{n,h,q}(x|\lambda)\right)\frac{t^n}{n!}\\ &= \frac{(q+1)}{q^{h+1}(1+\frac{1}{\lambda}\log(1+\lambda t))+1}\left(q^{h+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x\end{aligned}$$

$$= (q+1)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x}$$
$$= (q+1)\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(x)_m\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}. \tag{23}$$

By (23), we obtain the following theorem.

Theorem 2.3 *For each nonnegative integer n, we get*

$$q^{h+1}\operatorname{Ch}_{n,h,q}(x+1|\lambda)+\operatorname{Ch}_{n,h,q}(x|\lambda)=(q+1)\sum_{m=0}^{n}(x)_m\lambda^{n-m}S_1(n,m).$$

For positive integer d with $d\equiv 1 \pmod 2$, if we put $f(x)=q^{hx}(1+\frac{1}{\lambda}\log(1+\lambda t))^x$, then, by (2) and (7), we have

$$q^d\int_{\mathbb{Z}_p}q^{h(x+d)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x+d}d\mu_{-q}(x)+\int_{\mathbb{Z}_p}q^{hx}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x}d\mu_{-q}(x)$$
$$= (q+1)\sum_{l=0}^{d-1}(-1)^l q^{l(h+1)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{l}$$
$$= (q+1)\sum_{l=0}^{d-1}(-1)^l q^{l(h+1)}\sum_{k=0}^{\infty}\binom{l}{k}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^{k}$$
$$= (q+1)\sum_{l=0}^{d-1}(-1)^l q^{l(h+1)}\sum_{k=0}^{\infty}\binom{l}{k}\lambda^{-k}\sum_{r=k}^{\infty}S_1(r,k)\frac{t^r}{r!}$$
$$= \sum_{n=0}^{\infty}\left((q+1)\sum_{l=0}^{d-1}\sum_{k=0}^{n}(-1)^l q^{l(h+1)}\binom{l}{k}\lambda^{-k}S_1(n,k)\right)\frac{t^n}{n!} \tag{24}$$

and

$$q^d\int_{\mathbb{Z}_p}q^{h(x+d)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x+d}d\mu_{-q}(x)$$
$$+\int_{\mathbb{Z}_p}q^{hx}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x}d\mu_{-q}(x)$$
$$= \sum_{n=0}^{\infty}\left(q^{d(h+1)}\operatorname{Ch}_{n,q}^{(h)}(d|\lambda)+\operatorname{Ch}_{n,q}^{(h)}(\lambda)\right)\frac{t^n}{n!}. \tag{25}$$

By (24) and (25), we obtain the following theorem.

Theorem 2.4 *For each nonnegative integer n and odd integer d, we have*

$$q^{d(h+1)}\operatorname{Ch}_{n,h,q}(d|\lambda)+\operatorname{Ch}_{n,h,q}(\lambda)=(q+1)\sum_{l=0}^{d-1}\sum_{k=0}^{n}(-1)^l q^{l(h+1)}\binom{l}{k}\lambda^{-k}S_1(n,k).$$

3 *q*-Analog of higher order degenerate Changhee polynomials

In this section, we consider the *q-analog of higher order degenerate Changhee polynomials* which are defined by

$$\begin{aligned}
&\mathrm{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \\
&\quad = \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} q^{\sum_{i=1}^{r} h_i y_i} (x + y_1 \\
&\qquad + \cdots + y_r)_m \, d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r),
\end{aligned} \tag{26}$$

where n is a nonnegative integer, $h_1, \ldots, h_r \in \mathbb{Z}$ and $r \in \mathbb{N}$.

By (26), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i} (x + y_1 + \cdots + y_r)_m \, d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \right) \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda^{n-m} m! S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i} \\
&\qquad \times \binom{x + y_1 + \cdots + y_r}{m} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \frac{t^n}{n!} \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i} \sum_{n=0}^{\infty} \binom{x + y_1 + \cdots + y_r}{n} \lambda^{-n} n! \\
&\qquad \times \sum_{l=n}^{\infty} S_1(l,n) \frac{(\lambda t)^l}{l!} \, d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i} \sum_{n=0}^{\infty} \binom{x + y_1 + \cdots + y_r}{n} \lambda^{-n} \bigl(\log(1+\lambda t)\bigr)^n \, d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i} \left(1 + \frac{1}{\lambda} \log(1+\lambda t) \right)^{x + y_1 + \cdots + y_r} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \\
&\quad = \prod_{i=1}^{r} \left(\frac{1+q}{q^{h_i+1}(1 + \frac{1}{\lambda}\log(1+\lambda t)) + 1} \right) \left(1 + \frac{1}{\lambda} \log(1+\lambda t) \right)^x.
\end{aligned} \tag{27}$$

By (26) and (27), we see that

$$\sum_{n=0}^{\infty} \mathrm{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \frac{t^n}{n!} = \prod_{i=1}^{r} \left(\frac{1+q}{q^{h_i+1}(1 + \frac{1}{\lambda}\log(1+\lambda t)) + 1} \right) \left(1 + \frac{1}{\lambda} \log(1+\lambda t) \right)^x. \tag{28}$$

If we put

$$F_q^{(h_1,\ldots,h_r)}(x,t) = \prod_{i=1}^{r} \left(\frac{1+q}{q^{h_i+1}(1 + \frac{1}{\lambda}\log(1+\lambda t)) + 1} \right) \left(1 + \frac{1}{\lambda} \log(1+\lambda t) \right)^x,$$

then

$$F_q^{(0,\dots,0)}(x,t) = \left(\frac{[2]_q}{[2]_q + \frac{q}{\lambda}\log(1+\lambda t)}\right)^r \left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_{n,\lambda,q}^{(r)}(x)\frac{t^n}{n!}$$

and

$$\lim_{q\to 1} F_q^{(-1,\dots,-1)}(x,t) = \left(\frac{2}{\frac{1}{\lambda}\log(1+\lambda t)+2}\right)^r \left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.$$

Thus, $F_q^{(h_1,\dots,h_r)}(x,t)$ seems to be a new q-extension of the generating function for the degenerate Changhee polynomials of order r.

Note that

$$\begin{aligned}
&\prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}(1+\frac{1}{\lambda}\log(1+\lambda t))+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x \\
&= \left(\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(h_1,\dots,h_r)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{x}{m}\lambda^{n-m}m!S_1(n,m)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{x}{l}\binom{n}{m}\lambda^{n-m-l}l!\,\mathrm{Ch}_{m,q}^{(h_1,\dots,h_r)}(\lambda)S_1(n-m,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{29}$$

Since

$$\begin{aligned}
&(x+y_1+\cdots+y_r)_n \\
&= \sum_{l=0}^{n} S_1(n,l)(x+y_1+\cdots+y_r)^l \\
&= \sum_{l=0}^{n} S_1(n,l)\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\dots,l_r\ge 0}}\binom{n}{l_1,l_2,\dots,l_r}y_1^{l_1}y_2^{l_2}\cdots(x+y_r)^{l_r},
\end{aligned} \tag{30}$$

where $\binom{n}{l_1,l_2,\dots,l_r} = \frac{n!}{l_1!l_2!\cdots l_r!}$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_i}(x+y_1+\cdots+y_r)_m\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_i}\sum_{l=0}^{m}S_1(m,l)(x+y_1+\cdots+y_r)^l\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r}h_iy_i}\sum_{l=0}^{m}S_1(m,l)
\end{aligned}$$

$$\times \sum_{\substack{l_1+\cdots+l_r=l \\ l_1,\ldots,l_r\geq 0}} \binom{m}{l_1,l_2,\ldots,l_r} y_1^{l_1} y_2^{l_2} \cdots (x+y_r)^{l_r}\, d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)$$

$$= \sum_{l=0}^{m} S_1(m,l) \sum_{\substack{l_1+\cdots+l_r=l \\ l_1,\ldots,l_r\geq 0}} \binom{m}{l_1,l_2,\ldots,l_r} E_{l_1}(h_1,q)\cdots E_{l_{r-1}}(h_{r-1},q)E_{l_r,q}(x|h_r,q), \tag{31}$$

where $E_n(h,q) = E_n(0|h,q)$, which are called the (h,q)-*Euler numbers*.

Thus, by (26) and (31), we obtain the following theorem.

Theorem 3.1 *For $n \geq 0$, we have*

$$\begin{aligned} &\mathrm{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \\ &\quad = \sum_{m=0}^{n}\sum_{l=0}^{m} \sum_{\substack{l_1+\cdots+l_r=l \\ l_1,\ldots,l_r\geq 0}} \lambda^{n-m} S_1(n,m)S_1(m,l)\binom{m}{l_1,l_2,\ldots,l_r} \\ &\qquad \times E_{l_1}(h_1,q)\cdots E_{l_{r-1}}(h_{r-1},q)E_{l_r,q}(x|h_r,q). \end{aligned}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (28),

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathrm{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{1}{n!}\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^n \\ &\quad = \sum_{n=0}^{\infty} \mathrm{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{1}{n!}\lambda^{-n} n! \sum_{l=n}^{\infty} S_2(l,n)\frac{(\lambda t)^l}{l!} \\ &\quad = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \mathrm{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!} \end{aligned} \tag{32}$$

and, by (9),

$$\begin{aligned} &\prod_{i=1}^{r}\left(\frac{q+1}{q^{h_i+1}(1+t)+1}\right)(1+t)^x \\ &\quad = \left(\prod_{i=1}^{r-1}\left(\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(h_i)}\frac{t^n}{n!}\right)\right)\left(\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(h_r)}(x)\frac{t^n}{n!}\right) \\ &\quad = \sum_{n=0}^{\infty}\sum_{\substack{l_1+\cdots+l_r=n \\ l_1,\ldots,l_r\geq 0}}\binom{n}{l_1,\ldots,l_r}\mathrm{Ch}_{l_1,q}^{(h_1)}\cdots\mathrm{Ch}_{l_{r-1},q}^{(h_{r-1})}\mathrm{Ch}_{l_r,q}^{(h_r)}(x)\frac{t^n}{n!}. \end{aligned} \tag{33}$$

Thus, by (32) and (33), we obtain the following theorem.

Theorem 3.2 *For $n \geq 0$, we have*

$$\sum_{\substack{l_1+\cdots+l_r=n \\ l_1,\ldots,l_r\geq 0}}\binom{n}{l_1,\ldots,l_r}\mathrm{Ch}_{l_1,q}^{(h_1)}\cdots\mathrm{Ch}_{l_{r-1},q}^{(h_{r-1})}\mathrm{Ch}_{l_r,q}^{(h_r)}(x) = \sum_{m=0}^{n}\mathrm{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)\lambda^{n-m}S_2(n,m).$$

4 *q*-Analog of higher order degenerate Changhee polynomials of the second kind

In this section, we consider the *q-analog of higher order degenerate Changhee polynomials of the second kind* is defined as follows:

$$\begin{aligned}
&\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\\
&\quad=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}q^{\sum_{i=1}^{r}h_iy_i}(-x-y_1\\
&\qquad-\cdots-y_r)_m\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r),
\end{aligned}\tag{34}$$

where n is a nonnegative integer. In particular, $\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(0|\lambda)=\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(\lambda)$ are called the *q-analog of higher order degenerate Changhee numbers of the second kind*.

By (7) and (34), it leads to

$$\begin{aligned}
&\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\\
&\quad=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}(-x-y_1-\cdots-y_r)_m\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&\quad=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}(-1)^m\\
&\qquad\times(x+y_1+\cdots+y_r)^{(m)}\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&\quad=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}(-1)^m\\
&\qquad\times\sum_{l=0}^{m}\bigl|S_1(m,l)\bigr|(x+y_1+\cdots+y_r)^l\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&\quad=\sum_{m=0}^{n}\sum_{l=0}^{m}\lambda^{n-m}S_1(n,m)\bigl|S_1(m,l)\bigr|(-1)^m\\
&\qquad\times\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}\binom{l}{l_1,\ldots,l_r}\prod_{i=1}^{r-1}\int_{\mathbb{Z}_p}q^{h_iy_i}y_i^{l_i}\,d\mu_{-q}(y_i)\int_{\mathbb{Z}_p}q^{h_ry_r}(x+y_r)^{l_r}\,d\mu_{-q}(y_r)\\
&\quad=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}\binom{l}{l_1,\ldots,l_r}\lambda^{n-m}(-1)^l\\
&\qquad\times S_1(n,m)S_1(m,l)\left(\prod_{i=1}^{r-1}E_{l_i}^{(h_i)}(q)\right)E_{l_r}^{(h_r)}(x|q).
\end{aligned}\tag{35}$$

Thus, we state the following theorem.

Theorem 4.1 *For $n \geq 0$, we have*

$$\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}\binom{l}{l_1,\ldots,l_r}\lambda^{n-m}(-1)^l S_1(n,m)S_1(m,l) \times\left(\prod_{i=1}^{r-1}E_{l_i}(h_i,q)\right)E_{l_r}(x|h_r,q).$$

Now, we consider the generating function of the q-analog of higher order degenerate Changhee polynomials of the second kind as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_1}(-x-y_1\\
&\quad -\cdots-y_r)_n\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\frac{t^n}{n!}\\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\sum_{n=0}^{\infty}\binom{-x-y_1-\cdots-y_r}{n}\lambda^{-n}n!\\
&\quad\times\sum_{l=n}^{\infty}S_1(l,n)\frac{(\lambda t)^l}{l!}\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\sum_{n=0}^{\infty}\binom{-x-y_1-\cdots-y_r}{n}\\
&\quad\times\lambda^{-n}\bigl(\log(1+\lambda t)\bigr)^n\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{-x-y_1-\cdots-y_r}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&= \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}+1+\frac{1}{\lambda}\log(1+\lambda t)}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{r-x}.
\end{aligned}\tag{36}$$

In the special case $r = 1$,

$$\sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,h,q}(x|\lambda)\frac{t^n}{n!} = \frac{1+q}{q^{h+1}+1+\frac{1}{\lambda}\log(1+\lambda t)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{1-x}.\tag{37}$$

$\widehat{\mathrm{Ch}}_{n,q}^{(h)}(x|\lambda) = \widehat{\mathrm{Ch}}_{n,q}(x|h,\lambda)$ are called the *q-analog of degenerate Changhee polynomials of the second kind*.

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (36), we have

$$\begin{aligned}
&\left(\prod_{i=1}^{r}\frac{[2]_q}{q^{h_i+1}+(1+t)}\right)(1+t)^{r-x}\\
&= \sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{(\frac{1}{\lambda}(e^{\lambda t}-1))^n}{n!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{1}{n!}\lambda^{-n}n!\sum_{l=n}^{\infty} S_2(l,n)\frac{(\lambda t)^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \widehat{\mathrm{Ch}}_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!},
\end{aligned}
\tag{38}
$$

and, by (10), we get

$$
\begin{aligned}
&\left(\prod_{i=1}^{r}\frac{[2]_q}{q^{h_i+1}+(1+t)}\right)(1+t)^{r-x} \\
&= \left(\prod_{i=1}^{r-1}\left(\sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,q}^{(h_i)}\frac{t^n}{n!}\right)\right)\left(\sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,q}^{(h_r)}(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\ldots,i_r\geq 0}}\binom{n}{i_1,\ldots,i_r}\widehat{\mathrm{Ch}}_{i_1,q}^{(h_1)}\cdots\widehat{\mathrm{Ch}}_{i_{r-1},q}^{(h_{r-1})}\widehat{\mathrm{Ch}}_{i_r,q}^{(h_r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{39}
$$

By (38) and (39), we obtain the following theorem.

Theorem 4.2 *For $n \geq 0$, we have*

$$
\begin{aligned}
&\sum_{m=0}^{n}\widehat{\mathrm{Ch}}_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)\lambda^{n-m}S_2(n,m) \\
&= \sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\ldots,i_r\geq 0}}\binom{n}{i_1,\ldots,i_r}\widehat{\mathrm{Ch}}_{i_1,q}^{(h_1)}\cdots\widehat{\mathrm{Ch}}_{i_{r-1},q}^{(h_{r-1})}\widehat{\mathrm{Ch}}_{i_r,q}^{(h_r)}(x).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{t^n}{n!} \\
&= \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}+1+\frac{1}{\lambda}\log(1+\lambda t)}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{r-x} \\
&= \left(\prod_{i=1}^{r-1}\frac{(1+q)q^{-h_i-1}}{q^{-h_i-1}(1+\frac{1}{\lambda}\log(1+\lambda t))+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{r-1} \\
&\quad\times\left(\frac{(1+q)q^{-h_r-1}}{q^{-h_r-1}(1+\frac{1}{\lambda}\log(1+\lambda t))+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{1-x} \\
&= \left(\prod_{i=1}^{r-1}q^{-h_i-1}\right)\left(\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}^{(-h_1-2,\ldots,-h_{r-1}-2)}(r-1|\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\widehat{\mathrm{Ch}}_{n,-h_r-2,q}(x|\lambda)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\left(\prod_{i=1}^{r-1}q^{-h_i-1}\right)\sum_{m=0}^{\infty}\binom{n}{m}\mathrm{Ch}_{n-m,q}^{(-h_1-2,\ldots,-h_{r-1}-2)}(r-1|\lambda)\widehat{\mathrm{Ch}}_{m,-h_r-2,q}(x|\lambda)\right)\frac{t^n}{n!},
\end{aligned}
$$

and thus we see that

$$\widehat{\mathrm{Ch}}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) = \left(\prod_{i=1}^{r-1} q^{-h_i-1}\right)\sum_{m=0}^{\infty}\binom{n}{m}\mathrm{Ch}_{n-m,q}^{(-h_1-2,\ldots,-h_{r-1}-2)}(r-1|\lambda)\widehat{\mathrm{Ch}}_{m,-h_r-2,q}(x|\lambda).$$

5 Conclusion

The Changhee polynomials were defined by Kim, and have been attempted the various generalizations by many researchers (see [1, 6, 12, 13, 24–26, 28, 31]). The Changhee numbers (q-Changhee numbers, respectively) are closely relate with the Euler numbers (q-Euler numbers), the Stirling numbers of the first kind and second kind and the harmonic numbers, etc. which are interesting numbers of combinatorics, and pure and applied mathematics.

In this paper, we defined two types of the degenerate (h,q)-Changhee polynomials and number, and found the relationship between the Stirling numbers of the first kind and second kind, (h,q)-Euler numbers, q-Changhee numbers and those polynomials and numbers. It is a further problem to find the relationship between some special polynomials and degenerate (h,q)-Changhee polynomials.

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