Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Common fixed point theorems for rational $F_{\mathcal{R}}$-contractive pairs of mappings with applications

Mian Bahadur Zada[1] and Muhammad Sarwar[1*]

*Correspondence:
sarwarswati@gmail.com
[1]Department of Mathematics, University of Malakand, Chakdara Dir(L), Pakistan

Abstract

In this paper, we study the existence of solution for the following non-linear matrix equations:

$$X = Q + \sum_{i=1}^{n} A_i^* X A_i - \sum_{i=1}^{n} B_i^* X B_i,$$

$$X = Q + \sum_{i=1}^{n} A_i^* \Upsilon(X) A_i,$$

where Q is a Hermitian positive definite matrix, A_i, B_i are arbitrary $m \times m$ matrices and $\Upsilon : \mathcal{H}(m) \to \mathcal{P}(m)$ is an order preserving continuous map such that $\Upsilon(0) = 0$. To this aim, we establish several common fixed point theorems for two mapping satisfying a rational $F_{\mathcal{R}}$-contractive condition, where $\mathcal{R}$ is a binary relation.

MSC: 47H09; 54H25

Keywords: Fixed points; Binary relation; $F_{\mathfrak{R}}$-contraction; Non-linear matrix equations

1 Introduction

Non-linear matrix equations play an important role in several problems of engineering and applied mathematics. Various matrix equations are encountered in stability analysis [1], control theory [2, 3] and system theory [4–6]. To test the existence of solution to non-linear matrix equations, we can have a number of advanced methods. One of these methods is to use the tools of fixed point theory. Using fixed point results, many researchers checked the existence and uniqueness of solution of non-linear matrix equations [7–10].

An important result in fixed point theory, commonly known in the literature as the Banach principle, has been established by Banach [11]. This principle has been improved and generalized by several researchers for different kinds of contractions in various spaces. Wardowski [12] presented the concept of F-contraction and demonstrated fixed point theorems for this new type of contractions. Several authors generalized Wardowski's theorems by extending the concept of F-contraction. Recently, Sawangsup *et al.* [9] introduced the concept of $F_{\mathcal{R}}$-contraction and established fixed point results for such type of contractions.

Throughout this work we use the following notation:

$\mathcal{M}(m)$ = set of $m \times m$ complex matrices,

$\mathcal{H}(m)$ = set of $m \times m$ Hermitian matrices,

$\mathcal{P}(m)$ = set of $m \times m$ positive definite matrices,

$\mathcal{H}^+(m)$ = set of $m \times m$ positive semi-definite matrices.

Here $\mathcal{P}(m) \subseteq \mathcal{H}(m) \subseteq \mathcal{M}(m)$, $\mathcal{H}^+(m) \subseteq \mathcal{H}(m)$, $\Omega_1 \succ 0$ and $\Omega_1 \succeq 0$ means that $\Omega_1 \in \mathcal{P}(m)$ and $\Omega_1 \in \mathcal{H}^+(m)$, respectively; for $\Omega_1 - \Omega_2 \succeq 0$ and $\Omega_1 - \Omega_2 \succ 0$ we will use $\Omega_1 \succeq \Omega_2$ and $\Omega_1 \succ \Omega_2$, respectively. Moreover, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The main concern of this paper is to study the following non-linear matrix equations:

$$X = Q + \sum_{i=1}^{n} A_i^* X A_i - \sum_{i=1}^{n} B_i^* X B_i, \tag{1.1}$$

$$X = Q + \sum_{i=1}^{n} A_i^* \Upsilon(X) A_i, \tag{1.2}$$

where $Q \in \mathcal{P}(m)$, A_i, B_i are arbitrary $m \times m$ matrices and $\Upsilon : \mathcal{H}(m) \to \mathcal{P}(m)$ is a continuous order preserving map such that $\Upsilon(0) = 0$. The matrix equations (1.1) often occur in dynamic programming [13, 14], control theory [15, 16], ladder networks [17, 18], etc.

Berzig [7] used coupled fixed point results to prove the existence of the unique positive definite solutions of (1.1). Recently, Sawangsup *et al.* [9] established fixed point theorems for $F_{\mathcal{R}}$-contractions and proved the existence and uniqueness of a positive definite solution of the matrix equation (1.2).

The intention of this work is to introduce the concept of rational $F_{\mathcal{R}}$-contractive pair of mappings, under an arbitrary binary relation $\mathcal{R}$ and using this concept we prove fixed point results. By means of these results, we prove in the last section existence results for positive definite solutions of the two classes of non-linear matrix equations (1.1) and (1.2).

2 Preliminaries

In this section we recall some basic notions.

Definition 2.1 Let $\mathbb{F}$ be the class of all functions $\mathsf{f} : [0, \infty[\to \mathbb{R}$ satisfying the following properties:

(1) f is strictly increasing;

(2) for every sequence $\{s_n\}_{n\in\mathbb{N}}$ with $s_n > 0$, we have

$$\lim_{n\to\infty} s_n = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \mathsf{f}(s_n) = -\infty;$$

(3) there is $j \in]0, 1[$ such that $\lim_{s\to 0^+} s^j \mathsf{f}(s) = 0$.

Definition 2.2 ([19]) Let $\mathbb{X}$ be a non-empty set and $\mathcal{R}$ be a binary relation on $\mathbb{X}$. Then $\mathcal{R}$ is transitive if $(\gamma_2, \gamma_1) \in \mathcal{R}$ and $(\gamma_1, \gamma_3) \in \mathcal{R}$ implies that $(\gamma_2, \gamma_3) \in \mathcal{R}$, for all $\gamma_2, \gamma_1, \gamma_3 \in \mathbb{X}$.

Definition 2.3 ([20, 21]) Let $\mathbb{X}$ be a non-empty set and $\Phi : \mathbb{X} \to \mathbb{X}$. Then a binary relation $\mathcal{R}$ on $\mathbb{X}$ is called Φ-closed (equivalently Φ is $\mathcal{R}$-non-decreasing) if for any $\gamma_1, \gamma_2 \in \mathbb{X}$, we have

$$(\gamma_1, \gamma_2) \in \mathcal{R} \quad \Longrightarrow \quad (\Phi\gamma_1, \Phi\gamma_2) \in \mathcal{R}.$$

Definition 2.4 ([21]) Let $\gamma_1, \gamma_2 \in \mathbb{X}$ and $\mathcal{R}$ be a binary relation on a non-empty set $\mathbb{X}$. A path (of length $n \in \mathbb{N}$) in $\mathcal{R}$ from γ_1 to γ_2 is a sequence $\{t_0, t_1, t_2, \ldots, t_n\} \subseteq \mathbb{X}$ such that

(i) $t_0 = \gamma_1$ and $t_n = \gamma_2$;

(ii) $(t_j, t_{j+1}) \in \mathcal{R}$ for all $j \in \{0, 1, 2, \ldots, n-1\}$.

Note that $\Gamma(\gamma_1, \gamma_2, \mathcal{R})$ represents the class of all paths from γ_1 to γ_2 in $\mathcal{R}$.

Notice that a path of length n involves $n+1$ elements of $\mathbb{X}$, although they are not necessarily distinct.

Definition 2.5 ([22]) A metric space (M, d) equipped with a binary relation $\mathcal{R}$ is $\mathcal{R}$-non-decreasing-regular if for all sequences $\{\kappa_n\}$ in M,

$$\left.\begin{array}{ll} (\kappa_n, \kappa_{n+1}) \in \mathcal{R}, & \forall n \in \mathbb{N}, \\ \kappa_n \to \kappa \in M, & \end{array}\right\} \quad \Longrightarrow \quad (\kappa_n, \kappa) \in \mathcal{R}, \forall n \in \mathbb{N}.$$

Definition 2.6 ([9]) Let (M, d) be a metric space, $\mathcal{R}$ be a binary relation M and $\Psi : M \to M$ be a mapping. Let

$$\mathcal{W} = \big\{(\kappa_1, \kappa_2) \in \mathcal{R} : d(\Psi\kappa_1, \Psi\kappa_2) > 0\big\}.$$

Then Ψ is said to be an $\mathsf{F}_{\mathcal{R}}$-contraction if there exist $\xi > 0$ and $\mathsf{F} \in \mathbb{F}$ such that

$$\xi + \mathsf{F}\big(d(\Psi\kappa_1, \Psi\kappa_2)\big) \leq \mathsf{F}\big(d(\kappa_1, \kappa_2)\big), \quad \text{for all } (\kappa_1, \kappa_2) \in \mathcal{W}. \tag{2.1}$$

3 Main results

First we modify Definition 2.6 for two maps as follows.

Definition 3.1 Let Φ, Ψ be two self-mappings and $\mathcal{R}$ be a binary relation on a non-empty set $\mathbb{X}$. Then $\mathcal{R}$ is (Φ, Ψ)-closed if for each $a_1, a_2 \in \mathbb{X}$, we have

$$(a_1, a_2) \in \mathcal{R} \quad \Longrightarrow \quad (\Phi a_1, \Psi a_2), (\Psi a_1, \Phi a_2) \in \mathcal{R}.$$

Definition 3.2 Let (M, d) be a metric space, Φ, Ψ be self-mappings of M and $\mathcal{R}$ be a binary relation on M. Let

$$\mathcal{X} = \big\{(\kappa_1, \kappa_2) \in \mathcal{R} : d(\Phi\kappa_1, \Psi\kappa_2) > 0\big\}.$$

We say that (Φ, Ψ) is a rational $\mathsf{F}_{\mathcal{R}}$-contractive pair of mappings if there exist $\xi > 0$ and $\mathsf{F} \in \mathbb{F}$ such that

$$\begin{aligned} &\xi + \mathsf{F}\big(d(\Phi\kappa_1, \Psi\kappa_2)\big) \\ &\quad \leq \mathsf{F}\left(d(\kappa_1, \kappa_2) + \frac{d(\kappa_2, \Phi\kappa_1) d(\kappa_1, \Psi\kappa_2)}{1 + d(\kappa_1, \kappa_2)}\right), \quad \text{for all } (\kappa_1, \kappa_2) \in \mathcal{X}. \end{aligned} \tag{3.1}$$

Denote by $M((\Phi, \Psi); \mathcal{R})$ the set of all order pairs $(\kappa_1, \kappa_2) \in M \times M$ such that $(\Phi\kappa_1, \Psi\kappa_2) \in \mathcal{R}$.

Theorem 3.3 *Let (M,d) be a complete metric space, $\mathcal{R}$ be a binary relation on M and $\Phi,\Psi : M \to M$. Suppose that the following conditions hold:*

(C_1) *$M((\Phi,\Psi);\mathcal{R})$ is non-empty;*

(C_2) *$\mathcal{R}$ is (Φ,Ψ)-closed;*

(C_3) *Φ and Ψ are continuous;*

(C_4) *the pair (Φ,Ψ) is rational $\mathsf{F}_{\mathcal{R}}$-contractive.*

Then there is a common fixed point of Φ and Ψ.

Proof Let (κ_0,κ_1) be any element of $M((\Phi,\Psi);\mathcal{R})$, then $(\Phi\kappa_0,\Psi\kappa_1) \in \mathcal{R}$. Define the sequence $\{\kappa_n\}$ in M by

$$\left.\begin{aligned} \kappa_{2n+1} &= \Phi\kappa_{2n}, \\ \kappa_{2n+2} &= \Psi\kappa_{2n+1}, \end{aligned}\right\} \tag{3.2}$$

where $n \in \mathbb{N}_0$.

If $\kappa_{2n^*} = \kappa_{2n^*+1}$ for some $n^* \in \mathbb{N}_0$, then κ_{2n^*} is a common fixed point of Φ and Ψ. If $\kappa_{2n} \neq \kappa_{2n+1}$, for all $n \in \mathbb{N}_0$. Then $d(\Phi\kappa_{2n},\Psi\kappa_{2n+1}) > 0$, for all $n \in \mathbb{N}_0$ and using assumption (C_2), we obtain

$$\begin{aligned} (\kappa_1,\kappa_2) &= (\Phi\kappa_0,\Psi\kappa_1) \in \mathcal{R}, \\ (\kappa_2,\kappa_3) &= (\Psi\kappa_1,\Phi\kappa_2) \in \mathcal{R}, \\ (\kappa_3,\kappa_4) &= (\Phi\kappa_2,\Psi\kappa_3) \in \mathcal{R}, \\ (\kappa_4,\kappa_5) &= (\Psi\kappa_3,\Phi\kappa_4) \in \mathcal{R}, \\ &\vdots \end{aligned}$$

In general,

$$(\kappa_{2n},\kappa_{2n+1}) = (\Psi\kappa_{2n-1},\Phi\kappa_{2n}) \in \mathcal{R}.$$

Thus $(\kappa_{2n},\kappa_{2n+1}) \in \mathcal{X}$, for all $n \in \mathbb{N}_0$. Now, taking in (3.1) $\kappa_1 = \kappa_{2n}$ and $\kappa_2 = \kappa_{2n-1}$, we have

$$\begin{aligned} \mathsf{F}\bigl(d(\kappa_{2n},\kappa_{2n+1})\bigr) &= \mathsf{F}\bigl(d(\kappa_{2n+1},\kappa_{2n})\bigr) \\ &= \mathsf{F}\bigl(d(\Phi\kappa_{2n},\Psi\kappa_{2n-1})\bigr) \\ &\le \mathsf{F}\left(d(\kappa_{2n},\kappa_{2n-1}) + \frac{d(\kappa_{2n-1},\Phi\kappa_{2n})d(\kappa_{2n},\Psi\kappa_{2n-1})}{1+d(\kappa_{2n},\kappa_{2n-1})}\right) - \xi \\ &= \mathsf{F}\left(d(\kappa_{2n},\kappa_{2n-1}) + \frac{d(\kappa_{2n-1},\kappa_{2n+1})d(\kappa_{2n},\kappa_{2n})}{1+d(\kappa_{2n},\kappa_{2n-1})}\right) - \xi \\ &= \mathsf{F}\bigl(d(\kappa_{2n},\kappa_{2n-1}))\bigr) - \xi, \end{aligned}$$

for all $n \in \mathbb{N}$. Similarly, setting $\kappa_1 = \kappa_{2n}$ and $\kappa_2 = \kappa_{2n+1}$ in (3.1), we can write

$$\begin{aligned} \mathsf{F}\bigl(d(\kappa_{2n+1},\kappa_{2n+2})\bigr) &= \mathsf{F}\bigl(d(\Phi\kappa_{2n},\Psi\kappa_{2n+1})\bigr) \\ &\le \mathsf{F}\left(d(\kappa_{2n},\kappa_{2n+1}) + \frac{d(\kappa_{2n+1},\Phi\kappa_{2n})d(\kappa_{2n},\Psi\kappa_{2n+1})}{1+d(\kappa_{2n},\kappa_{2n+1})}\right) - \xi \end{aligned}$$

$$= \mathsf{F}\left(d(\kappa_{2n},\kappa_{2n+1}) + \frac{d(\kappa_{2n+1},\kappa_{2n+1})d(\kappa_{2n},\kappa_{2n+2})}{1+d(\kappa_{2n},\kappa_{2n+1})}\right) - \xi$$
$$= \mathsf{F}\big(d(\kappa_{2n},\kappa_{2n+1})\big)\big) - \xi.$$

In general,

$$\mathsf{F}\big(d(\kappa_n,\kappa_{n+1})\big) \le \mathsf{F}\big(d(\kappa_{n-1},\kappa_n)\big)\big) - \xi, \tag{3.3}$$

where $n \in \mathbb{N}$. Now, using inequality (3.3), we can write

$$\begin{aligned}\mathsf{F}\big(d(\kappa_n,\kappa_{n+1})\big) &\le \mathsf{F}\big(d(\kappa_{n-1},\kappa_n)\big) - \xi \\ &\le \mathsf{F}\big(d(\kappa_{n-2},\kappa_{n-1})\big) - 2\xi \\ &\le \mathsf{F}\big(d(\kappa_{n-3},\kappa_{n-2})\big) - 3\xi \\ &\le \mathsf{F}\big(d(\kappa_{n-4},\kappa_{n-3})\big) - 4\xi \\ &\vdots \\ &\le \mathsf{F}\big(d(\kappa_0,\kappa_1)\big) - n\xi,\end{aligned}$$

that is,

$$\mathsf{F}\big(d(\kappa_n,\kappa_{n+1})\big) \le \mathsf{F}\big(d(\kappa_0,\kappa_1)\big) - n\xi, \tag{3.4}$$

where $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} \mathsf{F}(d(\kappa_n,\kappa_{n+1})) = -\infty$, by condition (2) of Definition 2.1, we get

$$\lim_{n\to\infty} d(\kappa_n,\kappa_{n+1}) = 0 \quad \text{or} \quad \lim_{n\to\infty} d(\kappa_n,\kappa_{n+1}) = 0^+. \tag{3.5}$$

From condition (3) of Definition 2.1, we can find $\varepsilon \in \,]0,1[$ such that

$$\lim_{n\to\infty} \big(d(\kappa_n,\kappa_{n+1})\big)^{\varepsilon} \mathsf{F}\big(d(\kappa_n,\kappa_{n+1})\big) = 0. \tag{3.6}$$

Using (3.4), we have

$$\big(d(\kappa_n,\kappa_{n+1})\big)^{\varepsilon}\big(\mathsf{F}\big(d(\kappa_n,\kappa_{n+1})\big) - \mathsf{F}\big(d(\kappa_0,\kappa_1)\big)\big) \le -\big(d(\kappa_n,\kappa_{n+1})\big)^{\varepsilon} n\xi \le 0. \tag{3.7}$$

Taking the limit $n \to \infty$ in (3.7), and using (3.5) and (3.6), we get

$$\lim_{n\to\infty} n\big(d(\kappa_n,\kappa_{n+1})\big)^{\varepsilon} = 0. \tag{3.8}$$

Hence there exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$, $n(d(\kappa_n,\kappa_{n+1}))^{\varepsilon} \le 1$. Consequently, we have

$$d(\kappa_n,\kappa_{n+1}) \le \frac{1}{n^{\frac{1}{\varepsilon}}}, \quad \forall n \ge n_0. \tag{3.9}$$

Now, we show that $\{\kappa_n\}$ is a Cauchy sequence. For this purpose, using (3.9) and the triangular inequality, for all $m > n \geq n_1$, we have

$$\begin{aligned} d(\kappa_n,\kappa_m) &\leq d(\kappa_n,\kappa_{n+1}) + d(\kappa_{n+1},\kappa_{n+2}) + d(\kappa_{n+2},\kappa_{n+3}) + \cdots + d(\kappa_{m-1},\kappa_m) \\ &\leq \frac{1}{n^{\frac{1}{\varepsilon}}} + \frac{1}{(n+1)^{\frac{1}{\varepsilon}}} + \frac{1}{(n+2)^{\frac{1}{\varepsilon}}} + \cdots + \frac{1}{(m-1)^{\frac{1}{\varepsilon}}} \\ &= \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{\varepsilon}}}. \end{aligned}$$

Since $d(\kappa_n,\kappa_m) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{\varepsilon}}} < \infty$, the sequence $\{\kappa_n\}$ is Cauchy in M. Due to completeness of M, one can find $t \in M$ such that $\kappa_n \to t$ as $n \to \infty$.

Next, we show that $\Phi t = \Psi t = t$. Since Φ and Ψ are continuous and $\kappa_{2n}, \kappa_{2n-1} \to t$,

$$\kappa_{2n+1} = \Phi\kappa_{2n} \to \Phi t \quad \text{and} \quad \kappa_{2n} = \Psi\kappa_{2n-1} \to \Psi t.$$

Due to the limit uniqueness, we obtain $\Phi t = t$ and $\Psi t = t$, which implies that $\Phi t = \Psi t = t$ and hence there is a common fixed point of Φ and Ψ. □

The next result ensures the uniqueness of the common fixed point in Theorem 3.3.

Theorem 3.4 *Let $\mathcal{R}$ be a transitive relation and Φ, Ψ be the self-mappings on a complete metric space M. Assume that the following conditions hold*:

(C_0) *for all $(\kappa_1,\kappa_2) \in \mathcal{X}$, there exists $\mathsf{F} \in \mathbb{F}$ such that*

$$\xi + \mathsf{F}\big(d(\Phi\kappa_1,\Psi\kappa_2)\big) \leq \mathsf{F}\left(\frac{1}{2}d(\kappa_1,\kappa_2) + \frac{d(\kappa_2,\Phi\kappa_1)d(\kappa_1,\Psi\kappa_2)}{2[1+d(\kappa_1,\kappa_2)]}\right), \tag{3.10}$$

where $\xi > 0$;

(C_1) *$M((\Phi,\Psi);\mathcal{R})$ and $\Gamma(\kappa_1,\kappa_2,\mathcal{R})$ are non-empty*;

(C_2) *$\mathcal{R}$ is (Φ,Ψ)-closed*;

(C_3) *Φ and Ψ are continuous.*

Then there is a unique common fixed point of Φ and Ψ.

Proof Following the same steps as in the proof of Theorem 3.3, one can easily prove that there is a common fixed point of Φ and Ψ. Thus we have to show that there is a unique common fixed point of Φ and Ψ. For this purpose, assume that λ and λ^* are two distinct common fixed points of Φ and Ψ. Then since $\Gamma(\lambda,\lambda^*,\mathcal{R})$ is the class of paths in $\mathcal{R}$ from λ to λ^*, there is a path of finite length l, i.e. there is a sequence $\{z_0,z_1,z_2,\ldots,z_l\}$ in $\mathcal{R}$ from λ to λ^* with

$$z_0 = \lambda, \qquad z_l = \lambda^*, \qquad (z_j,z_{j+1}) \in \mathcal{R}, \quad \text{for every } j = 0,1,2,\ldots,(l-1).$$

But since $\mathcal{R}$ is transitive, we have

$$(\lambda,z_1) \in \mathcal{R}, (z_1,z_2) \in \mathcal{R}, \ldots, \big(z_{k-1},\lambda^*\big) \in \mathcal{R} \quad \Longrightarrow \quad \big(\lambda,\lambda^*\big) \in \mathcal{R}.$$

Now, setting $\lambda = \lambda$ and $\lambda^* = \lambda^*$ in contraction condition (3.10), we have

$$\begin{aligned}
\xi + \mathsf{F}\big(d(\Phi\lambda, \Psi\lambda^*)\big) &\leq \mathsf{F}\left(\frac{1}{2}d(\lambda,\lambda^*) + \frac{d(\lambda^*,\Phi\lambda)d(\lambda,\Psi\lambda^*)}{2[1+d(\lambda,\lambda^*)]}\right), \\
\xi + \mathsf{F}\big(d(\lambda,\lambda^*)\big) &\leq \mathsf{F}\left(\frac{1}{2}d(\lambda,\lambda^*) + \frac{d(\lambda^*,\lambda)d(\lambda,\lambda^*)}{2[1+d(\lambda,\lambda^*)]}\right) \\
&< \mathsf{F}\left(\frac{1}{2}d(\lambda,\lambda^*) + \frac{1}{2}d(\lambda^*,\lambda)\right) \\
&= \mathsf{F}\big(d(\lambda,\lambda^*)\big),
\end{aligned}$$

which is a contradiction. Thus $\lambda = \lambda^*$ and hence λ is the unique common fixed point of Φ and Ψ. □

Taking $\Phi = \Psi$ in Theorems 3.3 and 3.4, we get the following corollaries.

Corollary 3.5 *Let $\mathcal{R}$ be a binary relation and Ψ be the self-mappings on a complete metric space M. Assume that the following conditions hold*:

(C_0) *for all $(\kappa_1,\kappa_2) \in \mathcal{X}$, there exists $\mathsf{F} \in \mathbb{F}$ such that*

$$\xi + \mathsf{F}\big(d(\Psi\kappa_1,\Psi\kappa_2)\big) \leq \mathsf{F}\left(d(\kappa_1,\kappa_2) + \frac{d(\kappa_2,\Psi\kappa_1)d(\kappa_1,\Psi\kappa_2)}{1+d(\kappa_1,\kappa_2)}\right), \tag{3.11}$$

where $\xi > 0$;

(C_1) *$M(\Psi;\mathcal{R})$ is non-empty*;

(C_2) *$\mathcal{R}$ is Ψ-closed*;

(C_3) *Ψ is continuous.*

Then there is a fixed point of Ψ.

Corollary 3.6 *Let $\mathcal{R}$ be a transitive relation and Ψ be the self-mappings on a complete metric space M. Assume that the following conditions hold*:

(C_0) *for all $(\kappa_1,\kappa_2) \in \mathcal{X}$, there exists $\mathsf{F} \in \mathbb{F}$ such that*

$$\xi + \mathsf{F}\big(d(\Psi\kappa_1,\Psi\kappa_2)\big) \leq \mathsf{F}\left(\frac{1}{2}d(\kappa_1,\kappa_2) + \frac{d(\kappa_2,\Psi\kappa_1)d(\kappa_1,\Psi\kappa_2)}{2[1+d(\kappa_1,\kappa_2)]}\right), \tag{3.12}$$

where $\xi > 0$;

(C_1) *$M(\Psi;\mathcal{R})$ and $\Gamma(\kappa_1,\kappa_2,\mathcal{R})$ are non-empty*;

(C_2) *$\mathcal{R}$ is Ψ-closed*;

(C_3) *Ψ is continuous.*

Then there is a unique fixed point of Ψ.

To avoid the continuity of Φ and Ψ in Theorem 3.3, we present the following result.

Theorem 3.7 *Theorem 3.3 remains true if instead of condition (C_3), we assume that (M,d) is $\mathcal{R}$-non-decreasing regular.*

Proof In the proof of Theorem 3.3, we have seen that $(\kappa_n,\kappa_{n+1}) \in \mathcal{R}$ and $\kappa_n \to \gamma$ as $n \to \infty$, $\forall n \in \mathbb{N}$. Then since (M,d) is $\mathcal{R}$-non-decreasing regular, so $(\kappa_n,\gamma) \in \mathcal{R}$ for every $n \in \mathbb{N}$. Here we discuss two cases which depends on $\mathcal{M} = \{n \in \mathbb{N} : \Phi\kappa_{2n} = \Psi\gamma \text{ and } \Psi\kappa_{2n+1} = \Phi\gamma\}$.

Case (I): If $\mathcal{M}$ finite, there exist $n_0 \in \mathbb{N}$ and $\Phi\kappa_{2n} \neq \Psi\gamma$ and $\Psi\kappa_{2n+1} \neq \Phi\gamma$, for all $n \geq n_0$. Now since $\kappa_{2n} \neq \gamma$ and $\kappa_{2n+1} \neq \gamma$ implies that $d(\kappa_{2n},\gamma) > 0$, $d(\kappa_{2n+1},\gamma) > 0$ and $d(\Phi\kappa_{2n},\Psi\gamma) > 0$ and $d(\Psi\kappa_{2n+1},\Phi\gamma) > 0$, for all $n \geq n_0$.

Now, setting $\kappa_1 = \gamma$ and $\kappa_2 = \kappa_{2n+1}$ in the contractive condition (3.1), we have

$$\xi + \mathsf{F}\bigl(d(\Phi\gamma,\Psi\kappa_{2n+1})\bigr) \leq \mathsf{F}\left(d(\gamma,\kappa_{2n+1}) + \frac{d(\kappa_{2n+1},\Phi\gamma)d(\gamma,\Psi\kappa_{2n+1})}{1 + d(\gamma,\kappa_{2n+1})}\right)$$

$$\implies \quad \xi + \mathsf{F}\bigl(d(\Phi\gamma,\kappa_{2n+2})\bigr) \leq \mathsf{F}\left(d(\gamma,\kappa_{2n+1}) + \frac{d(\kappa_{2n+1},\Phi\gamma)d(\gamma,\kappa_{2n+2})}{1 + d(\gamma,\kappa_{2n+1})}\right).$$

But $\{\kappa_n\} = \{d(\gamma,\kappa_{2n+1}) + \frac{d(\kappa_{2n+1},\Phi\gamma)d(\gamma,\kappa_{2n+2})}{1+d(\gamma,\kappa_{2n+1})}\}$ is a sequence of positive terms with $\lim_{n\to\infty} \kappa_n = 0$, so by condition (2) of Definition 2.1, $\mathsf{F}(\kappa_n) \to -\infty$ implies that $\mathsf{F}(d(\Phi\gamma,\kappa_{2n+2})) \to -\infty$, again by condition (2) of Definition 2.1, $d(\Phi\gamma,\kappa_{2n+2}) \to 0$, that is, $\kappa_{2n+2} \to \Phi\gamma$ as $n \to \infty$. Also $\kappa_{2n+2} \to \gamma$ as $n \to \infty$, so by the uniqueness of the limit

$$\Phi\gamma = \gamma, \tag{3.13}$$

and hence γ is the fixed point of Φ.

Similarly, setting $\kappa_1 = \kappa_{2n}$ and $\kappa_2 = \gamma$ in contractive condition (3.1), we can easily show that $\mathsf{F}(d(\kappa_{2n+1},\Psi\gamma)) \to -\infty$. By condition (2) of Definition 2.1, $d(\kappa_{2n+1},\Psi\gamma) \to 0$, that is, $\kappa_{2n+1} \to \Psi\gamma$ as $n \to \infty$. Also $\kappa_{2n+1} \to \gamma$ as $n \to \infty$, so by the uniqueness of the limit

$$\Psi\gamma = \gamma, \tag{3.14}$$

and hence γ is the fixed point of Ψ.

From Eqs. (3.13) and (3.14), we get

$$\Phi\gamma = \Psi\gamma = \gamma. \tag{3.15}$$

Thus γ is a common fixed point of Φ and Ψ.

Case (II): If $\mathcal{M}$ is infinite, there exists a subsequence $\{\kappa_{2n(j)}\}$ of $\{\kappa_n\}$ with $\kappa_{2n(j)+1} = \Phi\kappa_{2n(j)} = \Psi\gamma$ such that $\kappa_{2n(j)+2} = \Psi\kappa_{2n(j)+1} = \Phi\gamma$ for all $j \in \mathbb{N}$. But $\kappa_{2n(j)+1}, \kappa_{2n(j)+2} \to \gamma$, so by the uniqueness of the limit $\Phi\gamma = \gamma$ and $\Psi\gamma = \gamma$ and hence γ is a common fixed point of Φ and Ψ.

In both cases, γ is a common fixed point of Φ and Ψ. □

Theorem 3.8 *Theorem 3.4 remains true if, instead of condition* (C_3), *we assume that* (M,d) *is* $\mathcal{R}$*-non-decreasing regular.*

Taking $\Phi = \Psi$ in Theorems 3.7 and 3.8, we get the following corollaries.

Corollary 3.9 *Let* $\mathcal{R}$ *be a binary relation and* Ψ *be the self-mappings on a complete metric space* M. *Assume that the following conditions hold*:

(C_0) *for all* $(\kappa_1,\kappa_2) \in \mathcal{X}$, *there exists* $\mathsf{F} \in \mathbb{F}$ *such that*

$$\xi + \mathsf{F}\bigl(d(\Psi\kappa_1,\Psi\kappa_2)\bigr) \leq \mathsf{F}\left(d(\kappa_1,\kappa_2) + \frac{d(\kappa_2,\Psi\kappa_1)d(\kappa_1,\Psi\kappa_2)}{1 + d(\kappa_1,\kappa_2)}\right), \tag{3.16}$$

where $\xi > 0$;

(C_1) *$M(\Psi;\mathcal{R})$ is non-empty;*
(C_2) *$\mathcal{R}$ is Ψ-closed;*
(C_3) *M is $\mathcal{R}$-non-decreasing regular.*
Then there is a fixed point of Ψ.

Corollary 3.10 *Let $\mathcal{R}$ be a transitive relation and Ψ be the self-mappings on a complete metric space M. Assume that the following conditions hold:*
(C_0) *for all $(\kappa_1,\kappa_2)\in\mathcal{X}$, there exists $\mathsf{F}\in\mathbb{F}$ such that*

$$\xi + \mathsf{F}\big(d(\Psi\kappa_1,\Psi\kappa_2)\big) \leq \mathsf{F}\left(\frac{1}{2}d(\kappa_1,\kappa_2) + \frac{d(\kappa_2,\Psi\kappa_1)d(\kappa_1,\Psi\kappa_2)}{2[1+d(\kappa_1,\kappa_2)]}\right), \tag{3.17}$$

where $\xi > 0$;
(C_1) *$M(\Psi;\mathcal{R})$ and $\Gamma(\kappa_1,\kappa_2,\mathcal{R})$ are non-empty;*
(C_2) *$\mathcal{R}$ is Ψ-closed;*
(C_3) *M is $\mathcal{R}$-non-decreasing-regular.*
Then there is a unique fixed point of Ψ.

4 Applications

In this section, by using the previous theorems, we obtain existence results for the solutions of the matrix equations (1.1) and (1.2). We use the metric which is induced by the norm $\|\aleph\|_{\mathrm{tr}} = \sum_{i=1}^{n}\theta_i(\aleph)$, where $\theta_i(\aleph)$, $i = 1,2,\dots,n$, are the singular values of $\aleph\in\mathcal{M}(m)$. The set $\mathcal{H}(m)$ equipped with the trace norm $\|\cdot\|_{\mathrm{tr}}$ is a complete metric space (see [7, 8, 23]) and partially ordered with partial ordering $\preceq$, where $\aleph_1\preceq\aleph_2 \Longleftrightarrow \aleph_2\succeq\aleph_1$. Also, for every $\aleph_1,\aleph_2\in\mathcal{H}(m)$ there is a glb and a lub (see [8]).

To establish the existence results we need the following lemmas.

Lemma 4.1 ([8]) *If $\aleph_1,\aleph_2\succeq O$ are $m\times m$ matrices, then*

$$0 \leq \mathrm{tr}(\aleph_1\aleph_2) \leq \|\aleph_2\|\,\mathrm{tr}(\aleph_1).$$

Lemma 4.2 ([24]) *If $\aleph\in\mathcal{H}(m)$ with $\aleph\prec I_n$, then $\|\aleph\| < 1$.*

Define the operator $\Psi:\mathcal{H}(m)\to\mathcal{H}(m)$ by

$$\Psi(\mathcal{X}) = \frac{1}{2}\big(\Psi_1(\mathcal{X}) + \Psi_2(\mathcal{X})\big),$$

where the operators $\Psi_1,\Psi_2:\mathcal{H}(m)\to\mathcal{H}(m)$ are given by

$$\Psi_1(\mathcal{X}) = Q + 2\sum_{i=1}^{n}A_i^*\mathcal{X}A_i$$

and

$$\Psi_2(\mathcal{X}) = Q - 2\sum_{i=1}^{n}B_i^*\mathcal{X}B_i.$$

Note that the solutions of the matrix equation (1.1) are the fixed points of the operator Ψ and the fixed points of the operator Ψ are the common fixed points of operators Ψ_1 and Ψ_2.

Theorem 4.3 *The class of non-linear matrix equations* (1.1) *has a solution under the following conditions*:

1. *there are two positive real numbers* M_1 *and* M_2 *such that* $\sum_{i=1}^{n} A_iA_i^* \prec M_1 I_n$ *and* $\sum_{i=1}^{n} B_iB_i^* \prec M_2 I_n$;
2. *for every* $\aleph_1, \aleph_2 \in \mathcal{H}(m)$ *such that* $(\aleph_1, \aleph_2) \in \preceq$, *we have*

$$\begin{aligned}&\|\aleph_1\|_{tr} + \|\aleph_2\|_{tr}\\ &\quad\le \bigl(\|\aleph_1 - \aleph_2\|_{tr}\bigl(1 + \|\aleph_1 - \aleph_2\|_{tr}\bigr) + \bigl\|\aleph_2 - \Psi_1(\aleph_1)\bigr\|_{tr}\bigl\|\aleph_1 - \Psi_2(\aleph_2)\bigr\|_{tr}\bigr)\\ &\qquad\Big/\Bigl(2M\Bigl(\xi\sqrt{\|\aleph_1 - \aleph_2\|_{tr}\bigl(1 + \|\aleph_1 - \aleph_2\|_{tr}\bigr) + \bigl\|\aleph_2 - \Psi_1(\aleph_1)\bigr\|_{tr}\bigl\|\aleph_1 - \Psi_2(\aleph_2)\bigr\|_{tr}}\\ &\qquad + \sqrt{1 + \|\aleph_1 - \aleph_2\|_{tr}}\Bigr)^2\Bigr),\end{aligned}$$

where $M = \max\{M_1, M_2\}$ *and* ξ *is positive real number.*

Proof Since Ψ_1 and Ψ_2 are well defined and $(\aleph_1, \aleph_2) \in \preceq$ implies that $(\Psi_1(\aleph_1), \Psi_2(\aleph_2))$, $(\Psi_2(\aleph_1), \Psi_1(\aleph_2)) \in \preceq$, so that $\preceq$ on $\mathcal{H}(m)$ is (Ψ_1, Ψ_2)-closed.

We have to show that the operators Ψ_1 and Ψ_2 satisfy the rational type $\mathsf{F}_{\preceq}$-contractive conditions. For this purpose, let us consider

$$\begin{aligned}\bigl\|\Psi_1(\aleph_1) - \Psi_2(\aleph_2)\bigr\|_{tr} &= \operatorname{tr}\bigl(\Psi_1(\aleph_1) - \Psi_2(\aleph_2)\bigr)\\ &= 2\operatorname{tr}\left(\sum_{i=1}^{n}\bigl(A_i^*\aleph_1 A_i + B_i^*\aleph_2 B_i\bigr)\right)\\ &= 2\sum_{i=1}^{n}\operatorname{tr}\bigl(A_i^*\aleph_1 A_i + B_i^*\aleph_2 B_i\bigr)\\ &= 2\left(\sum_{i=1}^{n}\operatorname{tr}\bigl(A_iA_i^*\aleph_1\bigr) + \sum_{i=1}^{n}\operatorname{tr}\bigl(B_iB_i^*\aleph_2\bigr)\right)\\ &= 2\left(\operatorname{tr}\left(\sum_{i=1}^{n}A_iA_i^*\aleph_1\right) + \operatorname{tr}\left(\sum_{i=1}^{n}B_iB_i^*\aleph_2\right)\right)\\ &\le 2\left(\left\|\sum_{i=1}^{n}A_iA_i^*\right\|\|\aleph_1\|_{tr} + \left\|\sum_{i=1}^{n}B_iB_i^*\right\|\|\aleph_2\|_{tr}\right)\\ &\le 2\bigl(M_1\|\aleph_1\|_{tr} + M_2\|\aleph_2\|_{tr}\bigr)\\ &\le 2M\bigl(\|\aleph_1\|_{tr} + \|\aleph_2\|_{tr}\bigr).\end{aligned}$$

From conditions (1) and (2) of Theorem 4.3 it follows that

$$\begin{aligned}&\bigl\|\Psi_1(\aleph_1) - \Psi_2(\aleph_2)\bigr\|_{tr}\\ &\quad\le \bigl(\|\aleph_1 - \aleph_2\|_{tr} + \|\aleph_1 - \aleph_2\|_{tr}^2 + \bigl\|\aleph_2 - \Psi_1(\aleph_1)\bigr\|_{tr}\bigl\|\aleph_1 - \Psi_2(\aleph_2)\bigr\|_{tr}\bigr)\end{aligned}$$

$$
\begin{aligned}
&\Big/\bigg(\bigg(\xi\sqrt{\|\aleph_1-\aleph_2\|_{tr}+\|\aleph_1-\aleph_2\|_{tr}^2+\|\aleph_2-\Psi_1(\aleph_1)\|_{tr}\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}} \\
&+\sqrt{1+\|\aleph_1-\aleph_2\|_{tr}}\bigg)^2\bigg) \\
\Longrightarrow\quad &\bigg(\xi\sqrt{\|\aleph_1-\aleph_2\|_{tr}+\|\aleph_1-\aleph_2\|_{tr}^2+\|\aleph_2-\Psi_1(\aleph_1)\|_{tr}\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}} \\
&\quad+\sqrt{1+\|\aleph_1-\aleph_2\|_{tr}}\bigg) \\
&\quad\Big/\bigg(\sqrt{\|\aleph_1-\aleph_2\|_{tr}+\|\aleph_1-\aleph_2\|_{tr}^2+\|\aleph_2-\Psi_1(\aleph_1)\|_{tr}\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}}\bigg) \\
&\le\frac{1}{\sqrt{\|\Psi_1(\aleph_1)-\Psi_2(\aleph_2)\|_{tr}}} \\
\Longrightarrow\quad &\xi+\frac{\sqrt{1+\|\aleph_1-\aleph_2\|_{tr}}}{\sqrt{\|-\aleph_2\|_{tr}+\|\aleph_1-\aleph_2\|_{tr}^2+\|\aleph_2-\Psi_1()\|\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}}} \\
&\le\frac{1}{\sqrt{\|\Psi_1(\aleph_1)-\Psi_2(\aleph_2)\|_{tr}}} \\
\Longrightarrow\quad &\xi-\frac{1}{\sqrt{\|\Psi_1(\aleph_1)-\Psi_2(\aleph_2)\|_{tr}}} \\
&\le-\frac{1}{\sqrt{\frac{\|\aleph_1-\aleph_2\|_{tr}+\|\aleph_1-\aleph_2\|_{tr}^2+\|\aleph_2-\Psi_1(\aleph_1)\|_{tr}\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}}{1+\|\aleph_1-\aleph_2\|_{tr}}}} \\
&=-\frac{1}{\sqrt{\|\aleph_1-\aleph_2\|_{tr}+\frac{\|\aleph_2-\Psi_1(\aleph_1)\|_{tr}\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}}{1+\|\aleph_1-\aleph_2\|_{tr}}}}.
\end{aligned}
$$

Let $\mathsf{F}:[0,\infty)\to\mathbb{R}$ be the mapping defined by $\mathsf{F}(\kappa_1)=-\frac{1}{\sqrt{\kappa_1}}$. Then $\mathsf{F}\in\mathbb{F}$ and the above inequality becomes

$$\xi+\mathsf{F}\big(\|\Psi_1(\aleph_1)-\Psi_2(\aleph_2)\|_{tr}\big)\le\mathsf{F}\bigg(\|\aleph_1-\aleph_2\|_{tr}+\frac{\|\aleph_2-\Psi_1(\aleph_1)\|_{tr}\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}}{1+\|\aleph_1-\aleph_2\|_{tr}}\bigg).$$

Thus

$$\xi+\mathsf{F}\big(d\big(\Psi_1(\aleph_1),\Psi_2(\aleph_2)\big)\le\mathsf{F}\bigg(d(\aleph_1,\aleph_2)+\frac{d(\aleph_2,\Psi_1(\aleph_1))d(\aleph_1,\Psi_2(\aleph_2))}{1+d(\aleph_1,\aleph_2)}\bigg).$$

That is, the pair (Ψ_1,Ψ_2) is rational $\mathsf{F}_{\mathcal{R}}$-contractive. Thus from Theorem 3.3, there is a common fixed point of Ψ_1 and Ψ_2, say $\aleph^*$, i.e., $\Psi_1(\aleph^*)=\Psi_2(\aleph^*)=\aleph^*$. Consequently, Ψ has a fixed point and hence the class of non-linear matrix equation (1.1) has a solution. □

The next existence result ensures the uniqueness of solution to the non-linear matrix equation (1.1) and the proof is similar to the proof of Theorem 4.3, so we omit it.

Theorem 4.4 *Under the condition* (1) *of Theorem* 4.3, *the class of non-linear matrix equations* (1.1) *has a unique solution if for every* $\aleph_1,\aleph_2\in\mathcal{H}(m)$ *such that* $(\aleph_1,\aleph_2)\in\preceq$, *we have*

$$
\begin{aligned}
&\|\aleph_1\|_{tr}+\|\aleph_2\|_{tr} \\
&\quad\le\big(\|\aleph_1-\aleph_2\|_{tr}\big(1+\|\aleph_1-\aleph_2\|_{tr}\big)+\|\aleph_2-\Psi_1(\aleph_1)\|_{tr}\|\aleph_1-\Psi_2(\aleph_2)\|_{tr}\big)
\end{aligned}
$$

$$/\Big(2M\Big(\xi\sqrt{\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big(1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big)+\big\|\aleph_2-\Psi_1(\aleph_1)\big\|_{\mathrm{tr}}\big\|\aleph_1-\Psi_2(\aleph_2)\big\|_{\mathrm{tr}}}$$
$$+2\sqrt{1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}}\Big)^2\Big),$$

where $M=\max\{M_1,M_2\}$ *and* ξ *is positive real number.*

Define the operator $\Phi:\mathcal{H}(m)\to\mathcal{H}(m)$ by

$$\Phi(\aleph)=Q+\sum_{i=1}^{n}A_i^*\Upsilon(\aleph)A_i.$$

Note that the solutions of the matrix equation (1.2) coincide with the fixed points of the operator $\Phi(\aleph)$.

Theorem 4.5 *The class of non-linear matrix equations* (1.2) *has a solution under the following conditions*:

(1) *there is a real positive real number* M *with* $\sum_{i=1}^n A_iA_i^*\prec MI_n$;

(2) *for every* $\aleph_1,\aleph_2\in\mathcal{H}(m)$ *such that* $(\aleph_1,\aleph_2)\in\preceq$ *and* $\sum_{i=1}^n A_i^*\Upsilon(\aleph_1)A_i\neq\sum_{i=1}^n A_i^*\Upsilon(\aleph_2)A_i$, *we have*

$$\big\|\mathrm{tr}\big(\Upsilon(\aleph_1)-\Upsilon(\aleph_2)\big)\big\|_{\mathrm{tr}}$$
$$\leq\Big(\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big(1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big)+\big\|\aleph_2-\Phi(\aleph_1)\big\|_{\mathrm{tr}}\big\|\aleph_1-\Phi(\aleph_2)\big\|_{\mathrm{tr}}\Big)$$
$$/\Big(M\Big(\xi\sqrt{\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big(1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big)+\big\|\aleph_2-\Phi(\aleph_1)\big\|_{\mathrm{tr}}\big\|\aleph_1-\Phi(\aleph_2)\big\|_{\mathrm{tr}}}$$
$$+\sqrt{1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}}\Big)^2\Big),$$

where ξ *is positive real number.*

Proof Since Φ is well defined and $(\aleph_1,\aleph_2)\in\preceq$ implies that $(\Phi(\aleph_1),\Phi(\aleph_2))\in\preceq$, $\preceq$ on $\mathcal{H}(m)$ is Φ-closed.

We have to show that the operator $\Phi(\aleph_1)$ satisfies the rational type $\mathsf{F}_{\preceq}$-contraction (3.16). Let $\aleph=\{(\aleph_1,\aleph_2)\in\preceq:\Upsilon(\aleph_1)\neq\Upsilon(\aleph_2)\}$. If $\aleph_1,\aleph_2\in\aleph$, then $\aleph_2\prec\aleph_1$. But Υ is an order preserving mapping, so that $\Upsilon(\aleph_2)\prec\Upsilon(\aleph_1)$. Therefore,

$$\begin{aligned}\big\|\Phi(\aleph_1)-\Phi(\aleph_2)\big\|_{\mathrm{tr}}&=\mathrm{tr}\big(\Phi(\aleph_1)-\Phi(\aleph_2)\big)\\&=\mathrm{tr}\left(\sum_{i=1}^{n}A_i^*\big(\Upsilon(\aleph_1)-\Upsilon(\aleph_2)\big)A_i\right)\\&=\sum_{i=1}^{n}\mathrm{tr}\big(A_i^*\big(\Upsilon(\aleph_1)-\Upsilon(\aleph_2)\big)A_i\big)\\&=\sum_{i=1}^{n}\mathrm{tr}\big(A_iA_i^*\big(\Upsilon(\aleph_1)-\Upsilon(\aleph_2)\big)\big)\\&=\mathrm{tr}\left(\left(\sum_{i=1}^{n}A_iA_i^*\right)\big(\Upsilon(\aleph_1)-\Upsilon(\aleph_2)\big)\right)\\&\leq\left\|\sum_{i=1}^{n}A_iA_i^*\right\|\big\|\Upsilon(\aleph_1)-\Upsilon(\aleph_2)\big\|_{\mathrm{tr}},\end{aligned}$$

using condition (2) of Theorem 4.3, we get

$$\begin{aligned}
&\big\|\Phi(\aleph_1)-\Phi(\aleph_2)\big\|_{\mathrm{tr}} \\
&\quad\leq \left(\left(\left\|\sum_{i=1}^{n}A_iA_i^*\right\|\right)\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big(1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big)+\big\|\aleph_2-\Phi(\aleph_1)\big\|_{\mathrm{tr}}\big\|\aleph_1-\Phi(\aleph_2)\big\|_{\mathrm{tr}}\right) \\
&\qquad\Big/\Big(M\Big(\xi\sqrt{\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big(1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}\big)+\big\|\aleph_2-\Phi(\aleph_1)\big\|_{\mathrm{tr}}\big\|\aleph_1-\Phi(\aleph_2)\big\|_{\mathrm{tr}}} \\
&\qquad+\sqrt{1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}}\Big)^2\Big).
\end{aligned}$$

From condition (1) of Theorem 4.3 it follows that

$$\begin{aligned}
&\big\|\Phi(\aleph_1)-\Phi(\aleph_2)\big\|_{\mathrm{tr}} \\
&\quad\leq \big(\|\aleph_1-\aleph_2\|_{\mathrm{tr}}+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}^2+\big\|\aleph_2-\Phi(\aleph_1)\big\|_{\mathrm{tr}}\big\|\aleph_1-\Phi(\aleph_2)\big\|_{\mathrm{tr}}\big) \\
&\qquad\Big/\Big(\Big(\xi\sqrt{\|\aleph_1-\aleph_2\|_{\mathrm{tr}}+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}^2+\big\|\aleph_2-\Phi(\aleph_1)\big\|_{\mathrm{tr}}\big\|\aleph_1-\Phi(\aleph_2)\big\|_{\mathrm{tr}}} \\
&\qquad+\sqrt{1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}}\Big)^2\Big) \\
&\Longrightarrow\quad \xi-\frac{1}{\sqrt{\|\Phi(\aleph_1)-\Phi(\aleph_2)\|_{\mathrm{tr}}}}\leq -\frac{1}{\sqrt{\|\aleph_1-\aleph_2\|_{\mathrm{tr}}+\frac{\|\aleph_2-\Phi(\aleph_1)\|_{\mathrm{tr}}\|\aleph_1-\Phi(\aleph_2)\|_{\mathrm{tr}}}{1+\|\aleph_1-\aleph_2\|_{\mathrm{tr}}}}}.
\end{aligned}$$

Using $\mathsf{F}(\kappa_1)=-\frac{1}{\sqrt{\kappa_1}}\in\mathbb{F}$, the above inequality becomes

$$\xi+\mathsf{F}\big(d\big(\Phi(\aleph_1),\Phi(\aleph_2)\big)\big)\leq \mathsf{F}\left(d(\aleph_1,\aleph_2)+\frac{d(\aleph_2,\Phi(\aleph_1))d(\aleph_1,\Phi(\aleph_2))}{1+d(\aleph_1,\aleph_2)}\right).$$

That is, Φ satisfies a rational type $\mathsf{F}_{\mathcal{R}}$-contraction (3.16). Thus from Corollary 3.9, there is a fixed point of Φ, say $\aleph$, i.e., $\Phi(\aleph)=\aleph$. Consequently, the class of non-linear matrix equations (1.2) has a solution. □

Theorem 4.6 *Under the conditions* (1) *and* (2) *of Theorem* 4.5, *the class of non-linear matrix equations* (1.2) *has a unique solution if* $\mathcal{R}$ *is transitive and* $\mathcal{H}(m)$ *is* $\mathcal{R}$*-non-decreasing-regular.*

Proof Using Corollary 3.10 and proceeding by the same arguments of Theorem 4.3, one can easily obtain a unique solution of the non-linear matrix equations (1.2). □

5 Conclusion

Non-linear matrix equations occur in several problems of engineering and applied mathematics. Some various matrix equations are faced in stability analysis, control theory and system theory. In the current work we obtain a common fixed point theorem via rational type $F_{\mathcal{R}}$-contractive conditions with applications to the existence of solutions to non-linear matrix equations.

Acknowledgements
The authors wish to thank the editor and anonymous referees for their comments and suggestions, which helped to improve this paper. The authors are also grateful to Springer International Publishing for granting full fee waiver.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
All authors contributed equally to the writing of this manuscript. All authors read and approved the final manuscript.

Authors' information
Department of Mathematics, University of Malakand, Chakdara, Dir(L), Khyber Pakhtunkhwa, Pakistan.

Publisher's Note

Received: 7 June 2018 Accepted: 20 December 2018 Published online: 16 January 2019

References

1. Wu, A.G., Fu, Y.M., Duan, G.R.: On solutions of matrix equations $V - AVF = BW$ and $V - A\bar{V}F = BW$. Math. Comput. Model. **47**(11–12), 1181–1197 (2008)
2. Zhou, B., Lam, J., Duan, G.R.: Convergence of gradient-based iterative solution of the coupled Markovian jump Lyapunov equations. Comput. Math. Appl. **56**(12), 3070–3078 (2008)
3. Wu, A.G., Feng, G., Duan, G.R., Liu, W.Q.: Iterative solutions to the Kalman–Yakubovich-conjugate matrix equation. Appl. Math. Comput. **217**(9), 4427–4438 (2011)
4. Zhang, H.M., Ding, F.: A property of the eigenvalues of the symmetric positive definite matrix and the iterative algorithm for coupled Sylvester matrix equations. J. Franklin Inst. B **351**(1), 340–357 (2014)
5. Zhang, H.M.: Reduced-rank gradient-based algorithms for generalized coupled Sylvester matrix equations and its applications. Comput. Math. Appl. **70**(8), 2049–2062 (2015)
6. Zhang, H.M., Ding, F.: Iterative algorithms for $X + A^T X^{-1} A = I$ by using the hierarchical identification principle. J. Franklin Inst. **353**(5), 1132–1146 (2016)
7. Berzig, M.: Solving a class of matrix equations via the Bhaskar–Lakshmikantham coupled fixed point theorem. Appl. Math. Lett. **25**, 1638–1643 (2012)
8. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435–1443 (2003)
9. Sawangsup, K., Sintunavarat, W., Roldán López de Hierro, A.F.: Fixed point theorems for $F_{\mathcal{R}}$-contractions with applications to solution of nonlinear matrix equations. J. Fixed Point Theory Appl. https://doi.org/10.1007/s11784-016-0306-z
10. Lim, Y.: Solving the nonlinear matrix equation $X = Q + M_i X^{\delta_i} M_i^*$ via a contraction principle. Linear Algebra Appl. **430**, 1380–1383 (2009)
11. Banach, S.: Sur les opérations dans les ensembles abstraits et leurs applications aux equations integrales. Fundam. Math. **3**, 133–181 (1922)
12. Wardowski, D.: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012)
13. Engwerda, J.C.: On the existence of a positive solution of the matrix equation $X + A^T X^{-1} A = I$. Linear Algebra Appl. **194**, 91–108 (1993)
14. Pusz, W., Woronowitz, S.L.: Functional calculus for sequilinear forms and the purification map. Rep. Math. Phys. **8**, 159–170 (1975)
15. Buzbee, B.L., Golub, G.H., Nielson, C.W.: On direct methods for solving Poisson's equations. SIAM J. Numer. Anal. **7**, 627–656 (1970)
16. Green, W.L., Kamen, E.: Stabilization of linear systems over a commutative normed algebra with applications to spatially distributed parameter dependent systems. SIAM J. Control Optim. **23**, 1–18 (1985)
17. Ando, T.: Limit of cascade iteration of matrices. Numer. Funct. Anal. Optim. **21**, 579–589 (1980)
18. Anderson, W.N., Morley, T.D., Trapp, G.E.: Ladder networks, fixed points and the geometric mean. Circuits Syst. Signal Process. **3**, 259–268 (1983)
19. Lipschutz, S.: Schaum's Outlines of Theory and Problems of Set Theory and Related Topics. McGraw-Hill, New York (1964)
20. Alam, A., Imdad, M.: Relation-theoretic contraction principle. Fixed Point Theory Appl. **17**(4), 693–702 (2015)
21. Kolman, B., Busby, R.C., Ross, S.: Discrete Mathematical Structures, 3rd edn. PHI Pvt., New Delhi (2000)
22. Roldan Lopez de Hierro, A.F.: A unified version of Ran and Reurings theorem and Nieto and Rodra iguez-LApezs theorem and low-dimensional generalizations. Appl. Math. Inf. Sci. **10**(2), 383–393 (2016)
23. Berzig, M., Samet, B.: Solving systems of nonlinear matrix equations involving Lipschitzian mappings. Fixed Point Theory Appl. **2011**, 89 (2011)
24. Long, J.H., Hu, X.Y., Zhang, L.: On the Hermitian positive definite solution of the nonlinear matrix equation $X + A^* x^{-1} A + B^* X^{-1} B = I$. Bull. Braz. Math. Soc. **39**(3), 317–386 (2015)