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A class of fourth-order parabolic equation with logarithmic nonlinearity

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Abstract

In this paper, we study a class of fourth-order parabolic equation with the logarithmic nonlinearity. By using the potential well method, we obtain the existence of the unique global weak solution. In addition, we also obtain results of decay and blow-up in the finite time for the weak solution.

MSC: Primary 35K35; secondary 35A01; 35B44; 35K55

Keywords: Global existence; Blow-up; Logarithmic source term

1 Introduction

In this paper, we study the following problem:

$$\begin{cases} u_t + \Delta^2 u = |u|^{q-2} u \log |u|, & x \in \Omega, t > 0, \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bound domain in $\mathbb{R}^n$ with smooth boundary, $2 < q < 2 + \frac{4}{n}$, $u_0(x) \in H_0^2(\Omega) \setminus \{0\}$.

Many papers have been devoted to the fourth-order parabolic equation. Qu and Zhou [1] studied the following fourth-order equation:

$$u_t + D^4 u = |u|^{p-1} u - \frac{1}{|\Omega|} \int_\Omega |u|^{p-1} u \, dx. \tag{1.2}$$

Using the method of potential wells, they established a threshold result for the global existence and blow-up for the sign-changing weak solutions. Zhou [2] proved new blow-up conditions and the maximum of the blow-up time for Eq. (1.2). Li, Gao and Han [3] considered

$$\begin{cases} u_t + D^4 u - (|u_x|^{p-2} u_x)_x = |u|^{p-1} u - \frac{1}{|\Omega|} \int_\Omega |u|^{p-1} u \, dx, & x \in \Omega, \\ Du = D^3 u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

They obtained the existence, uniqueness and blow-up of solutions. Liu and Liu [4] considered the following equation:

$$u_t - D^6 u + D\left(a(Du)D^3 u + \frac{a'(Du)}{2}\left(D^2 u\right)^2\right)$$
$$= |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,dx, \quad (x,t) \in \Omega \times (0,T).$$

They combine the potential well method, the classical Galerkin method and the energy method to give a threshold result for the global existence and non-existence of sign-changing weak solutions to the problem. The relevant equations have also been studied in [5, 6].

In this paper, we study a fourth-order parabolic equation with the logarithmic nonlinearity. The second-order parabolic equation with the logarithmic nonlinearity is diffusely studied. Chen, Luo and Liu [7] studied the heat equation with the logarithmic nonlinearity. Ji, Yin and Cao [8] established the existence of positive periodic solutions and discussed the instability of such solutions for the semilinear pseudo-parabolic equation with the logarithmic source. Nahn and Truong [9] studied the following nonlinear equation:

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\log\left(|u|\right). \tag{1.3}$$

They obtained results as regards the existence or non-existence of global weak solutions. He, Gao and Wang [10] considered the following equation:

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{q-2}u\log\left(|u|\right), \tag{1.4}$$

where $2 < p < q < p(1 + \frac{2}{n})$, they proved the decay and the finite time blow-up for weak solutions.

In this paper, we prove the existence of the unique global weak solution of the problem (1.1) based on the potential well method. In addition, we also obtain some properties of the solutions. This paper is organized as follows: in Sect. 2, we introduce some lemmas. In Sect. 3, we mainly introduce the existence of the unique local weak solution of the problem (1.1). In Sect. 4, under some conditions, we obtain the existence of the unique global weak solution of the problem (1.1). Meanwhile, we find that the solution is decaying. In the last section, we prove the blow-up theorem.

2 Some lemmas

We first consider the energy functional J and Nehari functional I defined on $H_0^2(\Omega) \setminus \{0\}$ as follows:

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{1}{q}\int_\Omega |u|^q \log|u|\,dx + \frac{1}{q^2}\int_\Omega |u|^q\,dx, \tag{2.1}$$

$$I(u) = \int_\Omega |\Delta u|^2\,dx - \int_\Omega |u|^q \log|u|\,dx. \tag{2.2}$$

We can see that J and I are continuous from the Gagliardo–Nirenberg multiplicative embedding inequality (see [11]).

By (2.1) and (2.2), we have

$$J(u)=\frac{1}{q}I(u)+\left(\frac{1}{2}-\frac{1}{q}\right)\int_{\Omega}|\Delta u|^{2}\,dx+\frac{1}{q^{2}}\int_{\Omega}|u|^{q}\,dx. \tag{2.3}$$

Let $\mathcal{N}=\{u\in H_0^2(\Omega)\setminus\{0\}:I(u)=0\}$. Lemma 2.1 indicates $\mathcal{N}$ is not empty. Thus, we can define

$$d=\inf_{u\in\mathcal{N}}J(u). \tag{2.4}$$

It is obvious that $d>0$ by (2.3), (2.4), $2<q<2+\frac{4}{n}$ and the definition of $\mathcal{N}$. For a fixed $u\in H_0^2(\Omega)\setminus\{0\}$, we consider the function $j:\lambda\to J(\lambda u)$ for $\lambda>0$.

Lemma 2.1 *Let $u\in H_0^2(\Omega)\setminus\{0\}$. Then the following results hold*:

(1) $\lim_{\lambda\to 0^+}j(\lambda)=0,\lim_{\lambda\to+\infty}j(\lambda)=0$;
(2) *there exists a unique* $\overline{\lambda}>0$ *such that* $j'(\overline{\lambda})=0$;
(3) $j(\lambda)$ *is increasing on* $(0,\overline{\lambda})$, *decreasing on* $(\overline{\lambda},+\infty)$ *and attains the maximum at* $\overline{\lambda}$;
(4) $I(\lambda u)>0$ *for* $0<\lambda<\overline{\lambda}$, $I(\lambda u)<0$ *for* $\overline{\lambda}<\lambda<+\infty$ *and* $I(\overline{\lambda}u)=0$.

Proof For $u\in H_0^2(\Omega)\setminus\{0\}$, by the definition of j, we have

$$\begin{aligned} j(\lambda)=J(\lambda u)&=\frac{1}{2}\int_{\Omega}\left|\Delta(\lambda u)\right|^{2}dx-\frac{1}{q}\int_{\Omega}|\lambda u|^{q}\log|\lambda u|\,dx+\frac{1}{q^{2}}\int_{\Omega}|\lambda u|^{q}\,dx\\ &=\frac{\lambda^{2}}{2}\int_{\Omega}|\Delta u|^{2}\,dx-\frac{\lambda^{q}}{q}\int_{\Omega}|u|^{q}\log|u|\,dx-\frac{\lambda^{q}}{q}\log\lambda\int_{\Omega}|u|^{q}\,dx\\ &\quad+\frac{\lambda^{q}}{q^{2}}\int_{\Omega}|u|^{q}\,dx. \end{aligned}$$

It is obvious that (1) holds due to $2<q<2+\frac{4}{n}$ and $\int_{\Omega}|u|^{q}\,dx\neq 0$. We have

$$\begin{aligned} j'(\lambda)&=\lambda\int_{\Omega}|\Delta u|^{2}\,dx-\lambda^{q-1}\int_{\Omega}|u|^{q}\log|u|\,dx-\lambda^{q-1}\log\lambda\int_{\Omega}|u|^{q}\,dx\\ &\quad-\frac{\lambda^{q-1}}{q}\int_{\Omega}|u|^{q}\,dx+\frac{\lambda^{q-1}}{q}\int_{\Omega}|u|^{q}\,dx\\ &=\lambda\int_{\Omega}|\Delta u|^{2}\,dx-\lambda^{q-1}\int_{\Omega}|u|^{q}\log|u|\,dx-\lambda^{q-1}\log\lambda\int_{\Omega}|u|^{q}\,dx. \end{aligned}$$

We construct a function $\varphi(\lambda)=\lambda^{-1}j'(\lambda)$, thus we obtain

$$\begin{aligned} \varphi(\lambda)&=\lambda^{-1}j'(\lambda)\\ &=\lambda^{-1}\left(\lambda\int_{\Omega}|\Delta u|^{2}\,dx-\lambda^{q-1}\int_{\Omega}|u|^{q}\log|u|\,dx-\lambda^{q-1}\log\lambda\int_{\Omega}|u|^{q}\,dx\right)\\ &=\int_{\Omega}|\Delta u|^{2}\,dx-\lambda^{q-2}\int_{\Omega}|u|^{q}\log|u|\,dx-\lambda^{q-2}\log\lambda\int_{\Omega}|u|^{q}\,dx. \end{aligned}$$

Then

$$\varphi'(\lambda)=-(q-2)\lambda^{q-3}\int_{\Omega}|u|^{q}\log|u|\,dx$$

$$
\begin{aligned}
&-(q-2)\lambda^{q-3}\log\lambda\int_\Omega |u|^q\,dx-\lambda^{q-3}\int_\Omega |u|^q\,dx\\
&=-\lambda^{q-3}\left((q-2)\int_\Omega |u|^q\log|u|\,dx+(q-2)\log\lambda\int_\Omega |u|^q\,dx+\int_\Omega |u|^q\,dx\right),
\end{aligned}
$$

which implies that there exists a $\lambda_1 > 0$ such that $\varphi(\lambda)$ is increasing on $(0,\lambda_1)$, decreasing on $(\lambda_1,+\infty)$. Using the Poincaré inequality and $u \in H_0^2(\Omega)\setminus\{0\}$, we have $0 < \int_\Omega |u|^2\,dx \le C_1\int_\Omega |Du|^2\,dx \le C_1C_2\int_\Omega |\Delta u|^2\,dx$, where C_1, C_2 is the Poincaré constants. Since $\lim_{\lambda\to 0^+}\varphi(\lambda) = \int_\Omega |\Delta u|^2\,dx > 0$, $\lim_{\lambda\to+\infty}\varphi(\lambda) = -\infty$, there exists a unique $\bar\lambda > 0$ such that $\varphi(\bar\lambda) = 0$, i.e. $j'(\bar\lambda) = 0$. So (2) holds. Thus $j'(\lambda) = \lambda\varphi(\lambda) > 0$ for $0 < \lambda < \bar\lambda$ and $j'(\lambda) < 0$ for $\bar\lambda < \lambda < +\infty$, which indicates $j(\lambda)$ is increasing on $(0,\bar\lambda)$, decreasing on $(\bar\lambda,+\infty)$ and attains the maximum at $\bar\lambda$. So (3) holds. From (2.2), we have

$$
\begin{aligned}
I(\lambda u) &= \int_\Omega \left|\Delta(\lambda u)\right|^2 dx-\int_\Omega |\lambda u|^q\log|\lambda u|\,dx\\
&=\lambda^2\int_\Omega |\Delta u|^2\,dx-\lambda^q\int_\Omega |u|^q\log|u|\,dx-\lambda^q\log\lambda\int_\Omega |u|^q\,dx\\
&=\lambda\left(\lambda\int_\Omega |\Delta u|^2\,dx-\lambda^{q-1}\int_\Omega |u|^q\log|u|\,dx-\lambda^{q-1}\log\lambda\int_\Omega |u|^q\,dx\right)\\
&=\lambda j'(\lambda).
\end{aligned}
$$

Thus, $I(\lambda u) > 0$ for $0 < \lambda < \bar\lambda$, $I(\lambda u) < 0$ for $\bar\lambda < \lambda < +\infty$ and $I(\bar\lambda u) = 0$. So (4) holds. □

Lemma 2.2 *There exists a $u > 0$ with $u \in \mathcal{N}$ such that $J(u) = d$.*

Proof By (2.4), we suppose $\{u_k\}_{k=1}^\infty \subset \mathcal{N}$ is a minimizing sequence of J. Since $\{|u_k|\}_{k=1}^\infty \subset \mathcal{N}$ is also a minimizing sequence of J, we consider the case where $u_k > 0$ a.e. in Ω, $k \in \mathbb{N}$ without loss of generality. Thus,

$$
\lim_{k\to\infty} J(u_k) = d, \tag{2.5}
$$

which implies that $\{J(u_k)\}_{k=1}^\infty$ is bounded, i.e. there exists a constant $C_3 > 0$ such that $|J(u_k)| \le C_3$. Using (2.3), $I(u_k) = 0$ and $|J(u_k)| \le C_3$, we have

$$
\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega |\Delta u_k|^2\,dx+\frac{1}{q^2}\int_\Omega |u_k|^q\,dx \le C_3. \tag{2.6}
$$

Combining $2 < q < 2+\frac{4}{n}$ with (2.6), we have

$$
\int_\Omega |\Delta u_k|^2\,dx \le \left(\frac{1}{2}-\frac{1}{q}\right)^{-1} C_3. \tag{2.7}
$$

Using (2.7) and the Poincaré inequality, we have

$$
\int_\Omega |u_k|^2\,dx \le C_4\int_\Omega |Du_k|^2\,dx \le C_4C_5\int_\Omega |\Delta u_k|^2\,dx \le \left(\frac{1}{2}-\frac{1}{q}\right)^{-1} C_3C_4C_5,
$$

where C_4, C_5 are the Poincaré constants. The above inequality implies that $\{u_k\}_{k=1}^\infty$ is bounded in $H_0^2(\Omega)$. Let $\mu_1 > 0$ be sufficiently small such that $q+\mu_1 < \frac{2n}{n-2}$. Since $H_0^2(\Omega) \hookrightarrow$

$L^{q+\mu_1}(\Omega)$ is a compact embedding, there exist a function u and a subsequence of $\{u_k\}_{k=1}^{\infty}$, still denoted by $\{u_k\}_{k=1}^{\infty}$, such that

$$u_k \to u, \quad \text{weakly in } H_0^2(\Omega),$$
$$u_k \to u, \quad \text{strongly in } L^{q+\mu_1}(\Omega),$$
$$u_k \to u(x), \quad \text{a.e. in } \Omega.$$

Then we have $u \geq 0$ a.e. in Ω. First, we prove $u \not\equiv 0$. Using the dominated convergence theorem, we obtain

$$\int_\Omega |u|^q \log|u|\,dx = \lim_{k\to\infty} \int_\Omega |u_k|^q \log|u_k|\,dx, \tag{2.8}$$
$$\int_\Omega |u|^q\,dx = \lim_{k\to\infty} \int_\Omega |u_k|^q\,dx. \tag{2.9}$$

It follows from the weak lower semicontinuity of the L^2 norm that

$$\int_\Omega |\Delta u|^2\,dx \leq \liminf_{k\to\infty} \int_\Omega |\Delta u_k|^2\,dx. \tag{2.10}$$

Using (2.1), (2.5), (2.8), (2.9) and (2.10), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{1}{q}\int_\Omega |u|^q \log|u|\,dx + \frac{1}{q^2}\int_\Omega |u|^q\,dx \\ &\leq \liminf_{k\to\infty} \frac{1}{2}\int_\Omega |\Delta u_k|^2\,dx - \lim_{k\to\infty} \frac{1}{q}\int_\Omega |u_k|^q \log|u_k|\,dx + \lim_{k\to\infty} \frac{1}{q^2}\int_\Omega |u_k|^q\,dx \\ &= \liminf_{k\to\infty} \left(\frac{1}{2}\int_\Omega |\Delta u_k|^2\,dx - \frac{1}{q}\int_\Omega |u_k|^q \log|u_k|\,dx + \frac{1}{q^2}\int_\Omega |u_k|^q\,dx \right) \\ &= \liminf_{k\to\infty} J(u_k) = d. \end{aligned} \tag{2.11}$$

Using (2.2), (2.8), (2.10) and $u_k \in \mathcal{N}$, we have

$$\begin{aligned} I(u) &= \int_\Omega |\Delta u|^2\,dx - \int_\Omega |u|^q \log|u|\,dx \\ &\leq \liminf_{k\to\infty} \int_\Omega |\Delta u_k|^2\,dx - \lim_{k\to\infty} \int_\Omega |u_k|^q \log|u_k|\,dx \\ &= \liminf_{k\to\infty} \left(\int_\Omega |\Delta u_k|^2\,dx - \int_\Omega |u_k|^q \log|u_k|\,dx \right) \\ &= \liminf_{k\to\infty} I(u_k) = 0. \end{aligned} \tag{2.12}$$

By $u_k \in \mathcal{N}$ and using the Sobolev embedding inequality and the Poincaré inequality, we have

$$\begin{aligned} \int_\Omega |\Delta u_k|^2\,dx &= \int_\Omega |u_k|^q \log|u_k|\,dx \\ &\leq \frac{e^{-1}}{\mu_1}\int_\Omega |u_k|^{q+\mu_1}\,dx \end{aligned}$$

$$\leq C_6^{q+\mu_1} \frac{e^{-1}}{\mu_1} \left(\int_\Omega |Du_k|^2 \,dx \right)^{\frac{q+\mu_1}{2}}$$
$$\leq C_6^{q+\mu_1} C_7^{\frac{q+\mu_1}{2}} \frac{e^{-1}}{\mu_1} \left(\int_\Omega |\Delta u_k|^2 \,dx \right)^{\frac{q+\mu_1}{2}}, \tag{2.13}$$

where C_6 is the Sobolev embedding constant, C_7 is the Poincaré constant.

By (2.13), we have, for some positive constant C_8,

$$\int_\Omega |u_k|^q \log |u_k| \,dx = \int_\Omega |\Delta u_k|^2 \,dx \geq C_8. \tag{2.14}$$

Using (2.8) and (2.14), we have

$$\frac{e^{-1}}{\mu_1} \int_\Omega |u|^{q+\mu_1} \,dx \geq \int_\Omega |u|^q \log |u| \,dx = \lim_{k\to\infty} \int_\Omega |u_k|^q \log |u_k| \,dx \geq C_8,$$

which indicates $u \neq 0$. Next, we will study $I(u) = 0$. If $I(u) < 0$, by Lemma 2.1, there exists a $\overline{\lambda}_1$ such that $I(\overline{\lambda}_1 u) = 0$ and $0 < \overline{\lambda}_1 < 1$. Thus, $\overline{\lambda}_1 u \in \mathcal{N}$. By (2.3), (2.4), (2.9) and (2.10), we have

$$\begin{aligned}
d &\leq J(\overline{\lambda}_1 u) \\
&= \frac{1}{q} I(\overline{\lambda}_1 u) + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \left|\Delta(\overline{\lambda}_1 u)\right|^2 \,dx + \frac{1}{q^2} \int_\Omega |\overline{\lambda}_1 u|^q \,dx \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \left|\Delta(\overline{\lambda}_1 u)\right|^2 \,dx + \frac{1}{q^2} \int_\Omega |\overline{\lambda}_1 u|^q \,dx \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \overline{\lambda}_1^2 \int_\Omega |\Delta u|^2 \,dx + \frac{\overline{\lambda}_1^q}{q^2} \int_\Omega |u|^q \,dx \\
&\leq \left(\frac{1}{2} - \frac{1}{q}\right) \overline{\lambda}_1^2 \int_\Omega |\Delta u|^2 \,dx + \frac{\overline{\lambda}_1^2}{q^2} \int_\Omega |u|^q \,dx \\
&= \overline{\lambda}_1^2 \left[\left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega |\Delta u|^2 \,dx + \frac{1}{q^2} \int_\Omega |u|^q \,dx \right] \\
&\leq \overline{\lambda}_1^2 \liminf_{k\to\infty} \left[\left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega |\Delta u_k|^2 \,dx + \frac{1}{q^2} \int_\Omega |u_k|^q \,dx \right] \\
&= \overline{\lambda}_1^2 \liminf_{k\to\infty} J(u_k) \\
&= \overline{\lambda}_1^2 d,
\end{aligned}$$

which indicates $\overline{\lambda}_1 \geq 1$ by $d > 0$. It contradicts $0 < \overline{\lambda}_1 < 1$. Then, by (2.12), we have $I(u) = 0$. Therefore, $u \in \mathcal{N}$. By (2.4), we have $J(u) \geq d$. By (2.11), we have $J(u) \leq d$. So, $J(u) = d$. □

Lemma 2.3 ([9]) *For any $u \in W_0^{1,p}(\Omega)$, $p \geq 1$, and $r \geq 1$, the inequality*

$$\|u\|_q \leq C \|Du\|_p^\theta \|u\|_r^{1-\theta},$$

is valid, where

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1},$$

and for $p \geq n = 1, r \leq q \leq \infty$*; for* $n > 1$ *and* $p < n$, $q \in [r, p^*]$ *if* $r \leq p^*$ *and* $q \in [p^*, r]$ *if* $r \geq p^*$*; for* $p = n > 1, r \leq q \leq \infty$*; for* $p > n > 1, r \leq q \leq \infty$.

Here, the constant C depends on n, p, q and r.

Lemma 2.4 ([12]) *Let* $h : \mathbb{R}^+ \to R^+$ *be a nonincreasing function. Assume that there is a constant* $A > 0$ *such that*

$$\int_s^{+\infty} h(t)\,dt \leq Ah(s), \quad 0 \leq s < +\infty.$$

Then $h(t) \leq h(0)e^{1-\frac{t}{A}}$*, for all* $t > 0$.

3 Local existence and uniqueness

Definition 3.1 (Weak solution) A function u is a solution of problem (1.1) over $[0, T]$ if $u \in L^\infty(0, T; H_0^2(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$, satisfies the initial condition $u(0) = u_0(x) \in H_0^2(\Omega) \setminus \{0\}$, and

$$\int_\Omega u_t w\,dx + \int_\Omega \Delta u \Delta w\,dx = \int_\Omega |u|^{q-2} u \log|u| w\,dx, \tag{3.1}$$

for any $w \in H_0^2(\Omega)$, and for a.e. $t \in [0, T]$.

Theorem 3.1 (Local existence) *Let* $u_0 \in H_0^2(\Omega) \setminus \{0\}$*. Then there exists a positive constant* T_0 *such that the problem* (1.1) *has a unique weak solution* $u(x,t)$ *on* $\Omega \times (0, T_0)$*. Furthermore,* $u(x,t)$ *satisfies the energy inequality*

$$\int_0^t \int_\Omega u_s^2\,dx\,ds + J\big(u(t)\big) \leq J(u_0), \quad t \in [0, T_0]. \tag{3.2}$$

Proof In the space of $H_0^2(\Omega)$, we take a basis $\{w_j\}_{j=1}^\infty$ and define the finite dimensional space

$$V_m = \operatorname{span}\{w_1, w_2, \dots, w_m\}.$$

Let u_{0m} be an element of V_m such that

$$u_{0m} = \sum_{j=1}^m a_{mj} w_j \to u_0 \quad \text{strongly in } H_0^2(\Omega), \tag{3.3}$$

as $m \to \infty$. We can find the approximate solution $u_m(x,t)$ of the problem (1.1) in the form

$$u_m(x,t) = \sum_{j=1}^m \alpha_{mj}(t) w_j(x), \tag{3.4}$$

where α_{mj} $(1 \leq j \leq m)$ satisfy the ordinary differential equations

$$\int_\Omega u_{mt} w_i\,dx + \int_\Omega \Delta u_m \Delta w_i\,dx = \int_\Omega |u_m|^{q-2} u_m \log|u_m| w_i\,dx, \tag{3.5}$$

for $i \in \{1, 2, \dots, m\}$, with

$$\alpha_{mj}(0) = a_{mj}, \quad i \in \{1, 2, \dots, m\}. \tag{3.6}$$

We find from Peano's theorem that (3.5)–(3.6) has a local solution α_{mj}, and there exists a positive $T_m > 0$ such that $\alpha_{mj} \in C^1([0, T_m])$, therefore $u_m \in C^1([0, T_m]; H_0^2(\Omega))$. Multiplying the ith equation in (3.5) by α_{mi}, summing over i from 1 to m, we have

$$\int_\Omega u_{mt} u_m \, dx + \int_\Omega |\Delta u_m|^2 \, dx = \int_\Omega |u_m|^q \log |u_m| \, dx. \tag{3.7}$$

Integrating the above formula with respect to s over $(0, t)$, we have

$$y_m(t) = y_m(0) + \int_0^t \int_\Omega |u_m|^q \log |u_m| \, dx \, ds, \tag{3.8}$$

where

$$y_m(t) = \frac{1}{2} \int_\Omega |u_m|^2 \, dx + \int_0^t \int_\Omega |\Delta u_m|^2 \, dx \, ds. \tag{3.9}$$

Choose μ_2 such that $0 < \mu_2 < 2 + \frac{4}{n} - q$. Using Lemma 2.3, the Poincaré inequality and the Young inequality, we have

$$\begin{aligned}
&\int_\Omega |u_m|^q \log |u_m| \, dx \\
&\quad \le \frac{e^{-1}}{\mu_2} \int_\Omega |u_m|^{q+\mu_2} \, dx \\
&\quad \le \frac{e^{-1}}{\mu_2} C_9^{q+\mu_2} \|Du_m\|_2^{\theta(q+\mu_2)} \|u_m\|_2^{(1-\theta)(q+\mu_2)} \\
&\quad \le \frac{e^{-1}}{\mu_2} C_9^{q+\mu_2} C_{10}^{\frac{\theta(q+\mu_2)}{2}} \|\Delta u_m\|_2^{\theta(q+\mu_2)} \|u_m\|_2^{(1-\theta)(q+\mu_2)} \\
&\quad \le \varepsilon \|\Delta u_m\|_2^2 + \left(\frac{\varepsilon \mu_2}{e^{-1} C_9^{q+\mu_2} C_{10}^{\frac{\theta(q+\mu_2)}{2}}} \right)^{-\frac{\theta(q+\mu_2)}{2-\theta(q+\mu_2)}} \|u_m\|_2^{\frac{2(1-\theta)(q+\mu_2)}{2-\theta(q+\mu_2)}},
\end{aligned} \tag{3.10}$$

where C_9 is the constant of Lemma 2.3, C_{10} is the Poincaré constant, $0 < \varepsilon < 1$, and $\theta = n(\frac{1}{2} - \frac{1}{q+\mu_2})$. Let $\gamma = \frac{(1-\theta)(q+\mu_2)}{2-\theta(q+\mu_2)}$ and $C_{11} = (\frac{\varepsilon \mu_2}{e^{-1} C_9^{q+\mu_2} C_{10}^{\frac{\theta(q+\mu_2)}{2}}})^{-\frac{\theta(q+\mu_2)}{2-\theta(q+\mu_2)}}$, thus (3.10) becomes

$$\int_\Omega |u_m|^q \log |u_m| \, dx \le \varepsilon \int_\Omega |\Delta u_m|^2 \, dx + C_{11} \left(\int_\Omega |u_m|^2 \, dx \right)^\gamma. \tag{3.11}$$

It is easy to check $\gamma > 1$ according to $2 < q < 2 + \frac{4}{n}$. Using (3.3), (3.8), (3.9) and (3.11), we have

$$\begin{aligned}
y_m(t) &= y_m(0) + \int_0^t \int_\Omega |u_m|^q \log |u_m| \, dx \, ds \\
&\le y_m(0) + \int_0^t \left[\varepsilon \int_\Omega |\Delta u_m|^2 \, dx + C_{11} \left(\int_\Omega |u_m|^2 \, dx \right)^\gamma \right] ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_\Omega \left|u_m(0)\right|^2 dx + \int_0^t \int_\Omega \left|\Delta u_m(0)\right|^2 dx\,ds \\
&\quad + \int_0^t \left[\varepsilon \int_\Omega |\Delta u_m|^2 dx + C_{11}\left(\int_\Omega |u_m|^2 dx\right)^\gamma\right] ds \\
&\le C_{12} + \int_0^t \left[\varepsilon \int_\Omega |\Delta u_m|^2 dx + C_{11}\left(\int_\Omega |u_m|^2 dx\right)^\gamma\right] ds \\
&\le C_{12} + \varepsilon \int_0^t \int_\Omega |\Delta u_m|^2 dx\,ds + C_{11}\int_0^t \left(\int_\Omega |u_m|^2 dx\right)^\gamma ds \\
&\le C_{12} + \frac{\varepsilon}{2}\int_\Omega |u_m|^2 dx + \varepsilon \int_0^t \int_\Omega |\Delta u_m|^2 dx\,ds \\
&\quad + C_{11}2^\gamma \int_0^t \left(\int_0^s \int_\Omega |\Delta u_m|^2 dx\,dy\right)^\gamma ds \\
&\quad + C_{11}2^\gamma \int_0^t \left(\frac{1}{2}\int_\Omega |u_m|^2 dx\right)^\gamma ds \\
&\le C_{12} + \varepsilon y_m(t) + C_{11}2^\gamma \int_0^t y_m(s)^\gamma\,ds. \qquad (3.12)
\end{aligned}
$$

Using $0 < \varepsilon < 1$ and (3.12),

$$
y_m(t) \le \frac{C_{12}}{1-\varepsilon} + \frac{C_{11}2^\gamma}{1-\varepsilon}\int_0^t y_m(s)^\gamma\,ds. \qquad (3.13)
$$

Using the integral inequality of Gronwall–Bellman–Bihari type and combining with (3.13), there exists T_0 such that

$$
y_m(t) \le C_{13}(T_0), \quad t \in [0, T_0], \qquad (3.14)
$$

where $C_{13}(T_0)$ is a positive constant dependent on T_0. Multiplying equation (3.5) by α'_{mi}, summing over i from 1 to m and integrating with respect to time variable on $[0,t]$, we have

$$
\int_0^t \int_\Omega u_{ms}^2\,dx\,ds + J\bigl(u_m(t)\bigr) = J\bigl(u_m(0)\bigr), \quad \text{for all } t \in [0, T_0]. \qquad (3.15)
$$

We find from (3.3) and the continuity of the J that there exists a constant $C_{14} > 0$ such that

$$
J\bigl(u_m(0)\bigr) \le C_{14}, \quad \text{for all } m. \qquad (3.16)
$$

Using (2.1), (3.9), (3.11), (3.14), (3.15) and (3.16), we have

$$
\begin{aligned}
C_{14} &\ge J\bigl(u_m(t)\bigr) \\
&= \frac{1}{2}\int_\Omega \left|\Delta u_m(t)\right|^2 dx - \frac{1}{q}\int_\Omega \left|u_m(t)\right|^q \log\left|u_m(t)\right| dx + \frac{1}{q^2}\int_\Omega \left|u_m(t)\right|^q dx \\
&\ge \frac{1}{2}\int_\Omega \left|\Delta u_m(t)\right|^2 dx - \frac{1}{q}\int_\Omega \left|u_m(t)\right|^q \log\left|u_m(t)\right| dx \\
&\ge \frac{1}{2}\int_\Omega \left|\Delta u_m(t)\right|^2 dx - \frac{\varepsilon}{q}\int_\Omega \left|\Delta u_m(t)\right|^2 dx - \frac{C_{11}}{q}\left(\int_\Omega \left|u_m(t)\right|^2 dx\right)^\gamma
\end{aligned}
$$

$$= \left(\frac{1}{2} - \frac{\varepsilon}{q}\right) \int_\Omega \left|\Delta u_m(t)\right|^2 dx - \frac{C_{11}}{q}\left(\int_\Omega \left|u_m(t)\right|^2 dx\right)^\gamma$$
$$\geq \left(\frac{1}{2} - \frac{\varepsilon}{q}\right) \int_\Omega \left|\Delta u_m(t)\right|^2 dx - \frac{C_{11}}{q}\left(2C_{13}(T_0)\right)^\gamma, \tag{3.17}$$

which implies that

$$\int_\Omega \left|\Delta u_m(t)\right|^2 dx \leq \left(\frac{1}{2} - \frac{\varepsilon}{q}\right)^{-1}\left[C_{14} + \frac{C_{11}}{q}\left(2C_{13}(T_0)\right)^\gamma\right]. \tag{3.18}$$

By the Poincaré inequality and (3.18), we obtain

$$\int_\Omega \left|u_m(t)\right|^2 dx \leq C_{15}\int_\Omega \left|Du_m(t)\right|^2 dx \leq C_{15}C_{16}\int_\Omega \left|\Delta u_m(t)\right|^2 dx$$
$$\leq C_{15}C_{16}\left(\frac{1}{2} - \frac{\varepsilon}{q}\right)^{-1}\left[C_{14} + \frac{C_{11}}{q}\left(2C_{13}(T_0)\right)^\gamma\right], \tag{3.19}$$

where C_{15}, C_{16} are the Poincaré constants. We can easily obtain from the above inequality

$$\|u_m\|_{L^\infty(0,T_0;H_0^2(\Omega))} \leq C_{17}(T_0), \tag{3.20}$$

where $C_{17}(T_0)$ is a positive constant dependent on T_0. Using (3.15)–(3.17), we have

$$\left(\frac{1}{2} - \frac{\varepsilon}{q}\right) \int_\Omega \left|\Delta u_m(t)\right|^2 dx - \frac{C_{11}}{q}\left(2C_{13}(T_0)\right)^\gamma + \int_0^t \int_\Omega u_{ms}^2 \, dx\, ds \leq C_{14}, \tag{3.21}$$

which implies that

$$\|u_{mt}\|_{L^2(0,T_0;L^2(\Omega))} \leq C_{18}(T_0), \tag{3.22}$$

where $C_{18}(T_0)$ is a positive constant dependent on T_0. It follows from (3.20) and (3.22) that there exist a function u and a subsequence of $\{u_m\}_{m=1}^\infty$ still denoted $\{u_m\}_{m=1}^\infty$ such that

$$u_m \to u \quad \text{weakly star in } L^\infty\left(0, T_0; H_0^2(\Omega)\right), \tag{3.23}$$
$$u_{mt} \to u_t \quad \text{weakly in } L^2\left(0, T_0; L^2(\Omega)\right). \tag{3.24}$$

We obtain from the Aubin–Lions–Simon lemma (see [13]) together with (3.23) and (3.24)

$$u_m \to u \quad \text{strongly in } C\left(0, T_0; L^2(\Omega)\right). \tag{3.25}$$

So, $u_m \to u$ a.e. $(x,t) \in \Omega \times (0, T_0)$. This implies that

$$|u_m|^{q-2}u_m \log|u_m| \to |u|^{q-2}u \log|u| \quad \text{a.e. } (x,t) \in \Omega \times (0, T_0). \tag{3.26}$$

According to $2 < q < 2 + \frac{4}{n}$, we can choose μ_3 such that $1 < \frac{q(q-1+\mu_3)}{q-1} < \frac{2n}{n-2}$. Then, using the Sobolev embedding inequality and combining (3.19), we have

$$\int_\Omega \left||u_m|^{q-2}u_m \log|u_m|\right|^{\frac{q}{q-1}} dx$$

$$
\begin{aligned}
&= \int_{\{x\in\Omega:|u_m|\leq 1\}} \big||u_m|^{q-2}u_m \log|u_m|\big|^{\frac{q}{q-1}}\,dx \\
&\quad + \int_{\{x\in\Omega:|u_m|\geq 1\}} \big||u_m|^{q-2}u_m \log|u_m|\big|^{\frac{q}{q-1}}\,dx \\
&\leq \big(e(q-1)\big)^{-\frac{q}{q-1}}|\Omega| + \left(\frac{e^{-1}}{\mu_3}\right)^{-\frac{q}{q-1}} \int_\Omega |u_m|^{\frac{q(q-1+\mu_3)}{q-1}}\,dx \\
&\leq \big(e(q-1)\big)^{-\frac{q}{q-1}}|\Omega| + \left(\frac{e^{-1}}{\mu_3}\right)^{-\frac{q}{q-1}} C_{19}^{\frac{q(q-1+\mu_3)}{q-1}} \left(\int_\Omega |Du_m|^2\,dx\right)^{\frac{q(q-1+\mu_3)}{2(q-1)}} \\
&\leq \big(e(q-1)\big)^{-\frac{q}{q-1}}|\Omega| + \left(\frac{e^{-1}}{\mu_3}\right)^{-\frac{q}{q-1}} C_{19}^{\frac{q(q-1+\mu_3)}{q-1}} \\
&\quad \times \left(C_{16}\left(\frac{1}{2}-\frac{\varepsilon}{q}\right)^{-1}\left[C_{14}+\frac{C_{11}}{q}\big(2C_{13}(T_0)\big)^{\gamma}\right]\right)^{\frac{q(q-1+\mu_3)}{2(q-1)}},
\end{aligned}
\tag{3.27}
$$

where C_{19} is the embedding constant. Using (3.26), (3.27) and Lion's lemma (see [13]), we obtain

$$
|u_m|^{q-2}u_m \log|u_m| \to |u|^{q-2}u\log|u| \quad \text{weakly}^* \text{ in } L^\infty\big(0,T_0;L^{\frac{q}{q-1}}(\Omega)\big). \tag{3.28}
$$

Passing to the limit in (3.5) and (3.6) as $m\to\infty$, by (3.23), (3.24) and (3.28), we see that u satisfies the initial condition $u(0)=u_0$ and

$$
\int_\Omega u_t(t)w\,dx + \int_\Omega \Delta u(t)\Delta w\,dx = \int_\Omega \big|u(t)\big|^{q-2}u(t)\log\big|u(t)\big|w\,dx, \tag{3.29}
$$

for all $w\in H_0^2(\Omega)$, and for a.e. $t\in[0,T_0]$. So, u is a desired solution of problem (1.1).

Next, we will study uniqueness of the solution. We obtain from (3.29) for any $v\in L^2(0,T_0;H_0^2(\Omega))$

$$
\int_\Omega u_t(t)v\,dx + \int_\Omega \Delta u(t)\Delta v\,dx = \int_\Omega \big|u(t)\big|^{q-2}u(t)\log\big|u(t)\big|v\,dx. \tag{3.30}
$$

We suppose there are two solutions u_1 and u_2. Let $w=u_1-u_2$, thus we have $w(0)=0$, $w\in L^2(0,T_0;H_0^2(\Omega))$ and $w_t\in L^2(0,T_0;L^2(\Omega))$. We set

$$
v(s) = \begin{cases} u_1(s)-u_2(s), & s\in[0,t], \\ 0, & s\in[t,T_0]. \end{cases}
$$

From (3.30), it follows that

$$
\begin{aligned}
&\int_0^t\int_\Omega w_s w\,dx\,ds + \int_0^t\int_\Omega |\Delta w|^2\,dx\,ds \\
&\quad = \int_0^t\int_\Omega \big(|u_1|^{q-2}u_1\log|u_1| - |u_2|^{q-2}u_2\log|u_2|\big)w\,dx\,ds.
\end{aligned}
\tag{3.31}
$$

According to $0\leq\int_0^t\int_\Omega|\Delta w|^2\,dx\,ds$, (3.31) becomes

$$
\int_0^t\int_\Omega w_s w\,dx\,ds \leq \int_0^t\int_\Omega \big(|u_1|^{q-2}u_1\log|u_1| - |u_2|^{q-2}u_2\log|u_2|\big)w\,dx\,ds. \tag{3.32}
$$

We construct a function $f : \mathbb{R}^* \to \mathbb{R}, f(s) = |s|^{q-2}s\log|s|$. Thus, we find that there exists a positive constant C_{20} such that

$$\big||u_1|^{q-2}u_1\log|u_1| - |u_2|^{q-2}u_2\log|u_2|\big| \leq C_{20}|w|. \tag{3.33}$$

By (3.32) and (3.33),

$$\int_0^t\int_\Omega w_s w\,dx\,ds \leq C_{20}\int_0^t\int_\Omega w^2\,dx\,ds,$$

i.e.,

$$\frac{1}{2}\int_\Omega w^2\,dx \leq \frac{1}{2}\int_\Omega w(0)^2\,dx + C_{20}\int_0^t\int_\Omega w^2\,dx\,ds \leq C_{20}\int_0^t\int_\Omega w^2\,dx\,ds. \tag{3.34}$$

Using Gronwall's inequality and combining with (3.34), we have

$$\int_\Omega w^2\,dx \leq 0.$$

So, the uniqueness is derived.

Finally, we will study (3.2). Let $\phi(t)$ is a nonnegative function which belongs to $C([0,T_0])$. From (3.15), we can obtain

$$\begin{aligned}&\int_0^{T_0}\phi(t)\,dt\int_0^t\int_\Omega u_{ms}^2\,dx\,ds + \int_0^{T_0}J\big(u_m(t)\big)\phi(t)\,dt\\&\quad=\int_0^{T_0}J\big(u_m(0)\big)\phi(t)\,dt.\end{aligned} \tag{3.35}$$

As $m \to \infty$,

$$\int_0^{T_0}J\big(u_m(0)\big)\phi(t)\,dt \to \int_0^{T_0}J(u_0)\phi(t)\,dt$$

and

$$\int_0^{T_0}\phi(t)\,dt\int_0^t\int_\Omega u_{ms}^2\,dx\,ds \to \int_0^{T_0}\phi(t)\,dt\int_0^t\int_\Omega u_s^2\,dx\,ds$$

hold. Since $\int_0^{T_0}J(u_m(t))\phi(t)\,dt$ is lower semi-continuous with respect to the weak topology of $L^2(0,T_0;H_0^2(\Omega))$, we know that

$$\int_0^{T_0}J\big(u(t)\big)\phi(t)\,dt \leq \liminf_{m\to\infty}\int_0^{T_0}J\big(u_m(t)\big)\phi(t)\,dt.$$

Hence, by (3.35), it follows that

$$\int_0^{T_0}\phi(t)\,dt\int_0^t\int_\Omega u_s^2\,dx\,ds + \int_0^{T_0}J\big(u(t)\big)\phi(t)\,dt \leq \int_0^{T_0}J(u_0)\phi(t)\,dt,$$

as $m \to \infty$. $\phi(t)$ is arbitrary nonnegative function, so we have

$$\int_0^t \int_\Omega u_s^2 \, dx \, ds + J\bigl(u(t)\bigr) \le J(u_0), \quad t \in [0, T_0]. \qquad \square$$

4 Global existence and decay estimates

Now as in [9], we introduce the following sets: $\mathscr{W}_1 = \{u \in H_0^2(\Omega) \setminus \{0\} : J(u) < d\}$, $\mathscr{W}_2 = \{u \in H_0^2(\Omega) \setminus \{0\} : J(u) = d\}$, $\mathscr{W}_1^+ = \{u \in \mathscr{W}_1 : I(u) > 0\}$, $\mathscr{W}_2^+ = \{u \in \mathscr{W}_2 : I(u) > 0\}$, $\mathscr{W}_1^- = \{u \in \mathscr{W}_1 : I(u) < 0\}$, $\mathscr{W}_2^- = \{u \in \mathscr{W}_2 : I(u) < 0\}$, $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$, $\mathscr{W}^+ = \mathscr{W}_1^+ \cup \mathscr{W}_2^+$, $\mathscr{W}^- = \mathscr{W}_1^- \cup \mathscr{W}_2^-$.

Definition 4.1 (Maximal existence time) Let $u(t)$ be a solution of problem (1.1). We define the maximal existence time $T_{\max}$ as follows:

$$T_{\max} = \sup\bigl\{T > 0 : u(t) \text{ exists on } [0, T]\bigr\}.$$

Then:

(i) if $T_{\max} < +\infty$, we say that $u(t)$ blows up in finite time and $T_{\max}$ is the blow-up time;
(ii) if $T_{\max} = +\infty$, we say that $u(t)$ is global.

Theorem 4.1 *Let $u_0 \in \mathscr{W}^+$. Then the problem of (1.1) admits a unique global weak solution such that*

$$u(t) \in \overline{\mathscr{W}^+}, \quad t \in [0, \infty),$$

and

$$\int_0^t \int_\Omega u_s^2 \, dx \, ds + J\bigl(u(t)\bigr) \le J(u_0), \quad t \in [0, \infty). \tag{4.1}$$

Furthermore, if $u_0 \in \mathscr{W}_1^+$, the solution $u(t)$ decays exponentially.

Proof We will consider the following two cases.

First we address the case of the initial data $u_0 \in \mathscr{W}_1^+$.

Let $\{w_j\}_{j=1}^\infty$, $\{u_{0m}\}_{m=1}^\infty$, and $\{u_m\}_{m=1}^\infty$ be the same as those stated in the proof of the local existence in the second section. Multiplying the (3.5) by $\alpha'_{mi}(t)$, summing over i from 1 to m and integrating with respect to time variable on $[0, t]$, we have

$$\int_0^t \int_\Omega u_{ms}^2 \, dx \, ds + J\bigl(u_m(t)\bigr) = J\bigl(u_m(0)\bigr), \quad t \in [0, T_{\max}), \tag{4.2}$$

where $T_{\max}$ is the maximal existence time of solution $u_m(x, t)$. We will prove that $T_{\max} = \infty$.

By (3.3), (3.6) and the continuity of J, we have

$$J\bigl(u_m(0)\bigr) \to J(u_0) \quad \text{as } m \to \infty. \tag{4.3}$$

Using (4.2) and (4.3) and combining with $J(u_0) < d$, we have

$$\int_0^t \int_\Omega u_{ms}^2 \, dx \, ds + J\bigl(u_m(t)\bigr) < d, \quad t \in [0, T_{\max}), \tag{4.4}$$

for sufficiently large m. Next, we will study

$$u_m(t) \in \mathscr{W}_1^+, \quad t \in [0, T_{\max}), \tag{4.5}$$

for sufficiently large m. We assume that (4.5) does not hold and think that there exists a smallest time t_0 such that $u_m(t_0) \notin \mathscr{W}_1^+$. Then, we have $u_m(t_0) \in \partial \mathscr{W}_1^+$. So, we have

$$J\big(u_m(t_0)\big) = d, \tag{4.6}$$

or

$$I\big(u_m(t_0)\big) = 0. \tag{4.7}$$

(4.6) contradicts with (4.4). If (4.7) holds, from (2.4) we can obtain

$$J\big(u_m(t_0)\big) \geq \inf_{u \in \mathscr{N}} J(u) = d,$$

which contradicts with (4.4). Hence, we have (4.5), i.e., $J(u_m(t)) < d$, and $I(u_m(t)) > 0$, for any $t \in [0, T_{\max})$, for sufficiently large m. Then, by (2.3), we have

$$\begin{aligned} d > J\big(u_m(t)\big) &= \frac{1}{q} I\big(u_m(t)\big) + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \big|\Delta u_m(t)\big|^2 \,dx + \frac{1}{q^2} \int_\Omega \big|u_m(t)\big|^q \,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \big|\Delta u_m(t)\big|^2 \,dx + \frac{1}{q^2} \int_\Omega \big|u_m(t)\big|^q \,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \big|\Delta u_m(t)\big|^2 \,dx. \end{aligned} \tag{4.8}$$

Using (4.8) and combining with the Poincaré inequality, we have

$$\begin{aligned} \int_\Omega \big|u_m(t)\big|^2 \,dx \leq C_{21} \int_\Omega \big|Du_m(t)\big|^2 \,dx \leq C_{21} C_{22} \int_\Omega \big|\Delta u_m(t)\big|^2 \,dx \\ \leq \left(\frac{1}{2} - \frac{1}{q}\right)^{-1} C_{21} C_{22} d, \end{aligned} \tag{4.9}$$

where C_{21} and C_{22} are the Poincaré constants. By (4.4) and (4.8), we have

$$\int_0^t \int_\Omega u_{ms}^2 \,dx\,ds + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \big|\Delta u_m(t)\big|^2 \,dx < d. \tag{4.10}$$

Equations (4.9) and (4.10) imply that $T_{\max} = \infty$. Then the rest is similar to the proof of the local existence, and we see that there exists a unique global weak solution $u(t) \in \mathscr{W}_1^+$ of the problem (1.1), and

$$\int_0^t \int_\Omega u_s^2 \,dx\,ds + J\big(u(t)\big) \leq J(u_0), \quad t \in [0, \infty).$$

Now we address the case of the initial data $u_0 \in \mathscr{W}_2^+$.

First, we can choose a sequence $\{\rho_m\}_{m=1}^{\infty} \subset (0,1)$ and $\lim_{m\to\infty} \rho_m = 1$. Next, we consider the following problem:

$$\begin{cases} u_t + \Delta^2 u = |u|^{q-2}u \log|u|, & x \in \Omega, t > 0, \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_{0m}(x), & x \in \Omega, \end{cases} \tag{4.11}$$

where $u_{0m} = \rho_m u_0$. By $I(u_0) > 0$ and Lemma 2.1, we see that there exists a $\overline{\lambda}_2 > 1$. Hence, $I(u_{0m}) = I(\rho_m u_0) > 0$ and $J(u_{0m}) = J(\rho_m u_0) < J(u_0) = d$ hold. So, we have $u_{0m} \in \mathscr{W}_1^+$. Similar to the previous case, we see that the problem (4.11) admits that, for any positive m, there exists a unique global u_m which satisfies $u_m \in L^\infty(0,\infty; H_0^2(\Omega))$, $u_{mt} \in L^2(0,\infty; L^2(\Omega))$, $u_m(0) = u_{0m} = \rho_m u_0 \to u_0$ strongly in $H_0^2(\Omega)$, and

$$\int_\Omega u_{mt} w\,dx + \int_\Omega \Delta u_m \Delta w\,dx = \int_\Omega |u_m|^{q-2} u_m \log|u_m| w\,dx, \tag{4.12}$$

for any $w \in H_0^2(\Omega)$, and for a.e. $t \in [0,\infty)$. Moreover, we have

$$u_m(t) \in \mathscr{W}_1^+, \quad t \in [0,\infty)$$

and

$$\int_0^t \int_\Omega u_{ms}^2\,dx\,ds + J\big(u_m(t)\big) \le J(u_{0m}) < d, \quad t \in [0,\infty).$$

The remainder of the proof can be processed as the previous case.

Finally, we discuss the decay results.

Since $u_0 \in \mathscr{W}_1^+$, similar to the first case, we obtain $u(t) \in \mathscr{W}_1^+$ for any $t \in [0,\infty)$. By (2.3) and (4.1), we obtain

$$\begin{aligned} J(u_0) > J\big(u(t)\big) &= \frac{1}{q} I\big(u(t)\big) + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \left|\Delta u(t)\right|^2 dx + \frac{1}{q^2} \int_\Omega \left|u(t)\right|^q dx \\ &\ge \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \left|\Delta u(t)\right|^2 dx + \frac{1}{q^2} \int_\Omega \left|u(t)\right|^q dx. \end{aligned} \tag{4.13}$$

By $I(u(t)) > 0$, (2.4) and Lemma 2.1, there exists a $\overline{\lambda}_3 > 1$ such that $I(\overline{\lambda}_3 u(t)) = 0$. We have

$$\begin{aligned} d &\le J\big(\overline{\lambda}_3 u(t)\big) \\ &= \frac{1}{q} I\big(\overline{\lambda}_3 u(t)\big) + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \left|\Delta\big(\overline{\lambda}_3 u(t)\big)\right|^2 dx + \frac{1}{q^2} \int_\Omega \left|\overline{\lambda}_3 u(t)\right|^q dx \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \left|\Delta\big(\overline{\lambda}_3 u(t)\big)\right|^2 dx + \frac{1}{q^2} \int_\Omega \left|\overline{\lambda}_3 u(t)\right|^q dx \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \overline{\lambda}_3^{\,2} \int_\Omega \left|\Delta u(t)\right|^2 dx + \frac{1}{q^2} \overline{\lambda}_3^{\,q} \int_\Omega \left|u(t)\right|^q dx \\ &= \overline{\lambda}_3^{\,q} \left(\left(\frac{1}{2} - \frac{1}{q}\right) \overline{\lambda}_3^{\,2-q} \int_\Omega \left|\Delta u(t)\right|^2 dx + \frac{1}{q^2} \int_\Omega \left|u(t)\right|^q dx \right) \end{aligned}$$

$$\leq \overline{\lambda}_3^{\,q}\left(\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega \left|\Delta u(t)\right|^2\,dx+\frac{1}{q^2}\int_\Omega \left|u(t)\right|^q\,dx\right). \tag{4.14}$$

Using (4.13) and (4.14), we have

$$d \leq \overline{\lambda}_3^{\,q} J(u_0),$$

which implies that

$$\overline{\lambda}_3 \geq \left(\frac{d}{J(u_0)}\right)^{\frac{1}{q}}. \tag{4.15}$$

It follows from (2.2) that

$$\begin{aligned}
0 = I(\overline{\lambda}_3 u(t)) &= \int_\Omega \left|\Delta(\overline{\lambda}_3 u(t))\right|^2\,dx-\int_\Omega \left|\overline{\lambda}_3 u(t)\right|^q \log\left|\overline{\lambda}_3 u(t)\right|\,dx \\
&= \overline{\lambda}_3^{\,2}\int_\Omega \left|\Delta u(t)\right|^2\,dx-\overline{\lambda}_3^{\,q}\log\overline{\lambda}_3\int_\Omega \left|u(t)\right|^q\,dx \\
&\quad -\overline{\lambda}_3^{\,q}\int_\Omega \left|u(t)\right|^q\log\left|u(t)\right|\,dx \\
&= \overline{\lambda}_3^{\,q} I(u(t))+\overline{\lambda}_3^{\,2}\int_\Omega \left|\Delta u(t)\right|^2\,dx-\overline{\lambda}_3^{\,q}\int_\Omega \left|\Delta u(t)\right|^2\,dx \\
&\quad -\overline{\lambda}_3^{\,q}\log\overline{\lambda}_3\int_\Omega \left|u(t)\right|^q\,dx \\
&= \overline{\lambda}_3^{\,q} I(u(t))+\left(\overline{\lambda}_3^{\,2}-\overline{\lambda}_3^{\,q}\right)\int_\Omega \left|\Delta u(t)\right|^2\,dx \\
&\quad -\overline{\lambda}_3^{\,q}\log\overline{\lambda}_3\int_\Omega \left|u(t)\right|^q\,dx.
\end{aligned} \tag{4.16}$$

Using (4.15) and (4.16), we have

$$\begin{aligned}
\overline{\lambda}_3^{\,q} I(u(t)) &= \left(\overline{\lambda}_3^{\,q}-\overline{\lambda}_3^{\,2}\right)\int_\Omega \left|\Delta u(t)\right|^2\,dx+\overline{\lambda}_3^{\,q}\log\overline{\lambda}_3\int_\Omega \left|u(t)\right|^q\,dx \\
&\geq \left(\overline{\lambda}_3^{\,q}-\overline{\lambda}_3^{\,2}\right)\int_\Omega \left|\Delta u(t)\right|^2\,dx,
\end{aligned}$$

which implies that

$$I(u(t)) \geq \left(1-\overline{\lambda}_3^{\,2-q}\right)\int_\Omega \left|\Delta u(t)\right|^2\,dx. \tag{4.17}$$

It follows from (4.15) and (4.17) that

$$\begin{aligned}
I(u(t)) &\geq \left(1-\overline{\lambda}_3^{\,2-q}\right)\int_\Omega \left|\Delta u(t)\right|^2\,dx \\
&\geq \left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right]\int_\Omega \left|\Delta u(t)\right|^2\,dx \\
&\geq C_{23}^{-1}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right]\int_\Omega \left|Du(t)\right|^2\,dx
\end{aligned}$$

$$\geq C_{23}^{-1}C_{24}^{-1}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right]\int_\Omega \left|u(t)\right|^2\,dx, \tag{4.18}$$

where C_{23} and C_{24} are the Poincaré inequality constants. Hence, by (4.18), we obtain

$$\begin{aligned} I\bigl(u(t)\bigr) &\geq \frac{1}{3}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right]\int_\Omega \left|\Delta u(t)\right|^2\,dx \\ &\quad + \frac{1}{3}C_{23}^{-1}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right]\int_\Omega \left|Du(t)\right|^2\,dx \\ &\quad + \frac{1}{3}C_{23}^{-1}C_{24}^{-1}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right]\int_\Omega \left|u(t)\right|^2\,dx \\ &\geq C_{25}\left(\int_\Omega \left|\Delta u(t)\right|^2\,dx + \int_\Omega \left|Du(t)\right|^2\,dx + \int_\Omega \left|u(t)\right|^2\,dx\right) \\ &= C_{25}\left\|u(t)\right\|^2_{H_0^2(\Omega)}, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} C_{25} = \max&\left\{\frac{1}{3}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right], \frac{C_{21}^{-1}}{3}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right],\right. \\ &\left.\frac{C_{21}^{-1}C_{22}^{-1}}{3}\left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2}{q}-1}\right]\right\}. \end{aligned}$$

Integrating the $I(u(s))$ with respect to s over (t,T) and using the embedding $H_0^2(\Omega)\hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned} \int_t^T I\bigl(u(s)\bigr)\,ds &= -\int_t^T\int_\Omega u_s(s)u(s)\,dx\,ds = -\frac{1}{2}\int_\Omega u(T)^2\,dx + \frac{1}{2}\int_\Omega u(t)^2\,dx \\ &\leq \frac{1}{2}\int_\Omega u(t)^2\,dx \\ &\leq \frac{1}{2}C_{26}^2\left\|u(t)\right\|^2_{H_0^2(\Omega)}, \end{aligned} \tag{4.20}$$

where C_{26} is the embedding constant. From (4.19) and (4.20), we have

$$\int_t^T C_{25}\left\|u(t)\right\|^2_{H_0^2(\Omega)}\,ds \leq \frac{1}{2}C_{26}^2\left\|u(t)\right\|^2_{H_0^2(\Omega)}, \quad \text{for all } t\in[0,T]. \tag{4.21}$$

Let $T\to\infty$ in (4.21), we can get

$$\int_t^\infty \left\|u(t)\right\|^2_{H_0^2(\Omega)}\,ds \leq \frac{1}{2}C_{25}^{-1}C_{26}^2\left\|u(t)\right\|^2_{H_0^2(\Omega)}. \tag{4.22}$$

From Lemma 2.4, we have

$$\left\|u(t)\right\|^2_{H_0^2(\Omega)} \leq \left\|u(0)\right\|^2_{H_0^2(\Omega)} e^{1-\frac{2t}{C_{25}^{-1}C_{26}^2}}, \quad t\in[0,\infty).$$

The above inequality implies that the solution $u(t)$ decays exponentially. □

5 Blow up

Theorem 5.1 *If $u_0 \in \mathscr{W}_1^-$, the unique local weak solution $u(t)$ of the problem (1.1) blows up in finite time, i.e., there exists a $T_* > 0$ such that*

$$\lim_{t\to T_*^-}\int_\Omega |u(t)|^2\,dx = \infty.$$

Proof Since $u_0 \in \mathscr{W}_1^-$, it follows from the local existence that there exists a unique local weak solution $u(t)$ of the problem (1.1) such that

$$\int_0^t\int_\Omega u_s^2\,dx\,ds + J(u(t)) \le J(u_0) < d, \quad t\in[0,T_{\max}]. \tag{5.1}$$

Next, we prove $u(t)\in\mathscr{W}_1^-$ for $t\in[0,T_{\max}]$. We assume $u(t)$ leaves $\mathscr{W}_1^+$ at time $t=t_1$, then there exists a sequence $\{t_n\}$ such that $I(u(t_n))\le 0$ as $t_n\to t_1^-$. It follows from lower semi-continuity of L^2 norm that

$$I(u(t_1)) \le \liminf_{n\to\infty} I(u(t_n)) \le 0. \tag{5.2}$$

We have $I(u(t_1))=0$ according to $u(t_1)\notin\mathscr{W}_1^+$. By (2.4) and (5.1), we have

$$d = \inf_{u\in\mathscr{N}} J(u) \le J(u(t_1)) < d,$$

which is a contradiction. So, $u(t)\in\mathscr{W}_1^-$ for $t\in[0,T_{\max}]$. Next, we will study that $u(t)$ blows up in finite time by contradiction. Thus, we assume $u(t)$ is global. We contract a function $\Phi:[0,\infty)\to\mathbb{R}^+$, and

$$\Phi(t) = \int_0^t\int_\Omega u^2\,dx\,ds. \tag{5.3}$$

We can easily obtain

$$\Phi'(t) = \int_\Omega u^2\,dx. \tag{5.4}$$

By (2.2) and (5.4), we have

$$\Phi''(t) = 2\int_\Omega uu_t\,dx = 2\int_\Omega |u|^{q-2}u\log|u|\,dx - 2\int_\Omega |\Delta u|^2\,dx = -2I(u). \tag{5.5}$$

From $u(t)\in\mathscr{W}_1^-$ and (5.5), we can obtain

$$\Phi''(t) > 0. \tag{5.6}$$

Thus, it follows from $u_0\in\mathscr{W}_1^-$ and (5.4) that

$$\Phi'(t) \ge \Phi'(0) = \int_\Omega u_0^2\,dx > 0. \tag{5.7}$$

Using the Hölder inequality and combining (5.5), we have

$$\begin{aligned}\frac{1}{4}\big(\Phi'(t)-\Phi'(0)\big)^2&=\frac{1}{4}\left(\int_0^t\Phi''(s)\,ds\right)^2=\left(\int_0^t\int_\Omega uu_s\,dx\,ds\right)^2\\&\le\int_0^t\int_\Omega u^2\,dx\,ds\int_0^t\int_\Omega u_s^2\,dx\,ds.\end{aligned}\tag{5.8}$$

By (2.3) and (5.5), we have

$$\begin{aligned}\Phi''(t)&=-2I(u)=-2qJ(u)+2q\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega|\Delta u|^2\,dx+\frac{2}{q}\int_\Omega|u|^q\,dx\\&=-2qJ(u_0)+2q\int_0^t\int_\Omega u_s^2\,dx\,ds\\&\quad+2q\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega|\Delta u|^2\,dx+\frac{2}{q}\int_\Omega|u|^q\,dx.\end{aligned}\tag{5.9}$$

Since $u(t)\in\mathscr{W}_1^-$, $I(u(t))<0$. By Lemma 2.1, there exists a $\overline{\lambda}_4$, $0<\overline{\lambda}_4<1$ such that $I(\overline{\lambda}_4u(t)))=0$. It follows from (2.3) and (2.4) that

$$\begin{aligned}d&=\inf_{u\in\mathcal{N}}J(u)\le J\big(\overline{\lambda}_4u(t)\big)\\&=\frac{1}{q}I\big(\overline{\lambda}_4u(t)\big)+\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega\big|\Delta\big(\overline{\lambda}_4u(t)\big)\big|^2\,dx+\frac{1}{q^2}\int_\Omega\big|\overline{\lambda}_4u(t)\big|^q\,dx\\&=\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega\big|\Delta\big(\overline{\lambda}_4u(t)\big)\big|^2\,dx+\frac{1}{q^2}\int_\Omega\big|\overline{\lambda}_4u(t)\big|^q\,dx\\&=\overline{\lambda}_4^2\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega\big|\Delta u(t)\big|^2\,dx+\overline{\lambda}_4^q\frac{1}{q^2}\int_\Omega\big|u(t)\big|^q\,dx\\&\le\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega\big|\Delta u(t)\big|^2\,dx+\frac{1}{q^2}\int_\Omega\big|u(t)\big|^q\,dx.\end{aligned}\tag{5.10}$$

Combining (5.9) with (5.10), we have

$$\begin{aligned}\Phi''(t)&=-2qJ(u_0)+2q\int_0^t\int_\Omega u_s^2\,dx\,ds+2q\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega|\Delta u|^2\,dx+\frac{2}{q}\int_\Omega|u|^q\,dx\\&=-2qJ(u_0)+2q\int_0^t\int_\Omega u_s^2\,dx\,ds\\&\quad+2q\left[\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega\big|\Delta u(t)\big|^2\,dx+\frac{1}{q^2}\int_\Omega\big|u(t)\big|^q\,dx\right]\\&\ge2q\big(d-J(u_0)\big)+2q\int_0^t\int_\Omega u_s^2\,dx\,ds.\end{aligned}\tag{5.11}$$

Using (5.3), (5.8) and (5.11), we have

$$\begin{aligned}\Phi(t)\Phi''(t)&=\int_0^t\int_\Omega u^2\,dx\,ds\Phi''(t)\\&\ge\int_0^t\int_\Omega u^2\,dx\,ds\left[2q\big(d-J(u_0)\big)+2q\int_0^t\int_\Omega u_s^2\,dx\,ds\right]\end{aligned}$$

$$\geq \Phi(t)2q\big(d-J(u_0)\big)+\frac{q}{2}\big(\Phi'(t)-\Phi'(0)\big)^2. \tag{5.12}$$

We fix a $t_2>0$. It follows from (5.7) that we have

$$\Phi(t)\geq \Phi(t_2)=\int_0^{t_2}\int_\Omega u^2\,dx\,ds\geq t_2\int_\Omega u_0^2\,dx>0,\quad \text{for } t\in[t_2,\infty). \tag{5.13}$$

Hence, by (5.12) and (5.13), we have

$$\begin{aligned}\Phi(t)\Phi''(t)-\frac{q}{2}\big(\Phi'(t)-\Phi'(0)\big)^2 &\geq \Phi(t)2q\big(d-J(u_0)\big)\\ &\geq t_2\int_\Omega u_0^2\,dx>0,\quad \text{for } t\in[t_2,\infty).\end{aligned} \tag{5.14}$$

We choose $T>t_2$ sufficiently large and contract a function $\Psi(t)$ as follows:

$$\Psi(t)=\Phi(t)+(T-t)\int_\Omega u_0^2\,dx,\quad t\in[t_2,T]. \tag{5.15}$$

From (5.13) and (5.15), we can easily see that for any $t\in[t_2,T]$, $\Psi(t)\geq\Phi(t)>0$ holds. It follows from (5.4) and (5.14) that, for any $t\in[t_2,T]$, $\Psi'(t)=\Phi'(t)-\Phi'(0)$ holds, thus we also have $\Psi''(t)=\Phi''(t)>0$ from (5.6). Thus, we can obtain from (5.14)

$$\begin{aligned}\Psi(t)\Psi''(t)-\frac{q}{2}\Psi'(t)^2 &\geq \Phi(t)\Phi''(t)+\frac{q}{2}\big(\Phi'(t)-\Phi'(0)\big)^2\\ &\geq \Phi(t)2q\big(d-J(u_0)\big)\\ &\geq t_2\int_\Omega u_0^2\,dx>0,\end{aligned} \tag{5.16}$$

for $t\in[t_2,T]$. Let $\chi(t)=\Psi(t)^{-\frac{q-2}{2}}$. Thus,

$$\chi'(t)=-\frac{q-2}{2}\Psi(t)^{-\frac{q}{2}}\Psi'(t). \tag{5.17}$$

From (5.16) and (5.17), we have

$$\begin{aligned}\chi''(t)&=\frac{q(q-2)}{4}\Psi(t)^{-\frac{q+2}{2}}\Psi'(t)^2-\frac{q-2}{2}\Psi(t)^{-\frac{q}{2}}\Psi''(t)\\ &=\frac{q-2}{2}\Psi(t)^{-\frac{q+2}{2}}\left[\frac{q}{2}\Psi'(t)^2-\Psi(t)\Psi''(t)\right]<0,\end{aligned} \tag{5.18}$$

for $t\in[t_2,T]$. This shows that, for any sufficiently large $T>t_2$, $\chi(t)$ is a concave function in $[t_2,T]$. $\chi(t_2)>0$ and $\chi''(t_2)<0$, so there exists a finite time $T_*>t_2>0$ such that

$$\lim_{t\to T_*^-}\chi(t)=0,$$

which implies

$$\lim_{t\to T_*^-}\Psi(t)=\infty.$$

Hence, we have

$$\lim_{t\to T_*^-}\int_0^t\int_\Omega u^2\,dx\,ds=\infty,$$

i.e.,

$$\lim_{t\to T_*^-}\int_\Omega u^2\,dx=\infty.$$

This is a contradiction to our assumption. □

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